On the spectral decomposition of the complex Robin Laplacian

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Abstract

The mathematical properties of the Laplacian on a bounded domain are well-known when the boundary condition is of the first type (Dirichlet), or second type (Neumann). In both cases, this operator is self-adjoint and, therefore, diagonalizable, its spectrum is discrete, and the set of eigenfunctions can be chosen to form an orthonormal basis of the Hilbert space of square-integrable functions on the domain. However, in the case of the third type (Robin) boundary condition, the same is true only when the parameter is real-valued. On the contrary, when this parameter is complex-valued, the Laplacian may not even be diagonalizable. In this paper, the spectral decomposition of the complex Robin Laplacian is investigated in the most general case possible, and a formula that decomposes any square-integrable function on the set of its (generalized) eigenfunctions is provided. This result is applied to the Green's function of the Helmholtz equation, whose existence, unicity and closed-form expression are established in this general setting, and the statistical wave field theory, which provides the statistical laws of waves propagating in a bounded domain.

Keywords: Robin Laplacian, Helmholtz equation, Green's function, Statistical wave field theory

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1. Introduction

In an old book dating back to 1953, Morse and Feshback showed that various linear partial differential equations in physics, including the Helmholtz, diffusion, and wave equations, could be expressed in a unified abstract operator form. This approach permitted them to investigate the Green's functions of these equations through a common mathematical framework, including cases of self-adjoint and non-self-adjoint linear operators (Morse and Feshbach, 1953, Sec. 7.5). Unfortunately, it turned out that in the case of non-self-adjoint operators, their mathematical developments, which were based on supposedly convergent series expansions on biorthogonal sets of eigenfunctions, were flawed, therefore, so was their closed-form expression of the Green's function (Morse and Feshbach, 1953, p. 884). Indeed, Kostenbauder *et al.* (1997) exhibited examples in physics of diverging series expansions involving non-self-adjoint linear operators. This possible pathological behavior of eigenmode expansions has somewhat restrained the use of the modal approach for many years in acoustics. One of our objectives here is, thus, to rehabilitate this approach by showing that all mathematical issues can be simply overcome in the case of the complex Robin Laplacian.

Let us quickly summarize the known mathematical properties of the Robin Laplacian. Given a bounded domain $\Omega \subset \mathbb{R}^d$, with $d \geq 1$, whose boundary $\partial\Omega$ is Lipschitz continuous, and an essentially bounded function $\alpha : \partial\Omega \to \mathbb{C}$, the Robin Laplacian $-\Delta_\Omega^\alpha$ is a linear operator defined on a dense subset $D(-\Delta_\Omega^\alpha)$ of the Hilbert space $L^2(\Omega)$ of square-integrable functions on Ω, which is such that $\forall \psi \in D(-\Delta_\Omega^\alpha)$, $\frac{\partial\psi(\boldsymbol{x})}{\partial\boldsymbol{n}(\boldsymbol{x})} + \alpha(\boldsymbol{x})\psi(\boldsymbol{x}) = 0$ on $\partial\Omega$, where $\frac{\partial\psi(\boldsymbol{x})}{\partial\boldsymbol{n}(\boldsymbol{x})}$ denotes the outer normal derivative of function ψ. When function $\alpha(\boldsymbol{x})$ is real-valued, then it is well-known that the Robin Laplacian is self-adjoint and, therefore, diagonalizable[1], its spectrum $\{\lambda_n\}_{n\in\mathbb{N}}$ is discrete, consisting of real eigenvalues of finite multiplicity, and its real-valued eigenfunctions $\{\varphi_n\}_{n\in\mathbb{N}}$, which are such that $\Delta_\Omega^\alpha\varphi_n + \lambda_n\varphi_n = 0$, can be chosen to form an orthonormal basis of $L^2(\Omega)$, which can be written as

$$\forall \boldsymbol{x}, \boldsymbol{y} \in \Omega, \ \sum_{n\in\mathbb{N}} \varphi_n(\boldsymbol{x})\varphi_n(\boldsymbol{y}) = \delta(\boldsymbol{x} - \boldsymbol{y}). \tag{1}$$

However, when function $\alpha(\boldsymbol{x})$ is complex-valued, the Robin Laplacian is no longer self-adjoint, hence its eigenvalues are generally complex, and it may not be diagonalizable. The purpose of this paper is, then, to investigate its *Jordan* decomposition, i.e., to consider the possible existence of generalized complex eigenfunctions, which are such that $(\Delta_\Omega^\alpha + \lambda I)^m\varphi = 0$ for some $m > 1$ (where I denotes the identity), but $(\Delta_\Omega^\alpha + \lambda I)\varphi \neq 0$. In this complex case,

[1] In this context, *diagonalizable* means that there is a unitary operator U such that $U\,(-\Delta_\Omega^\alpha)\,U^{-1}$ is a diagonal operator on $L^2(\Omega)$ (with respect to a fixed orthonormal basis of that space).

we will observe that the set of (generalized) eigenfunctions can never be orthonormal with respect to the Hermitian inner product, and it may not even form a basis of $L^2(\Omega)$. However, it has been recently proved in Bögli *et al.* (2022) that the set of (generalized) eigenfunctions of the complex Robin Laplacian can always be chosen to form an *Abel basis* of $L^2(\Omega)$, a notion that involves a weaker form of convergence than the usual convergence in $L^2(\Omega)$.

Yet, it is not clear whether Eq. (1) still holds when $\alpha(\boldsymbol{x})$ is complex-valued, i.e., whether

$$\forall \psi \in L^2(\Omega),\ \psi(\boldsymbol{x}) = \sum_{n\in\mathbb{N}} \langle \psi, \overline{\varphi_n} \rangle \, \varphi_n(\boldsymbol{x}), \tag{2}$$

where $\langle \psi_1, \psi_2 \rangle = \int_\Omega \psi_1(\boldsymbol{x})\overline{\psi_2(\boldsymbol{x})} d\boldsymbol{x}$ denotes the Hermitian inner product on the Hilbert space $L^2(\Omega)$. In particular, we do not know if

1. The series in Eq. (2) converges in $L^2(\Omega)$;
2. the dual set of $\{\varphi_n\}_{n\in\mathbb{N}}$ [i.e., $\{\overline{\varphi_n}\}_{n\in\mathbb{N}}$ in Eq. (2)] is unique; and
3. for any simple eigenvalue with eigenfunction φ_n, $\int_\Omega \varphi_n(\boldsymbol{x})^2 d\boldsymbol{x} \neq 0$ [the contrary would prevent the term $\varphi_n(\boldsymbol{x})\varphi_n(\boldsymbol{y})$ in Eq. (1) from being a projection onto the vector space spanned by φ_n].

Nevertheless, in Badeau (2025a, Proposition 1), we were able to prove the following result (we focused on the case $d = 3$, but the same proof actually holds $\forall d \geq 1$).

Proposition 1. *Let $\Omega \subset \mathbb{R}^d$ where $d \geq 1$ be a bounded domain whose boundary $\partial\Omega$ is Lipschitz continuous, and $\alpha \in L^\infty(\partial\Omega)$ be a complex-valued function. Assume that*

a) The Robin Laplacian is diagonalizable; and

b) the set of eigenfunctions $\{\varphi_n(\boldsymbol{x})\}_{n\in\mathbb{N}}$ forms a basis[2] *of $L^2(\Omega)$.*

Then, without loss of generality, the set $\{\varphi_n(\boldsymbol{x})\}_{n\in\mathbb{N}}$ can be chosen to form a pseudo-orthonormal[3] *basis of $L^2(\Omega)$, which means that Eq.* (1) *holds.*

However, the current state of mathematical knowledge does not tell us whether the two assumptions in Proposition 1 hold in all generality.

- Even though a) is a very reasonable assumption that certainly holds almost surely in some mathematical sense, Bögli *et al.* (2022, p. 12) exhibited a counterexample of non-diagonalizable Robin Laplacian in one dimension based on the study in Krejčiřík *et al.* (2006), in which two different Neumann eigenvalues are mapped to the same Robin eigenvalue, creating a nontrivial eigennilpotent; and

[2]See Definition 1.
[3]See Definition 7.

- in the same way, assumption b) seems to be reasonable, because it is known to hold when $\alpha(\boldsymbol{x}) \in \mathbb{R}$ for any domain Ω (hence, we expect it to still hold at least when $\alpha(\boldsymbol{x}) \in \mathbb{C}$ stays close to the real axis, as the eigenprojections are holomorphic functions of α, as shown in Bögli *et al.* (2022, Theorem 1.1)), and also for any $\alpha(\boldsymbol{x}) \in \mathbb{C}$ for some particular geometries of Ω, including the rectangular cuboid. Yet, although we are not aware of any counterexample of a set of eigenfunctions of the Robin Laplacian that would not form a basis of $L^2(\Omega)$, there exists no mathematical guarantee that assumption b) holds in the general case.

Therefore, in this paper, to address the most general case possible given the current state of mathematical knowledge, we consider the possibility that the Robin Laplacian is not diagonalizable, and the set of (generalized) eigenfunctions is not a basis of $L^2(\Omega)$ but only an Abel basis as proved in Bögli *et al.* (2022). We will, thus, in our Theorem 2, introduce a generalized version of Eq. (1) that is guaranteed to hold in all cases, and this theorem will also provide answers to the three previous questions.

1. We still do not know whether the series in Eq. (2) converges in all cases, but introducing proper weights in front of each term of this series guarantees its convergence;
2. yes, the dual set is unique; and
3. yes, for any simple eigenvalue with eigenfunction φ_n, $\int_\Omega \varphi_n(\boldsymbol{x})^2 d\boldsymbol{x} \neq 0$ (this assertion answers the Open Problem 4.9 in Bögli *et al.* (2022)).

Then we will show how this new mathematical result applies to the Green's function of the Helmholtz equation, and in our Proposition 2, we will provide the correct closed-form expression of this Green's function that holds in the general case. We will also show how this result applies to the statistical wave field theory recently proposed by the author, which establishes the statistical laws of waves propagating in a bounded domain (Badeau, 2024, 2025a,b).

This paper is organized as follows. In Sec. 2, we introduce mathematical notations that will be used in the rest of the paper. Then Sec. 3 summarizes the main definitions and mathematical results that are needed to introduce our main theorem. This theorem is then stated and proved in Sec. 4. The applications of this theorem to the Green's function of the Helmholtz equation and the statistical wave field theory are presented in Sec. 5. Finally, in Sec. 6, we summarize the main contributions of this paper, and propose perspectives for future work.

