Statistical wave field theory: Curvature term

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Abstract

In a recent research paper, we introduced the statistical wave field theory, which establishes the statistical laws of waves propagating in a bounded volume. These laws hold after many reflections on the boundary surface and at high frequency. The statistical wave field theory is the first statistical theory of reverberation that provides the closed-form expression of the power distribution and the correlations of the wave field jointly over time, frequency and space, in terms of the geometry and the specific admittance of the boundary surface. In this paper, we refine the theory predictions, by investigating the impact of a curved boundary surface on the wave field statistics. In particular, we provide an improved closed-form expression of the reverberation time in room acoustics that holds at lower frequency.

Keywords: Statistical physics, wave equation, Helmholtz equation, reverberation

Contents

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1. Introduction

In a recent publication (Badeau, 2024), we introduced the foundations of the statistical wave field theory, which for the first time establishes mathematically the statistical properties of the solutions to the wave equation in a bounded volume, after many reflections on the boundary surface, in terms of power distribution and correlations, jointly over time, frequency, and space. The first and second order statistics of the wave field were expressed in closed-form, via asymptotic expansions that hold at high frequency, with respect to (w.r.t.) the geometry and the specific admittance of the boundary surface. This theory may find applications in various science fields, including room acoustics, electromagnetic theory, and nuclear physics. In room acoustics, this theory has permitted us to retrieve the well-known statistical properties of reverberation that hold in mixing rooms, which provides a first confirmation of the theory predictions.

The statistical wave field theory is based on the following physical assumptions: first, the medium has to be lossless, homogeneous and at rest (Kuttruff, 2014, Chap. 1), so that the wave equation holds exactly inside the bounded volume. Second, the physical source is assumed punctual. In addition, all parameters of the problem, including the source position and the boundaries of the domain, are assumed constant over time. Last, the boundary surface is closed and bounded, and its shape has to meet the mathematical conditions of a *mixing* dynamical billiard (Tabachnikov, 1995), a notion that is related to *diffusion*. Concretely, this means that most geometric shapes are allowed, especially those including irregular (rough) surfaces producing wave scattering. Indeed, even though there is no simple mathematical characterization of mixing billiards, there exist general results that show basically that the more the boundary is irregular, the more it is mixing, and on the contrary, the more