2. Mathematical notations

- $\triangleq$: equal by definition to;
- $\overline{z}$: conjugate of the complex number z;
- $\boldsymbol{x}^\top$: transpose of vector $\boldsymbol{x}$;

- $A \subseteq B$: A is a subset of B, possibly equal to B;
- $\|.\|_E$: norm in the Banach space E;
- $l^2(\mathbb{N})$: Hilbert space of sequences $\{h_n\}_{n\in\mathbb{N}}$ such that $\|\{h_n\}_{n\in\mathbb{N}}\|^2_{l^2(\mathbb{N})} \triangleq \sum_{n\in\mathbb{N}} |h_n|^2 < +\infty$;
- $\partial\Omega$: boundary of the bounded open set $\Omega \subset \mathbb{R}^d$;
- $\overline{\Omega}$: closure of the bounded open set $\Omega \subset \mathbb{R}^d$;
- S: $(d-1)$-dimensional Hausdorff measure on $\partial\Omega$;
- $\frac{\partial\psi(\boldsymbol{x})}{\partial\boldsymbol{n}(\boldsymbol{x})}$: outer normal derivative of ψ on $\partial\Omega$;
- $C^0(\Omega)$: class of continuous functions on Ω;
- $L^2(\Omega)$: Hilbert space of square-integrable functions on Ω: $\forall\psi \in L^2(\Omega)$, $\|\psi\|^2_{L^2(\Omega)} \triangleq \int_\Omega |\psi(\boldsymbol{x})|^2 d\boldsymbol{x} < +\infty$;
- $H^1(\Omega)$: Sobolev space of functions in $L^2(\Omega)$ whose gradient is also in $L^2(\Omega)$: $\forall\psi \in H^1(\Omega)$, $\|\psi\|^2_{H^1(\Omega)} \triangleq \|\psi\|^2_{L^2(\Omega)} + \|\nabla\psi\|^2_{L^2(\Omega)} < +\infty$;
- $L^\infty(\partial\Omega)$: Lebesgue space of essentially bounded functions[4] on $\partial\Omega$: $\|\alpha\|_{L^\infty(\partial\Omega)} \triangleq \operatorname{ess\,sup}_{\boldsymbol{x}\in\partial\Omega} |\alpha(\boldsymbol{x})| < +\infty$;
- I: identity operator on $L^2(\Omega)$;
- $\langle\psi_1, \psi_2\rangle = \int_\Omega \psi_1(\boldsymbol{x})\overline{\psi_2(\boldsymbol{x})} d\boldsymbol{x}$: Hermitian inner product on $L^2(\Omega)$;
- $-\Delta^\alpha_\Omega$: Robin Laplacian on Ω of parameter $\alpha : \partial\Omega \to \mathbb{C}$;
- $D(T)$: domain of definition of operator T;
- $\{\lambda_n\}_{n\in\mathbb{N}}$: eigenvalues of the Robin Laplacian;
- $\{\varphi_n\}_{n\in\mathbb{N}}$: eigenfunctions of the Robin Laplacian; and
- δ: Dirac delta function.

3. Mathematical framework

In this section, we first define different kinds of bases of separable Hilbert spaces that are related to our problem (Sec. 3.1), and then we summarize the known properties of the complex Robin Laplacian that are needed to introduce our main theorem (Sec. 3.2).

[4]An *essentially bounded* function $\alpha : \partial\Omega \to \mathbb{C}$ is a measurable function that is bounded on a measurable subset E of $\partial\Omega$ such that the set $\partial\Omega\backslash E$ is negligible. Then the notation "ess sup" is defined as $\operatorname{ess\,sup}_{\boldsymbol{x}\in\partial\Omega} |\alpha(\boldsymbol{x})| = \sup_{\boldsymbol{x}\in E} |\alpha(\boldsymbol{x})|$.

3.1. Bases of separable Hilbert spaces

Let us introduce the standard notion of *basis* of a separable complex Hilbert space (Bögli *et al.*, 2022, Sec. 5).

Definition 1 (Basis)**.** *Let H be a separable complex Hilbert space. A set of vectors $\mathcal{B} = \{\boldsymbol{e}_n\}_{n\in\mathbb{N}}$ is a* basis *of H if and only if every vector $\boldsymbol{h} \in H$ admits a unique convergent series representation*

$$\boldsymbol{h} = \textstyle\sum_{n\in\mathbb{N}} h_n \boldsymbol{e}_n, \tag{3}$$

where $h_n \in \mathbb{C}$ $\forall n \in \mathbb{N}$.

The next definition is given in Hussein *et al.* (2015, Sec. 5).

Definition 2 (Riesz basis with brackets)**.** *Let H be a separable complex Hilbert space. A set of vectors $\mathcal{B} = \{\boldsymbol{e}_n\}_{n\in\mathbb{N}}$ is called a* Riesz basis with brackets *(or a* Riesz basis of subspaces*) of H if and only if there is an increasing sequence of natural numbers N_m (with $N_0 = 0$) such that every vector $\boldsymbol{h} \in H$ admits a unique* unconditionally *convergent series representation*

$$\boldsymbol{h} = \textstyle\sum_{m\in\mathbb{N}} \left[\sum_{n=N_m}^{N_{m+1}-1} h_n \boldsymbol{e}_n\right]$$

(i.e., all reorderings of the series over m converge to the same limit), where $h_n \in \mathbb{C}$ $\forall n \in \mathbb{N}$. If $N_m = m$ $\forall m \in \mathbb{N}$, we retrieve the standard notion of a Riesz basis.

Considering Definition 2, we must understand that the usual notion of *basis* introduced in Definition 1 may involve a series that is only *conditionally* convergent [i.e., the limit of the series in Eq. (3) may depend on the particular ordering of the basis vectors $\boldsymbol{e}_n$]. Therefore all Riesz bases are bases, but not all bases are Riesz bases. Nevertheless, it is well-known that all *orthonormal* bases are Riesz bases (Bögli *et al.*, 2022, Sec. 5).

In contrast to the notions of bases and Riesz bases, an *Abel basis* is always defined with respect to the eigenvectors and generalized eigenvectors $\{\boldsymbol{e}_n\}_{n\in\mathbb{N}}$ of a densely defined operator T (i.e., an operator defined on a dense subset of H). The intuitive idea is that the formal series expansion of a vector $\boldsymbol{h} \in H$ in Eq. (3) may not converge, but if every coefficient h_n is multiplied by a suitable weight $e^{-w_n\varepsilon} \in \mathbb{C}$, then the series $\boldsymbol{h}(\varepsilon) = \sum_{n\in\mathbb{N}} h_n e^{-w_n\varepsilon}\boldsymbol{e}_n$ does converge $\forall\varepsilon > 0$, and then $\boldsymbol{h}(\varepsilon) \to \boldsymbol{h}$ when $\varepsilon \to 0^+$. Note that an Abel basis is generally not a basis in the sense of Definition 1 because the unweighted series expansion $\sum_{n\in\mathbb{N}} h_n\boldsymbol{e}_n$ is not required to converge.

Definition 3 (Abel basis with brackets)**.** *Let H be a separable complex Hilbert space and $T : D(T) \to H$ be a linear operator defined on a dense subset $D(T) \subset H$ with purely discrete spectrum, such that all but finitely many of its eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ lie in the sector $T_\theta^+ \triangleq \{z \in \mathbb{C} : |\arg z| < \theta\}$ for some $\theta \in (0, \pi)$. Then the set of (generalized) eigenvectors $\{\boldsymbol{e}_n\}_{n\in\mathbb{N}}$ forms an* Abel basis with brackets *of H of order $\gamma \geq 0$ if and only if $\gamma\theta < \pi/2$ and if there is an increasing sequence of natural numbers N_m (with $N_0 = 0$) such that $\forall\boldsymbol{h} \in H$, there is a sequence of coefficients $h_n \in \mathbb{C}$ such that the series*

$$\boldsymbol{h}(\varepsilon) \triangleq \textstyle\sum_{m\in\mathbb{N}} \left[\sum_{n=N_m}^{N_{m+1}-1} h_n e^{-w_n\varepsilon} \boldsymbol{e}_n\right] \tag{4}$$

converges $\forall \varepsilon > 0$, *and* $\lim_{\varepsilon \to 0^+} \boldsymbol{h}(\varepsilon) = \boldsymbol{h}$. *In Eq. (4), the weights* $w_n \in \mathbb{C}$ *are such that* $\forall \lambda_n \in T_\theta^+$, $w_n = \lambda_n^\gamma$, *and* $\forall \lambda_n \notin T_\theta^+$, $w_n = 0$. *If* $N_m = m$ $\forall m \in \mathbb{Z}$, *we get the standard notion of an* Abel basis.

Note that in Eq. (4), the series over m may be only conditionally convergent, as in Definition 1. In other respects, it follows from Definitions 2 and 3 that a Riesz basis, if it consists of the generalized eigenfunctions of a suitable operator, is always an Abel basis of order zero.

3.2. Known properties of the complex Robin Laplacian

We consider a bounded *Lipschitz domain* $\Omega \subset \mathbb{R}^d$ with $d \geq 1$, which means that Ω is a non-empty bounded open set with a finite number of connected components, and its boundary $\partial\Omega$ is a Lipschitz continuous submanifold of $\mathbb{R}^d$, i.e., it is locally the graph of a Lipschitz function[5]. We then consider the Sobolev space $H^1(\Omega)$ of functions in $L^2(\Omega)$ whose gradient is also in $L^2(\Omega)$. Before defining the Robin Laplacian, we first need to introduce the notions of *trace* and *normal derivative*, as defined in Arendt and ter Elst (2011).

Definition 4 (Trace)**.** *Let* $\Omega \subset \mathbb{R}^d$ *with* $d \geq 1$ *be a bounded Lipschitz domain. Then, every function* $\psi \in H^1(\Omega)$ *admits a unique trace on* $\partial\Omega$*, which is defined as a function* $\varphi \in L^2(\partial\Omega)$ *such that there is a sequence of functions* $\psi_n \in H^1(\Omega) \cap C^0(\overline{\Omega})$ *such that* $\psi_n \to \psi$ *in* $H^1(\Omega)$ *and* $\psi_n|_{\partial\Omega} \to \varphi$ *in* $L^2(\partial\Omega)$.

From now on, the trace of any function $\psi \in H^1(\Omega)$ will be simply denoted with the same notation ψ.

Definition 5 (Normal derivative)**.** *Let* $\Omega \subset \mathbb{R}^d$ *with* $d \geq 1$ *be a bounded Lipschitz domain. Then every function* $\psi \in H^1(\Omega)$ *such that* $\Delta\psi \in L^2(\Omega)$ *admits a unique* normal derivative $\frac{\partial \psi(\boldsymbol{x})}{\partial \boldsymbol{n}(\boldsymbol{x})} \in L^2(\partial\Omega)$, *which is defined in the weak sense as* $\forall \varphi \in H^1(\Omega)$,

$$\int_{\partial\Omega} \frac{\partial \psi(\boldsymbol{x})}{\partial \boldsymbol{n}(\boldsymbol{x})} \overline{\varphi(\boldsymbol{x})} dS(\boldsymbol{x}) = \int_\Omega \left(\nabla\psi(\boldsymbol{x})^\top \overline{\nabla\varphi(\boldsymbol{x})} + \Delta\psi(\boldsymbol{x}) \overline{\varphi(\boldsymbol{x})} \right) d\boldsymbol{x}. \tag{5}$$

We can now define the Robin Laplacian, following the discussion in Bögli *et al.* (2022, Sec. 3), which is based on the mathematical framework of Kato (1976).

Definition 6 (Robin Laplacian)**.** *Let* $\Omega \subset \mathbb{R}^d$ *with* $d \geq 1$ *be a bounded Lipschitz domain, and let* $\alpha \in L^\infty(\partial\Omega)$ *be a complex-valued function. The sesquilinear form* $a_\alpha : H^1(\Omega) \times H^1(\Omega) \to \mathbb{C}$ *defined as*

$$a_\alpha[\psi_1, \psi_2] = \int_{\partial\Omega} \alpha(\boldsymbol{x}) \psi_1(\boldsymbol{x}) \overline{\psi_2(\boldsymbol{x})} dS(\boldsymbol{x}) + \int_\Omega \nabla\psi_1(\boldsymbol{x}) \overline{\nabla\psi_2(\boldsymbol{x})} d(\boldsymbol{x}), \tag{6}$$

[5] That means that for any point $\boldsymbol{x} \in \partial\Omega$, there is a neighborhood $\mathcal{V} \subset \mathbb{R}^d$ of $\boldsymbol{x}$, a local Cartesian coordinate system $(x_1, \ldots, x_d)$, and a Lipschitz function $f : \mathbb{R}^{d-1} \to \mathbb{R}$, such that $\forall \boldsymbol{y} \in \partial\Omega \cap \mathcal{V}$, $y_d = f(y_1, \ldots, y_{d-1})$.

will be referred to as the Robin form. *Then the* Robin Laplacian $-\Delta_\Omega^\alpha$ *is a linear operator defined on a dense subset* $D(-\Delta_\Omega^\alpha)$ *of the separable Hilbert space* $H = L^2(\Omega)$*, which is such that* $\forall \psi_1 \in D(-\Delta_\Omega^\alpha) \subset H^1(\Omega)$*,* $\forall \psi_2 \in H^1(\Omega)$*,*

$$a_\alpha[\psi_1, \psi_2] = \int_\Omega (-\Delta_\Omega^\alpha \psi_1)(\boldsymbol{x})\, \overline{\psi_2(\boldsymbol{x})} d\boldsymbol{x}, \tag{7}$$

where

$$D(-\Delta_\Omega^\alpha) = \left\{ \psi \in H^1(\Omega) : \Delta\psi \in L^2(\Omega) \text{ and } \frac{\partial \psi(\boldsymbol{x})}{\partial \boldsymbol{n}(\boldsymbol{x})} + \alpha(\boldsymbol{x})\psi(\boldsymbol{x}) = 0 \text{ on } \partial\Omega \right\}. \tag{8}$$

Finally, a function $\varphi \in D(-\Delta_\Omega^\alpha)$ *is an eigenfunction of the Robin Laplacian of eigenvalue* $\lambda \in \mathbb{C}$ *if and only if* $\Delta_\Omega^\alpha \varphi + \lambda\varphi = 0$ *in* $L^2(\Omega)$*.*

The following theorem summarizes the main results presented in Bögli *et al.* (2022), which we need in this paper.

Theorem 1. *Let* $\Omega \subset \mathbb{R}^d$*,* $d \geq 1$*, be a bounded Lipschitz domain, and* $\alpha \in L^\infty(\partial\Omega)$*. Then,*

1. *The spectrum of the Robin Laplacian is discrete, consisting of eigenvalues of finite algebraic multiplicity, without any finite point of accumulation;*

2. *the Robin Laplacian is self-adjoint if and only if function* α *is real-valued. In this case, it is diagonalizable and its eigenfunctions can be chosen to form an orthonormal basis of* $L^2(\Omega)$*; and*

3. *if function* α *is not real-valued, then without loss of generality, the set of (generalized) eigenfunctions can be chosen so as to form an Abel basis with brackets of* $L^2(\Omega)$ *of order* $\gamma = (d-1)/2 + \eta$ *for any* $\eta > 0$ *and any* $\theta \in (0, \frac{\pi}{2\gamma}\min(\gamma, 1))$*, and even a Riesz basis with brackets if* $d = 1$*. However, they cannot be chosen to form an orthonormal basis of* $L^2(\Omega)$*.*

Remark 1. *The definition of an Abel basis with brackets (Definition 3) involves a linear operator* T*. In assertion 3 of Theorem 1, this linear operator is actually* $T = -\Delta_\Omega^\alpha + \omega I$ *for a well-chosen constant* $\omega \geq 0$*, as explained in the proof of Bögli* et al. *(2022, Theorem 5.7), which is based on the main theorem in Agranovich (1994). Indeed, this theorem deals with sesquilinear forms in Hilbert spaces that are continuous and coercive*[6] *(Agranovich, 1994, p. 151), and Bögli* et al. *(2022, p. 17) show that it is always possible to choose a real constant* $\omega \geq 0$ *high enough such that the sesquilinear form defined by the operator* T *satisfies this condition. Note that this operator has the same (generalized) eigenfunctions as the Robin Laplacian* $-\Delta_\Omega^\alpha$*, but its spectrum is shifted by the constant* ω*. In a few places in the remainder of this paper, we will cite directly the reference Agranovich (1994), to refer to specific results and proofs that do not appear in Bögli* et al. *(2022).*

[6] A *coercive* sesquilinear form $a_\alpha : H^1(\Omega) \times H^1(\Omega) \to \mathbb{C}$ is such that there is a constant $\varepsilon > 0$ such that $\forall \psi \in H^1(\Omega)$, $\varepsilon \|\psi\|^2_{H^1(\Omega)} \leq \mathrm{Re}(a_\alpha[\psi, \psi])$.

Remark 2. *The assertions in Theorem 1 were formulated in Bögli* et al. *(2022) by assuming that the complex function* α *is constant, but the authors explicitly stated that assertions 1 and 2 still hold when* $\alpha \in L^\infty(\partial\Omega)$*. Moreover, assertion 3 also holds when* $\alpha \in L^\infty(\partial\Omega)$ *because the proof of Bögli* et al. *(2022, Theorem 5.7) trivially generalizes to this case*[7]*. In other respects, note that the original statement of Theorem 5.7 in Bögli* et al. *(2022) neglected to refer to bases* with brackets*, but the original theorem in Agranovich (1994) and its proof explicitly deal with such bases. That is why the statement of our Theorem 1 does involve bases with brackets.*

4. Spectral decomposition of the complex Robin Laplacian

Before introducing our main theorem, let us first introduce the notion of *pseudo-orthogonality*:

Definition 7 (Pseudo-orthogonality)**.** *Let* $\Omega \subset \mathbb{R}^d$*,* $d \geq 1$*, be a bounded Lipschitz domain. Two functions* $\psi_1, \psi_2 \in L^2(\Omega)$ *are said to be* pseudo-orthogonal *if and only if* $\int_\Omega \psi_1(\boldsymbol{x})\psi_2(\boldsymbol{x})d\boldsymbol{x} = 0$ *(which means that* ψ_1 *and* $\overline{\psi_2}$ *are orthogonal). In the same way, a function* $\psi \in L^2(\Omega)$ *is said to be* pseudo-unitary *if and only if* $\int_\Omega \psi(\boldsymbol{x})^2 d\boldsymbol{x} = 1$*. Finally, a pseudo-orthogonal set of pseudo-unitary functions will be said to be* pseudo-orthonormal*.*

We can now state our main theorem:

Theorem 2 (Spectral decomposition of the complex Robin Laplacian)**.** *Let* $\Omega \subset \mathbb{R}^d$*,* $d \geq 1$*, be a bounded Lipschitz domain, and* $\alpha \in L^\infty(\partial\Omega)$*. Then,*

1. *Every (generalized) eigenfunction of the Robin Laplacian is an analytic function of* $\boldsymbol{x}$ *in* Ω*;*

2. *for any* simple *eigenvalue* λ_n*, the corresponding eigenfunction* φ_n *is such that* $\int_\Omega \varphi_n(\boldsymbol{x})^2 d\boldsymbol{x} \neq 0$*; moreover, for any (simple or multiple) eigenvalue* λ_n*, the set of eigenfunctions* φ_n *that correspond to a one-dimensional Jordan block can be* chosen *such that* $\int_\Omega \varphi_n(\boldsymbol{x})^2 d\boldsymbol{x} = 1$*; however, any eigenfunction that corresponds to a Jordan block of dimension greater than one is such that* $\int_\Omega \varphi_n(\boldsymbol{x})^2 d\boldsymbol{x} = 0$*; and*

3. *the set of eigenfunctions and generalized eigenfunctions can be* chosen *to form a pseudo-orthonormal Abel basis with brackets of* $L^2(\Omega)$*, i.e., there is an increasing sequence of natural numbers* N_m *(with* $N_0 = 0$*) such that*

$$\lim_{\varepsilon \to 0^+} \sum_{m \in \mathbb{N}} \left[\sum_{n=N_m}^{N_{m+1}-1} e^{-w_n \varepsilon} \varphi_n(\boldsymbol{x}) \varphi_n(\boldsymbol{y}) \right] = \delta(\boldsymbol{x} - \boldsymbol{y}). \tag{9}$$

In Eq. (9)*, the weights* $w_n \in \mathbb{C}$ *are defined as follows: if* $\omega + \lambda_n \in T_\theta^+$*, then* $w_n = (\omega + \lambda_n)^\gamma$*, otherwise* $w_n = 0$*, where*

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[7]Note that this proof contains a minor typographical error that should be corrected: everywhere, "$d/2$" should be replaced by "$2/d$". This typographical error appears only in the proof and does not alter the conclusions of Bögli *et al.* (2022, Theorem 5.7).

- *The constant $\omega \geq 0$ is such that the sesquilinear form defined by the operator $-\Delta_\Omega^\alpha + \omega I$ is coercive (*cf.* Remark 1);*
- *$\gamma = (d-1)/2 + \eta$ for any $\eta > 0$; and*
- *$T_\theta^+ = \{z \in \mathbb{C} : |\arg z| < \theta\}$ for any $\theta \in (0, \frac{\pi}{2\gamma} \min(\gamma, 1))$.*

In particular, Eq. (9) shows that the dual set of $\{\varphi_n\}_{n\in\mathbb{N}}$ is unique and equal to $\{\overline{\varphi_n}\}_{n\in\mathbb{N}}$.

Remark 3. *For a Jordan block of dimension greater than one, the eigenfunction φ_n is such that $\int_\Omega \varphi_n(\boldsymbol{x})^2 d\boldsymbol{x} = 0$, as stated in assertion 2 of Theorem 2. However, in assertion 3, the* chosen *basis of generalized eigenfunctions associated with this Jordan block is obtained by means of a pseudo-orthonormalization process similar to Gram–Schmidt (see the proof of Theorem 2). This means that every function φ_m in this pseudo-orthonormal basis is actually a linear combination of the eigenfunction and at least one generalized eigenfunction, such that $\int_\Omega \varphi_m(\boldsymbol{x})^2 d\boldsymbol{x} = 1$.*

Remark 4. *Equation (9) generalizes Eq. (1), which holds in the real case $\alpha \in \mathbb{R}$, and the complex case $\alpha \in \mathbb{C}$ when the set of (generalized) eigenfunctions $\{\varphi_n\}_{n\in\mathbb{N}}$ forms a basis of $L^2(\Omega)$ as in Proposition 1. Equation (9) should be interpreted in the following sense: $\forall\varepsilon > 0$, the linear operator $I_\Omega^\alpha(\varepsilon)$ introduced in Eq. (10) is well-defined as follows*

$$I_\Omega^\alpha(\varepsilon) : \begin{array}{ccc} L^2(\Omega) & \to & L^2(\Omega) \\ \psi & \mapsto & \sum_{m\in\mathbb{N}} \left[\sum_{n=N_m}^{N_{m+1}-1} e^{-w_n\varepsilon} \langle \psi, \overline{\varphi_n} \rangle \varphi_n \right] \end{array} \tag{10}$$

*[i.e., the series over m in Eq. (10) converges in $L^2(\Omega)$], and it is bounded (*cf.* the proof of the Abel basis property in Agranovich (1994, p. 162)), therefore, Lipschitz continuous. Then the sequence of operators $I_\Omega^\alpha(\varepsilon)$ converges, in turn, to the identity I on $L^2(\Omega)$ when $\varepsilon \to 0^+$.*

Proof of Theorem 2. First, assertion 1 of Theorem 2 comes from the fact that $\forall\lambda \in \mathbb{C}$, $\forall m \geq 1$, the differential operator $P \triangleq (\Delta + \lambda I)^m$ is elliptic, and it is well-known that any solution φ of the differential equation $P\varphi = 0$ on an open set Ω with P elliptic is analytic (Hörmander, 2015, Theorem 4.4.3).

Then, note that assertion 1 of Theorem 1 shows that every eigenvalue of the Robin Laplacian has finite algebraic multiplicity. For convenience, here, we will use an enumeration of the eigenvalues and (generalized) eigenfunctions that is different from that in Eq. (9). Therefore, let $\{\lambda_n\}_{n\in\mathbb{N}}$ be the set of eigenvalues, where each eigenvalue is repeated successively according to its finite *geometric* multiplicity, i.e., its number of distinct eigenfunctions (or equivalently its number of distinct Jordan blocks). Then for each $n \in \mathbb{N}$, let $M_n < +\infty$ be the dimension of the corresponding Jordan block, and let $\varphi_n^{(M_n)}$ be a generalized eigenfunction of the Robin Laplacian such that $(\Delta_\Omega^\alpha + \lambda_n I)^{M_n} \varphi_n^{(M_n)} = 0$ (where I denotes the identity) and $(\Delta_\Omega^\alpha + \lambda_n I)^{M_n - 1} \varphi_n^{(M_n)} \neq 0$. Then, $\varphi_n^{(M_n)}$ generates a Jordan chain of M_n linearly independent generalized eigenfunctions $\varphi_n^{(m)}$ of rank $m \in \{1 \dots M_n\}$, such that

$$\varphi_n^{(m-1)} = -(\Delta_\Omega^\alpha + \lambda_n I)\varphi_n^{(m)}, \tag{11}$$

which starts at rank $m = M_n > 0$ with the generator $\varphi_n^{(M_n)}$ and ends at rank $m = 1$ with the eigenfunction $\varphi_n^{(1)}$ [if we push the recursion one step further, we get $\varphi_n^{(0)} = 0$]. Finally, $\forall n_1, n_2 \in \mathbb{N}$, $\forall m_1 \in \{1 \dots M_{n_1}\}$, $\forall m_2 \in \{1 \dots M_{n_2}\}$, let us define the $M_{n_1} \times M_{n_2}$ matrix $\boldsymbol{H}_{n_1,n_2}$ of entries $[\boldsymbol{H}_{n_1,n_2}]_{m_1,m_2} = \int_\Omega \varphi_{n_1}^{(m_1)}(\boldsymbol{x})\varphi_{n_2}^{(m_2)}(\boldsymbol{x})d\boldsymbol{x}$. Then $\forall n_1, n_2 \in \mathbb{N}$, $\forall m_1 \in \{1 \dots M_{n_1}\}$, $\forall m_2 \in \{1 \dots M_{n_2}\}$, by substituting Eq. (8) applied to $\psi(\boldsymbol{x}) = \varphi_{n_1}^{(m_1)}(\boldsymbol{x})$ and Eq. (11) applied to $n = n_1$ and $m = m_1$ into Eq. (5) applied to $\psi(\boldsymbol{x}) = \varphi_{n_1}^{(m_1)}(\boldsymbol{x})$ and $\varphi(\boldsymbol{x}) = \varphi_{n_2}^{(m_2)}(\boldsymbol{x})$, we get

$$\begin{array}{rl} & \int_\Omega \varphi_{n_1}^{(m_1-1)}(\boldsymbol{x})\varphi_{n_2}^{(m_2)}(\boldsymbol{x})d\boldsymbol{x} + \lambda_{n_1}\int_\Omega \varphi_{n_1}^{(m_1)}(\boldsymbol{x})\varphi_{n_2}^{(m_2)}(\boldsymbol{x})d\boldsymbol{x} \\ = & \int_{\partial\Omega} \alpha(\boldsymbol{x})\varphi_{n_1}^{(m_1)}(\boldsymbol{x})\varphi_{n_2}^{(m_2)}(\boldsymbol{x})dS(\boldsymbol{x}) + \int_\Omega \nabla\varphi_{n_1}^{(m_1)}(\boldsymbol{x})^\top\nabla\varphi_{n_2}^{(m_2)}(\boldsymbol{x})d\boldsymbol{x}. \end{array}$$

In the same way, we also get

$$\begin{array}{rl} & \int_\Omega \varphi_{n_1}^{(m_1)}(\boldsymbol{x})\varphi_{n_2}^{(m_2-1)}(\boldsymbol{x})d\boldsymbol{x} + \lambda_{n_2}\int_\Omega \varphi_{n_1}^{(m_1)}(\boldsymbol{x})\varphi_{n_2}^{(m_2)}(\boldsymbol{x})d\boldsymbol{x} \\ = & \int_{\partial\Omega} \alpha(\boldsymbol{x})\varphi_{n_1}^{(m_1)}(\boldsymbol{x})\varphi_{n_2}^{(m_2)}(\boldsymbol{x})dS(\boldsymbol{x}) + \int_\Omega \nabla\varphi_{n_1}^{(m_1)}(\boldsymbol{x})^\top\nabla\varphi_{n_2}^{(m_2)}(\boldsymbol{x})d\boldsymbol{x}. \end{array}$$

By subtracting the two equalities, we get

$$\begin{array}{rl} & (\lambda_{n_1} - \lambda_{n_2})\int_\Omega \varphi_{n_1}^{(m_1)}(\boldsymbol{x})\varphi_{n_2}^{(m_2)}(\boldsymbol{x})d\boldsymbol{x} \\ = & \int_\Omega \varphi_{n_1}^{(m_1)}(\boldsymbol{x})\varphi_{n_2}^{(m_2-1)}(\boldsymbol{x})d\boldsymbol{x} - \int_\Omega \varphi_{n_1}^{(m_1-1)}(\boldsymbol{x})\varphi_{n_2}^{(m_2)}(\boldsymbol{x})d\boldsymbol{x}. \end{array} \tag{12}$$

If $\lambda_{n_1} \neq \lambda_{n_2}$, Eq. (12) proves by induction that $\forall m_1 \in \{1 \dots M_{n_1}\}$, $\forall m_2 \in \{1 \dots M_{n_2}\}$, $\varphi_{n_1}^{(m_1)}$ is pseudo-orthogonal to $\varphi_{n_2}^{(m_2)}$ because $\varphi_{n_1}^{(0)}$ and $\varphi_{n_2}^{(0)}$ are zero. Therefore the two invariant subspaces related to λ_{n_1} and λ_{n_2} are pseudo-orthogonal, i.e., $\boldsymbol{H}_{n_1,n_2} = \boldsymbol{0}$. If on the contrary $\lambda_{n_1} = \lambda_{n_2}$, then the two functions $\varphi_{n_1}^{(m_1)}$ and $\varphi_{n_2}^{(m_2)}$ belong to the same invariant subspace, and Eq. (12) yields $\int_\Omega \varphi_{n_1}^{(m_1)}(\boldsymbol{x})\varphi_{n_2}^{(m_2-1)}(\boldsymbol{x})d\boldsymbol{x} = \int_\Omega \varphi_{n_1}^{(m_1-1)}(\boldsymbol{x})\varphi_{n_2}^{(m_2)}(\boldsymbol{x})d\boldsymbol{x}$. Again, this equality proves by induction that $\varphi_{n_1}^{(m_1)}$ is pseudo-orthogonal to $\varphi_{n_2}^{(m_2)}$ as $\varphi_{n_1}^{(0)}$ and $\varphi_{n_2}^{(0)}$ are zero, but only for $m_1 + m_2 \leq \max(M_{n_1}, M_{n_2})$, therefore, in general, $\boldsymbol{H}_{n_1,n_2} \neq \boldsymbol{0}$. For instance, if $n_1 = n_2 = n$ and if $M_n > 1$, then $\boldsymbol{H}_{n,n}$ is a lower anti-triangular Hankel matrix[8]. In this particular case, we note that $[\boldsymbol{H}_{n,n}]_{1,1} = 0$, i.e., the eigenfunction $\varphi_n^{(1)}$ is pseudo-orthogonal to itself, which proves the last part of assertion 2 of Theorem 2: *any eigenfunction that corresponds to a Jordan block of dimension greater than one is such that* $\int_\Omega \varphi_n(\boldsymbol{x})^2 d\boldsymbol{x} = 0$.

In the remainder of this proof, for convenience we will use the same enumeration of the eigenvalues and (generalized) eigenfunctions as that in Eq. (9): $\{\lambda_n\}_{n\in\mathbb{N}}$ will now denote the set of eigenvalues, where each eigenvalue λ is repeated successively according to its finite *algebraic* multiplicity, which is defined as $M \triangleq \sum_{\lambda_n=\lambda} M_n$.

Now, let us prove the first part of assertion 2 of Theorem 2. *For any* simple *eigenvalue* λ_n, *the corresponding eigenfunction* φ_n *is such that* $\int_\Omega \varphi_n(\boldsymbol{x})^2 d\boldsymbol{x} \neq 0$. Indeed, if λ_n is a simple eigenvalue (i.e., $M = 1$), then we have already proved that $\varphi_n(\boldsymbol{x})$ is pseudo-orthogonal

[8] A *Hankel* matrix $\boldsymbol{H}$ is such that every entry $[\boldsymbol{H}]_{m_1,m_2}$ depends only on $m_1 + m_2$. In addition, $\boldsymbol{H}$ is *lower anti-triangular* if it is square of dimension M, and $\forall m_1, m_2$ such that $m_1 + m_2 \leq M$, $[\boldsymbol{H}]_{m_1,m_2} = 0$.

to all of the other (generalized) eigenfunctions. By *reductio ad absurdum*, suppose that $\int_\Omega \varphi_n(\boldsymbol{x})^2 d\boldsymbol{x} = 0$. Then $\psi(\boldsymbol{x}) \triangleq \overline{\varphi_n(\boldsymbol{x})}$ is orthogonal to $\varphi_{n'}(\boldsymbol{x})$ $\forall n' \in \mathbb{N}$. However, Theorem 1 proves that the set of (generalized) eigenfunctions can be chosen so as to form an Abel basis with brackets of $L^2(\Omega)$ of order $\gamma = (d-1)/2 + \eta$ for any $\eta > 0$ and any $\theta \in (0, \frac{\pi}{2\gamma}\min(\gamma, 1))$. Following Definition 3 with $T = -\Delta_\Omega^\alpha + \omega I$, there is an increasing sequence of natural numbers N_m (with $N_0 = 0$) such that $\psi(\boldsymbol{x}, \varepsilon) \triangleq \sum_{m\in\mathbb{N}} \left[\sum_{n'=N_m}^{N_{m+1}-1} \psi_{n'} e^{-w_{n'}\varepsilon} \varphi_{n'}(\boldsymbol{x})\right]$ converges to $\psi(\boldsymbol{x})$ in $L^2(\Omega)$ when $\varepsilon \to 0^+$, where the weights $w_{n'} \in \mathbb{C}$ are such that if $\omega + \lambda_{n'} \in T_\theta^+$, then $w_{n'} = (\omega + \lambda_{n'})^\gamma$, otherwise $w_{n'} = 0$. Because $\psi(\boldsymbol{x})$ is orthogonal to $\varphi_{n'}(\boldsymbol{x})$ $\forall n' \in \mathbb{N}$, by continuity of the inner product it is also orthogonal to $\psi(\boldsymbol{x}, \varepsilon)$ $\forall \varepsilon > 0$. Then, when $\varepsilon \to 0^+$, still by continuity of the inner product, we conclude that $\psi(\boldsymbol{x})$ is orthogonal to itself, therefore it is zero. As $\psi(\boldsymbol{x}) = \overline{\varphi_n(\boldsymbol{x})}$, this contradicts the fact that $\varphi_n(\boldsymbol{x})$ is nonzero. We, thus, conclude that $\int_\Omega \varphi_n(\boldsymbol{x})^2 d\boldsymbol{x} \neq 0$. Hence, without loss of generality, we can assume that $\int_\Omega \varphi_n(\boldsymbol{x})^2 d\boldsymbol{x} = 1$.

In the same way, if λ_n is a multiple eigenvalue, let M be its algebraic multiplicity. Then there is a basis $\{\varphi_n(\boldsymbol{x}), \ldots, \varphi_{n+M-1}(\boldsymbol{x})\}$ of eigenfunctions and generalized eigenfunctions of the corresponding invariant subspace. Let us next define the $M \times M$ matrix $\boldsymbol{H}$ of entries $[\boldsymbol{H}]_{i,j} = \int_\Omega \varphi_{n+i}(\boldsymbol{x})\varphi_{n+j}(\boldsymbol{x})d\boldsymbol{x}$ for $i, j \in \{0 \ldots M-1\}$. If we assume that $\boldsymbol{H}$ is singular, then the same line of reasoning as in the previous case shows that the matrix $\boldsymbol{G}$ of entries $[\boldsymbol{G}]_{i,j} = \int_\Omega \varphi_{n+i}(\boldsymbol{x})\overline{\varphi_{n+j}(\boldsymbol{x})}d\boldsymbol{x}$ is also singular, which is in contradiction with the fact that the set $\{\varphi_n(\boldsymbol{x}), \ldots, \varphi_{n+M-1}(\boldsymbol{x})\}$ is linearly independent. We, thus, conclude that matrix $\boldsymbol{H}$ is non-singular, therefore, we can apply a Gram–Schmidt-like process to make the set $\{\varphi_n(\boldsymbol{x}), \ldots, \varphi_{n+M-1}(\boldsymbol{x})\}$ pseudo-orthonormal. Hence, without loss of generality, we can assume that the set $\{\varphi_n(\boldsymbol{x}), \ldots, \varphi_{n+M-1}(\boldsymbol{x})\}$ is pseudo-orthonormal.

Note that, still when λ_n is a multiple eigenvalue, the exact same reasoning but applied to the subset of eigenfunctions whose Jordan blocks are one-dimensional, instead of the whole set of eigenfunctions and generalized eigenfunctions, proves the remaining part of assertion 2 of Theorem 2. *The set of eigenfunctions can be* chosen *such that* $\int_\Omega \varphi_n(\boldsymbol{x})^2 d\boldsymbol{x} = 1$ *for all* φ_n *that correspond to a one-dimensional Jordan block.*

Let us now move on to assertion 3 of Theorem 2. So far, we have proved that, without loss of generality, the whole set of eigenfunctions and generalized eigenfunctions $\{\varphi_n(\boldsymbol{x})\}_{n\in\mathbb{N}}$ can be chosen to form a pseudo-orthonormal Abel basis with brackets of $L^2(\Omega)$. In particular, the Abel basis with brackets property shows that $\forall \psi \in L^2(\Omega)$, $\psi(\boldsymbol{x}) = \lim_{\varepsilon\to 0^+} \psi(\boldsymbol{x}, \varepsilon)$ where $\forall \varepsilon > 0$, $\psi(\boldsymbol{x}, \varepsilon) = \sum_{m\in\mathbb{N}} \left[\sum_{n=N_m}^{N_{m+1}-1} \psi_n e^{-w_n\varepsilon} \varphi_n(\boldsymbol{x})\right]$, and the pseudo-orthonormality property further implies that $\forall n \in \mathbb{N}$, $\langle \psi(\boldsymbol{x}, \varepsilon), \overline{\varphi_n(\boldsymbol{x})}\rangle = \psi_n e^{-w_n\varepsilon}$. When $\varepsilon \to 0^+$, the continuity of the inner product then implies that $\langle \psi, \overline{\varphi_n}\rangle = \psi_n$. We, thus, get $\forall \psi \in L^2(\Omega)$, $\psi = \lim_{\varepsilon\to 0^+} \sum_{m\in\mathbb{N}} \left[\sum_{n=N_m}^{N_{m+1}-1} e^{-w_n\varepsilon} \langle \psi, \overline{\varphi_n}\rangle\, \varphi_n\right]$, which can be rewritten as Eq. (9) (*cf.* Remark 4).

Let us now study the uniqueness of this decomposition. If λ_n is a simple eigenvalue, then its two possible pseudo-unitary eigenfunctions are $\varphi_n(\boldsymbol{x})$ and $-\varphi_n(\boldsymbol{x})$, therefore, the product $\varphi_n(\boldsymbol{x})\varphi_n(\boldsymbol{y})$ is unique. In the same way, if λ_n is a multiple eigenvalue, then any pseudo-orthonormal basis of eigenfunctions of the corresponding invariant subspace will result in the same sum of products $\sum_{i=0}^{M} \varphi_{n+i}(\boldsymbol{x})\varphi_{n+i}(\boldsymbol{y})$.

Finally, let us investigate the uniqueness of the dual set of the pseudo-orthonormal Abel basis with brackets $\{\varphi_n(\boldsymbol{x})\}_{n\in\mathbb{N}}$. A set $\{\psi_n(\boldsymbol{x})\}_{n\in\mathbb{N}}$ is dual to $\{\varphi_n(\boldsymbol{x})\}_{n\in\mathbb{N}}$ if and only if $\forall n, n' \in \mathbb{N}$, $\langle\psi_n(\boldsymbol{x}), \varphi_{n'}(\boldsymbol{x})\rangle = \delta_{n,n'}$. Consequently,

$$\begin{aligned}\overline{\psi_n(\boldsymbol{x})} &= \lim_{\varepsilon\to 0^+} \sum_{m\in\mathbb{N}} \left[\sum_{n'=N_m}^{N_{m+1}-1} e^{-w_{n'}\varepsilon}\overline{\langle\psi_n(\boldsymbol{x}), \varphi_{n'}(\boldsymbol{x})\rangle}\varphi_{n'}(\boldsymbol{x})\right] \\ &= \lim_{\varepsilon\to 0^+} e^{-w_n\varepsilon}\varphi_n(\boldsymbol{x}) = \varphi_n(\boldsymbol{x}).\end{aligned}$$

Therefore, the dual set of $\{\varphi_n\}_{n\in\mathbb{N}}$ is unique and equal to $\{\overline{\varphi_n}\}_{n\in\mathbb{N}}$, which ends the proof of assertion 3 of Theorem 2. □

5. Applications

In this section, we present applications of Theorem 2 to the Green's function of the Helmholtz equation (Sec. 5.1) and the statistical wave field theory (Sec. 5.2).

5.1. Green's function of the Helmholtz equation

Let us first summarize a few fundamental notions regarding wave propagation, which are presented in more detail in Badeau (2024) and Badeau (2025a) (we refer to Morse and Ingard (1968) and Kuttruff (2014) for an in-depth introduction to theoretical acoustics and room acoustics). In a simply connected domain $\Omega \subseteq \mathbb{R}^d$, the homogeneous Helmholtz equation states that

$$\forall \boldsymbol{x} \in \Omega,\ \Delta\varphi(\boldsymbol{x}) + 4\pi^2\kappa^2\varphi(\boldsymbol{x}) = 0, \tag{13}$$

where $\kappa = \frac{\nu}{c}$, where ν is the frequency and c is the propagation speed. Then, given a punctual source position $\boldsymbol{x}_0 \in \Omega$ and wave number k, a Green's function G of the Helmholtz equation is a solution of the following inhomogeneous Helmholtz equation: $\forall \boldsymbol{x} \in \Omega$,

$$\Delta G(\boldsymbol{x}, \boldsymbol{x}_0, k) + 4\pi^2k^2G(\boldsymbol{x}, \boldsymbol{x}_0, k) = -\delta(\boldsymbol{x} - \boldsymbol{x}_0). \tag{14}$$

If the domain Ω admits a Lipschitz continuous boundary $\partial\Omega$ characterized by its *specific admittance* $\widehat{\beta}(\boldsymbol{x}, k) \in L^\infty(\partial\Omega)$, then the Robin boundary condition is written

$$\forall \boldsymbol{x} \in \partial\Omega,\ \frac{\partial\psi(\boldsymbol{x}, k)}{\partial\boldsymbol{n}(\boldsymbol{x})} + \imath 2\pi k\widehat{\beta}(\boldsymbol{x}, k)\psi(\boldsymbol{x}, k) = 0, \tag{15}$$

and this boundary condition applies to Eqs. (13) and (14). In the case of *non-rigid* boundaries, which absorb a part of the energy of the incident wave, $\widehat{\beta}(\boldsymbol{x}, k)$ is complex and the real part of $\widehat{\beta}(\boldsymbol{x}, k)$ is positive.

Because the boundary condition depends on the wave number k, the solutions of the homogeneous Helmholtz equation also depend on k, thus, Eq. (13) has to be rewritten as

$$\forall \boldsymbol{x} \in \Omega,\ \Delta\varphi(\boldsymbol{x}, k) + 4\pi^2\kappa(k)^2\varphi(\boldsymbol{x}, k) = 0, \tag{16}$$

where $\kappa(k) \in \mathbb{C}$. If, in addition, the domain Ω is bounded, then these definitions fit in the general mathematical framework introduced in Sec. 3.2, with $\alpha(\boldsymbol{x}) \triangleq \imath 2\pi k\widehat{\beta}(\boldsymbol{x}, k) \in \mathbb{C}$.

Therefore, the conclusions of Theorems 1 and 2 hold. In particular, the set of eigenvalues $\lambda_n \triangleq 4\pi^2\kappa_n(k)^2$ and eigenfunctions $\varphi_n(\boldsymbol{x}) \triangleq \varphi_n(\boldsymbol{x}, k)$, which are solutions of Eqs. (15) and (16), is discrete and can be indexed by $n \in \mathbb{N}$. Then the Green's function of the Helmholtz equation satisfies the following properties:

Proposition 2 (Green's function of the Helmholtz equation)**.** *Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, be a bounded Lipschitz domain, $\boldsymbol{x}_0 \in \Omega$, and $\widehat{\beta}(\boldsymbol{x}, k) \in L^\infty(\partial\Omega)$ for a given $k \in \mathbb{C}$.*

1. *Suppose that the value $4\pi^2k^2$ does not belong to the complex Robin spectrum $\{4\pi^2\kappa_n(k)^2\}_{n\in\mathbb{N}}$ of parameter $\alpha(\boldsymbol{x}) = \imath 2\pi k\widehat{\beta}(\boldsymbol{x}, k)$. Then, there exists a unique Green's function $G(\boldsymbol{x}, \boldsymbol{x}_0, k)$ that is the solution to Eqs.* (14) *and* (15)*, and G is an analytic function of $\boldsymbol{x}$ in $\Omega\backslash\{\boldsymbol{x}_0\}$; and*

2. *suppose, in addition, that the value $4\pi^2k^2$ does not belong to the real Robin spectrum of parameter* $\mathrm{Re}(\alpha)$*. Then, the Green's function $G(\boldsymbol{x}, \boldsymbol{x}_0, k)$ can be written as*

$$G(\boldsymbol{x}, \boldsymbol{x}_0, k) = \lim_{\varepsilon\to 0^+} \sum_{m\in\mathbb{N}} \left[\sum_{n=N_m}^{N_{m+1}-1} e^{-w_n\varepsilon} \frac{\varphi_n(\boldsymbol{x}_0,k)\,\varphi_n(\boldsymbol{x},k)}{4\pi^2(\kappa_n(k)^2-k^2)} \right], \tag{17}$$

where the natural numbers N_m, the (generalized) eigenfunctions φ_n, and the weights $w_n \in \mathbb{C}$ are the same as those in assertion 3 of Theorem 2.

Remark 5. *Equation* (17) *generalizes the following expression of the Green's function:*

$$G(\boldsymbol{x}, \boldsymbol{x}_0, k) = \sum_{n\in\mathbb{N}} \frac{\varphi_n(\boldsymbol{x}_0, k)\,\varphi_n(\boldsymbol{x}, k)}{4\pi^2(\kappa_n(k)^2 - k^2)},$$

which holds in the real case $\alpha \in \mathbb{R}$ and the complex case $\alpha \in \mathbb{C}$ when the set of (generalized) eigenfunctions $\{\varphi_n\}_{n\in\mathbb{N}}$ forms a basis of $L^2(\Omega)$ as in Proposition 1.

Remark 6. *Because $4\pi^2k^2$ does not belong to the Robin spectrum, the linear operator $-(\Delta_\Omega^\alpha + 4\pi^2k^2I) : D(-\Delta_\Omega^\alpha) \to L^2(\Omega)$ is invertible. Therefore, the Green's operator $G_\Omega^\alpha : L^2(\Omega) \to D(-\Delta_\Omega^\alpha)$ is uniquely defined as $G_\Omega^\alpha = -(\Delta_\Omega^\alpha + 4\pi^2k^2I)^{-1}$. Note that the existence of the Green's operator G_Ω^α does not imply the existence of a function $G(\boldsymbol{x}, \boldsymbol{x}_0, k)$ such that this operator can be written as*

$$G_\Omega^\alpha : \begin{array}{ccc} L^2(\Omega) & \to & D(-\Delta_\Omega^\alpha) \\ \psi & \mapsto & \int_\Omega G(\boldsymbol{x}, \boldsymbol{x}_0, k)\psi(\boldsymbol{x}_0)d\boldsymbol{x}_0. \end{array} \tag{18}$$

Therefore, the existence of the Green's function $G(\boldsymbol{x}, \boldsymbol{x}_0, k)$ such that Eq. (18) *holds will be proved hereunder.*

Proof of Proposition 2. Let us prove the first assertion. We start with the uniqueness of the Green's function. Let us consider two solutions $G_1(\boldsymbol{x}, \boldsymbol{x}_0, k)$ and $G_2(\boldsymbol{x}, \boldsymbol{x}_0, k)$ of Eqs. (14) and (15). Then their difference is such that $(\Delta_\Omega^\alpha + 4\pi^2k^2)(G_1(\boldsymbol{x}, \boldsymbol{x}_0, k) - G_2(\boldsymbol{x}, \boldsymbol{x}_0, k)) = 0$. Because $4\pi^2k^2$ does not belong to the spectrum of the Robin Laplacian $-\Delta_\Omega^\alpha$, we deduce that $G_1(\boldsymbol{x}, \boldsymbol{x}_0, k) - G_2(\boldsymbol{x}, \boldsymbol{x}_0, k) = 0$. Therefore a solution of Eqs. (14) and (15), if it exists, is unique.

Let us now prove the existence of the Green's function. To do so, we consider a Green's function G_0 in the free field, i.e., a fundamental solution $G_0(\boldsymbol{x}, k)$ of the inhomogeneous Helmholtz equation (14) with $\boldsymbol{x}_0 = \mathbf{0}$ in $\mathbb{R}^d$ (without boundary condition). Note that, in general, the function $G_0(\boldsymbol{x} - \boldsymbol{x}_0, k)$ does not satisfy Robin's boundary condition in Eq. (15), hence, it is different from the Green's function $G(\boldsymbol{x}, \boldsymbol{x}_0, k)$ that we are looking for. In other respects, as the differential operator $P \triangleq \Delta + 4\pi^2 k^2 I$ is elliptic, it is well-known that any solution G_0 of the differential equation $PG_0 = 0$ on the open set $\mathbb{R}^d \backslash \{\mathbf{0}\}$ with P elliptic is analytic (Hörmander, 2015, Theorem 4.4.3), therefore, $G_0(\boldsymbol{x}, k)$ is an analytic function in $\mathbb{R}^d \backslash \{\mathbf{0}\}$.

Now, suppose that the Green's function $G(\boldsymbol{x}, \boldsymbol{x}_0, k)$, which is the solution to Eqs. (14) and (15), exists. Because functions $G(\boldsymbol{x}, \boldsymbol{x}_0, k)$ and $G_0(\boldsymbol{x} - \boldsymbol{x}_0, k)$ satisfy the inhomogeneous Helmholtz equation (14), then function $u(\boldsymbol{x}) \triangleq G(\boldsymbol{x}, \boldsymbol{x}_0, k) - G_0(\boldsymbol{x} - \boldsymbol{x}_0, k)$, which is defined on Ω, satisfies $\forall \boldsymbol{x} \in \Omega$, $(\Delta + 4\pi^2 k^2) u(\boldsymbol{x}) = 0$, which implies that $\forall v \in H^1(\Omega)$,

$$\int_\Omega (\Delta + 4\pi^2 k^2) u(\boldsymbol{x}) \overline{v(\boldsymbol{x})} d\boldsymbol{x} = 0. \tag{19}$$

As $G(\boldsymbol{x}, \boldsymbol{x}_0, k)$ satisfies the Robin boundary condition [Eq. (15)], Eq. (19) is equivalent to $\forall v \in H^1(\Omega)$,

$$\begin{aligned} & -a_\alpha[u, v] + 4\pi^2 k^2 \textstyle\int_\Omega u(\boldsymbol{x}) \overline{v(\boldsymbol{x})} d(\boldsymbol{x}) \\ = & \textstyle\int_{\partial\Omega} \left(\alpha(\boldsymbol{x}) G_0(\boldsymbol{x} - \boldsymbol{x}_0, k) + \frac{\partial G_0(\boldsymbol{x} - \boldsymbol{x}_0, k))}{\partial \boldsymbol{n}(\boldsymbol{x})} \right) \overline{v(\boldsymbol{x})} dS(\boldsymbol{x}), \end{aligned} \tag{20}$$

where the Robin form a_α was defined in Eq. (6). Reciprocally, if u is a solution of Eq. (20), then by applying Eq. (20) to all functions $v \in H^1(\Omega)$ whose trace on $\partial\Omega$ is zero, we prove that function $G(\boldsymbol{x}, \boldsymbol{x}_0, k) \triangleq G_0(\boldsymbol{x} - \boldsymbol{x}_0, k) + u(\boldsymbol{x})$ is a solution to Eq. (14). Then by applying Eq. (20) again to the larger set of all functions $v \in H^1(\Omega)$, we deduce that $G(\boldsymbol{x}, \boldsymbol{x}_0, k)$ is also a solution to Eq. (15). Therefore proving the existence of the Green's function $G(\boldsymbol{x}, \boldsymbol{x}_0, k)$ is equivalent to proving the existence of a solution u to Eq. (20). Moreover, because we have already proved that the Green's function $G(\boldsymbol{x}, \boldsymbol{x}_0, k)$ is unique, the solution u of Eq. (20) is also necessarily unique, if it exists.

Let us now prove the existence (and confirm the uniqueness) of function u. First, the trace inequality (6.15) in Bögli *et al.* (2022) shows that there exists a constant $C(\Omega) > 0$ that only depends on the bounded Lipschitz domain Ω, such that $\forall v \in H^1(\Omega)$,

$$\left| \int_{\partial\Omega} |v(\boldsymbol{x})|^2 dS(\boldsymbol{x}) \right| \leq C(\Omega) \|v\|_{L^2(\Omega)} \|v\|_{H^1(\Omega)}. \tag{21}$$

The trace inequality Eq. (21) shows that the left member of Eq. (20) is a continuous sesquilinear form on $H^1(\Omega) \times H^1(\Omega)$, and the right member of Eq. (20) is a continuous antilinear form on $H^1(\Omega)$. Thus, the Riesz representation theorem implies that there is a unique bounded linear operator $A : H^1(\Omega) \to H^1(\Omega)$ and a unique vector $w \in H^1(\Omega)$ such that $\forall v \in H^1(\Omega)$,

$$a_\alpha[u, v] - 4\pi^2 k^2 \textstyle\int_\Omega u(\boldsymbol{x}) \overline{v(\boldsymbol{x})} d(\boldsymbol{x}) = \langle Au, v \rangle_{H^1(\Omega)} \tag{22}$$

and

$$\int_{\partial\Omega}\left(\alpha(\boldsymbol{x})G_0(\boldsymbol{x}-\boldsymbol{x}_0,k)+\frac{\partial G_0(\boldsymbol{x}-\boldsymbol{x}_0,k))}{\partial \boldsymbol{n}(\boldsymbol{x})}\right)\overline{v(\boldsymbol{x})}dS(\boldsymbol{x})=-\langle w,v\rangle_{H^1(\Omega)}.$$

Therefore, Eq. (20) can be rewritten in the more compact form: $\forall v\in H^1(\Omega)$, $\langle Au,v\rangle_{H^1(\Omega)}=\langle w,v\rangle_{H^1(\Omega)}$, which is equivalent to $Au=w$ in $H^1(\Omega)$. Now we need to prove that the linear operator A is bijective, which will permit us to conclude that there exists a unique vector $u\in H^1(\Omega)$ such that $Au=w$. First, by substituting Eq. (7) into Eq. (22), we get $\forall v\in D(-\Delta_\Omega^\alpha)$,

$$\langle Av,v\rangle_{H^1(\Omega)}=\int_\Omega -(\Delta_\Omega^\alpha+4\pi^2k^2)v(\boldsymbol{x})\overline{v(\boldsymbol{x})}d(\boldsymbol{x}).$$

Consequently, as $4\pi^2k^2$ does not belong to the spectrum of the Robin Laplacian $-\Delta_\Omega^\alpha$ and this spectrum has no finite point of accumulation (as shown in Theorem 1), there is a constant $c_0>0$ such that $\forall v\in D(-\Delta_\Omega^\alpha)$,

$$\left|\langle Av,v\rangle_{H^1(\Omega)}\right|\geq c_0\|v\|^2_{L^2(\Omega)}, \tag{23}$$

and because $D(-\Delta_\Omega^\alpha)$ is dense in $H^1(\Omega)$, by continuity, this inequality also holds $\forall v\in H^1(\Omega)$. In particular, $\forall v\in H^1(\Omega)$ such that $Av=0$, Eq. (23) implies that $v=0$, therefore A is injective. In the same way, $\forall v\in \mathrm{span}(A)^\perp$, $\langle Av,v\rangle_{H^1(\Omega)}=0$, therefore Eq. (23) implies that $v=0$, which proves that $\mathrm{span}(A)$ is dense in $H^1(\Omega)$. Now we have yet to prove that $\mathrm{span}(A)$ is closed, which will permit us to conclude that $\mathrm{span}(A)=H^1(\Omega)$, i.e., A is surjective, thus bijective. To do so, first, we need to establish the following stronger variant of Eq. (23). There exists a constant $c_1>0$ such that $\forall v\in H^1(\Omega)$,

$$\left|\langle Av,v\rangle_{H^1(\Omega)}\right|\geq c_1\|v\|^2_{H^1(\Omega)}. \tag{24}$$

By *reductio ad absurdum*, suppose that Eq. (24) is false. Then, there is a sequence of vectors $v_n\in H^1(\Omega)$ such that $\|v_n\|_{H^1(\Omega)}=1$ $\forall n\in\mathbb{N}$ and $\langle Av_n,v_n\rangle_{H^1(\Omega)}\to 0$ when $n\to+\infty$. However, Eq. (23) proves that $\left|\langle Av_n,v_n\rangle_{H^1(\Omega)}\right|\geq c_0\|v_n\|^2_{L^2(\Omega)}$, therefore, $\|v_n\|_{L^2(\Omega)}\to 0$ when $n\to+\infty$. Because $\|v_n\|_{H^1(\Omega)}=1$ $\forall n\in\mathbb{N}$, we also deduce that $\|\nabla v_n\|_{L^2(\Omega)}\to 1$ when $n\to+\infty$. However, Eqs. (22) and (6) yield

$$\langle Av_n,v_n\rangle_{H^1(\Omega)}=\int_{\partial\Omega}\alpha(\boldsymbol{x})|v_n(\boldsymbol{x})|^2dS(\boldsymbol{x})+\|\nabla v_n\|^2_{L^2(\Omega)}-4\pi^2k^2\|v_n\|^2_{L^2(\Omega)}. \tag{25}$$

In addition, the trace inequality [Eq. (21)] shows that

$$\left|\int_{\partial\Omega}\alpha(\boldsymbol{x})|v_n(\boldsymbol{x})|^2dS(\boldsymbol{x})\right|\leq C(\Omega)\|v_n\|_{L^2(\Omega)}\|v_n\|_{H^1(\Omega)}\|\alpha\|_{L^\infty(\Omega)},$$

therefore, $\int_{\partial\Omega}\alpha(\boldsymbol{x})|v_n(\boldsymbol{x})|^2dS(\boldsymbol{x})\to 0$ when $n\to+\infty$. Consequently, when $n\to+\infty$, the right member in Eq. (25) tends to one, whereas the left member tends to zero, which is a contradiction. We have thus proved Eq. (24). Then, applying the Cauchy-Schwarz inequality to Eq. (24) yields

$$c_1\|v\|^2_{H^1(\Omega)}\leq\left|\langle Av,v\rangle_{H^1(\Omega)}\right|\leq\|Av\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)},$$

which implies that

$$c_1\|v\|_{H^1(\Omega)}\leq\|Av\|_{H^1(\Omega)}. \tag{26}$$

We can now prove that $\text{span}(A)$ is closed. Indeed, suppose that a sequence of vectors Av_n converges to a vector b in $H^1(\Omega)$; we want to prove that $b \in \text{span}(A)$. However, $\{Av_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $H^1(\Omega)$. Then Eq. (26) proves that $\{v_n\}_{n\in\mathbb{N}}$ is also a Cauchy sequence in $H^1(\Omega)$. Because the Hilbert space $H^1(\Omega)$ is complete, the sequence $\{v_n\}_{n\in\mathbb{N}}$ converges to a vector $v_\infty \in H^1(\Omega)$. Finally, as the operator A is bounded, it is continuous, therefore, $Av_\infty = \lim_{n\to+\infty} Av_n = b$, which proves that $b \in \text{span}(A)$.

In conclusion, we have proved that $\text{span}(A)$ is closed, which proves that A is surjective, thus bijective, which proves that there exists a unique vector $u \in H^1(\Omega)$ such that $Au = w$, which finally proves the existence of the unique Green's function $G(\boldsymbol{x}, \boldsymbol{x}_0, k) = G_0(\boldsymbol{x} - \boldsymbol{x}_0, k) + u(\boldsymbol{x})$, which is the solution to Eqs. (14) and (15). In other respects, function u is a solution of the differential equation $PG = 0$ on the open set Ω with $P \triangleq (\Delta + 4\pi^2k^2)$ elliptic, which proves that u is actually an analytic function in Ω (Hörmander, 2015, Theorem 4.4.3). Because function G_0 is analytic in $\Omega\backslash\{\mathbf{0}\}$, we conclude that the Green's function $G(\boldsymbol{x}, \boldsymbol{x}_0, k)$ is analytic in $\Omega\backslash\{\boldsymbol{x}_0\}$. We have, thus, finally proved the first assertion of Proposition 2.

The second assertion can be proved by adapting the proof of the Abel basis property in Agranovich (1994, p. 162). Agranovich's theorem deals with sesquilinear forms in Hilbert spaces that are continuous and coercive (Agranovich, 1994, p. 151). As already explained in Remark 1, Bögli *et al.* (2022, p. 17) show that it is always possible to choose a constant $\omega \geq 0$ high enough such that the sesquilinear forms defined by the operators $A \triangleq -\Delta_\Omega^\alpha + \omega I$ and $B \triangleq -\Delta_\Omega^{\text{Re}(\alpha)} + \omega I$ satisfy this condition. Then, the linear operators

$$P_p(\varepsilon) = -\tfrac{1}{2\pi\imath} \int_{\Gamma_p} e^{-\lambda^\gamma \varepsilon} (A - \lambda I)^{-1} d\lambda$$

and

$$Q_p(\varepsilon) = -\tfrac{1}{2\pi\imath} \int_{\Gamma_p} e^{-\lambda^\gamma \varepsilon} (B - \lambda I)^{-1} d\lambda$$

introduced in Eq. (5.1) of Agranovich (1994) (with different notations) have to be replaced by

$$P_p(\varepsilon) = \tfrac{1}{2\pi\imath} \int_{\Gamma_p} e^{-\lambda^\gamma \varepsilon} \tfrac{(A-\lambda I)^{-1}}{\lambda-\omega-4\pi^2k^2} d\lambda$$

and

$$Q_p(\varepsilon) = \tfrac{1}{2\pi\imath} \int_{\Gamma_p} e^{-\lambda^\gamma \varepsilon} \tfrac{(B-\lambda I)^{-1}}{\lambda-\omega-4\pi^2k^2} d\lambda,$$

with well-chosen contours Γ_p in the complex plane (including one around $\omega + 4\pi^2k^2$) and the same convention that $e^{-\lambda^\gamma \varepsilon}$ has to be replaced by one for the contours that contain the finitely many eigenvalues such that $|\arg(\omega + \lambda_n)| \geq \theta$, and also for the contour that contains $\omega + 4\pi^2k^2$. In particular, every pair of contours Γ_{p-1} and Γ_p share the same vertical line segment I_p in the complex plane, and Agranovich's proof can be easily adapted to show that these segments I_p can be chosen such that the sequence $\int_{I_p} e^{-\lambda^\gamma \varepsilon} \frac{(A-\lambda I)^{-1} - (B-\lambda I)^{-1}}{\lambda-\omega-4\pi^2k^2} d\lambda$ tends to zero when $p \to +\infty$. With this modification, the same arguments as in Agranovich (1994, p. 162) show that $\forall\varepsilon > 0$, the linear operator $O_\Omega^\alpha(\varepsilon)$ introduced in Eq. (27) is well-defined as follows

$$O_\Omega^\alpha(\varepsilon) : \begin{array}{ccc} L^2(\Omega) & \to & L^2(\Omega) \\ \psi & \mapsto & G_\Omega^\alpha \psi - \sum\limits_{m\in\mathbb{N}} \left[\sum\limits_{n=N_m}^{N_{m+1}-1} \frac{e^{-w_n \varepsilon} \langle \psi, \overline{\varphi_n} \rangle \varphi_n}{\lambda_n - 4\pi^2k^2} \right] \end{array} \tag{27}$$

[i.e., the series over m in Eq. (27) converges in $L^2(\Omega)$], where $G_\Omega^\alpha = -(\Delta_\Omega^\alpha + 4\pi^2 k^2 I)^{-1}$ as in Remark 6. Moreover, $\forall \varepsilon > 0$ the operator $O_\Omega^\alpha(\varepsilon)$ is bounded, therefore, Lipschitz continuous, the sequence of operators $O_\Omega^\alpha(\varepsilon)$ converges, in turn, to an operator O_Ω^α on $L^2(\Omega)$ when $\varepsilon \to 0^+$, and if the complex Robin Laplacian $-\Delta_\Omega^\alpha$ is replaced by the real Robin Laplacian $-\Delta_\Omega^{\mathrm{Re}(\alpha)}$, then the sequence of operators $O_\Omega^{\mathrm{Re}(\alpha)}(\varepsilon)$ converges to the same limit $O_\Omega^{\mathrm{Re}(\alpha)} = O_\Omega^\alpha$ when $\varepsilon \to 0^+$. However, in the real case, the series in Eq. (27) converges in $L^2(\Omega)$ for $\varepsilon = 0$, and its limit is $G_\Omega^{\mathrm{Re}(\alpha)}\,\psi$ [*cf.* Remark 5 and Eq. (18)], therefore, the sequence $O_\Omega^{\mathrm{Re}(\alpha)}(\varepsilon)$ converges to the zero operator $O_\Omega^{\mathrm{Re}(\alpha)} = 0$ when $\varepsilon \to 0^+$. We, thus, conclude that $O_\Omega^\alpha = O_\Omega^{\mathrm{Re}(\alpha)} = 0$, which, considering Eqs. (27) and (18), finally proves Eq. (17). □

5.2. Statistical wave field theory

In Badeau (2024), we introduced the foundations of the statistical wave field theory, which establishes the statistical laws of waves propagating in a bounded domain, and focused on the particular case of three space dimensions ($d = 3$). We introduced the *B-function* as the main calculation tool of this theory, and presented a closed-form expression of this B-function that holds when the two assumptions of Proposition 1 hold, i.e., when the Robin Laplacian is diagonalizable and its eigenfunctions form a basis of $L^2(\Omega)$, as explained in Badeau (2025a). Here, we generalize the definition of the B-function to an arbitrary space dimension d, get rid of these two assumptions as a result of Theorem 2, and provide a modified expression of the B-function that is guaranteed to hold in the general case. We also show that the fundamental property of the B-function, which is needed to perform the mathematical developments in Badeau (2024), continues to hold in this general case. Contrary to the other sections of this paper, we will only present the main lines of reasoning and not present detailed mathematical proofs here, as it would take us too far from our main topic.

First, Theorem 2 shows that every (generalized) eigenfunction $\varphi_n(\boldsymbol{x}, k)$ of the Robin Laplacian is an analytic function of $\boldsymbol{x}$ in Ω. In the same way, Proposition 2 shows that the Green's function $G(\boldsymbol{x}, \boldsymbol{x}_0, k)$, introduced in Sec. 5.1, is an analytic function of $\boldsymbol{x}$ in $\Omega \backslash \{\boldsymbol{x}_0\}$ as well as an analytic function of $\boldsymbol{x}_0$ in $\Omega \backslash \{\boldsymbol{x}\}$ because the expression of G in Eq. (17) is symmetric. Now, suppose that the analytic functions $\varphi_n(\boldsymbol{x}, k)$ and $G(\boldsymbol{x}, \boldsymbol{x}_0, k)$ can be analytically continued on a mathematical neighborhood $\mathcal{D}$ of Ω such that Eq. (17) still holds on $\mathcal{D} \subseteq \mathbb{R}^d$. The B-function on $\mathcal{D} \times \mathcal{D}$ is then defined as

$$B(\boldsymbol{x}, \boldsymbol{x}_0, k) = -\left(\Delta G(\boldsymbol{x}, \boldsymbol{x}_0, k) + 4\pi^2 k^2 G(\boldsymbol{x}, \boldsymbol{x}_0, k)\right). \tag{28}$$

By definition of the Green's function G in Eq. (14), the restriction of the B-function to $\Omega \times \Omega$ is $\delta(\boldsymbol{x} - \boldsymbol{x}_0)$, where $\boldsymbol{x}_0 \in \Omega$ is the original source position. Elsewhere, the B-function in Eq. (28) can be interpreted as a distribution of image sources located outside of Ω (Badeau, 2024, Sec. III.B). Then, substituting Eq. (17) into Eq. (28), the expression of the B-function on $\mathcal{D} \times \mathcal{D}$ is $B(\boldsymbol{x}, \boldsymbol{x}_0, k) = \lim_{\varepsilon \to 0^+} B(\boldsymbol{x}, \boldsymbol{x}_0, k, \varepsilon)$, with

$$B(\boldsymbol{x}, \boldsymbol{x}_0, k, \varepsilon) \triangleq \sum_{m \in \mathbb{N}} \left[\sum_{n=N_m}^{N_{m+1}-1} e^{-w_n \varepsilon} \varphi_n(\boldsymbol{x}_0, k) \varphi_n(\boldsymbol{x}, k) \right]. \tag{29}$$

Note that Eq. (9) confirms that the restriction of $B(\boldsymbol{x}, \boldsymbol{x}_0, k)$ to $\Omega\times\Omega$ is $\delta(\boldsymbol{x}-\boldsymbol{x}_0)$. In addition, as the Abel basis with brackets $\{\varphi_n(\boldsymbol{x}, k)\}_{n\in\mathbb{N}}$ is pseudo-orthonormal (*cf.* assertion 3 in Theorem 2), Eq. (29) yields $\forall\varepsilon_1, \varepsilon_2 > 0$,

$$\int_{\boldsymbol{x}_0\in\Omega} B(\boldsymbol{x}_1, \boldsymbol{x}_0, k, \varepsilon_1)B(\boldsymbol{x}_2, \boldsymbol{x}_0, k, \varepsilon_2)d\boldsymbol{x}_0 = B(\boldsymbol{x}_1, \boldsymbol{x}_2, \varepsilon_1 + \varepsilon_2, k).$$

When $\varepsilon_1 \to 0^+$ and $\varepsilon_2 \to 0^+$, we, thus, get

$$\int_{\boldsymbol{x}_0\in\Omega} B(\boldsymbol{x}_1, \boldsymbol{x}_0, k)B(\boldsymbol{x}_2, \boldsymbol{x}_0, k)d\boldsymbol{x}_0 = B(\boldsymbol{x}_1, \boldsymbol{x}_2, k). \tag{30}$$

In Badeau (2024), Eq. (30) was derived from Eq. (1), which holds under the restrictive assumptions of Proposition 1. Here, Eq. (30) has been derived from Eq. (9), which holds under the relaxed assumptions of Theorem 2. Note that Eq. (30) is the fundamental equation that permitted us to derive Eq. (61) in Badeau (2024), which provides the expression of the pseudo-covariance function (PCF) of the randomized B-function, on which the mathematical developments of the general statistical wave field theory rely.

6. Conclusion

In this paper, we have investigated the spectral decomposition of the complex Robin Laplacian on a bounded Lipschitz domain, and stated a theorem that shows that the set of its (generalized) eigenfunctions can be chosen to form a pseudo-orthonormal Abel basis with brackets of the Hilbert space of square-integrable functions on this domain. This theorem has then been applied to the Green's function of the Helmholtz equation. To the best of our knowledge, this is the first time that the existence, unicity and general closed-form expression of the Green's function of the Helmholtz equation are established mathematically in this general setting. The theorem has also been applied to the B-function introduced in the statistical wave field theory to show that the mathematical grounds of this theory also hold in the most general case possible. We hope that this new mathematical result will encourage further spreading of the modal approach in acoustics, now that the series expansions on biorthogonal sets of (generalized) eigenfunctions of the complex Robin Laplacian are proved to converge in the weak sense of Abel bases with brackets.

In future work, the approach developed in this paper could be applied to other non-self-adjoint linear operators that also fit in the mathematical framework of Agranovich (1994).

Acknowledgments

The author would like to thank the anonymous reviewers for their very helpful comments. He would also like to thank J.-D. Polack and T. Hélie for bringing the convergence issues with the Robin Laplacian to his attention. Finally, he would like to thank A. Deleforge for informing him of the existence of the reference Bögli *et al.* (2022), and the authors of this reference, without whom this paper would never have existed.

Author Declarations

Conflict of Interest: The author has no conflict of interest to disclose.

Data Availability

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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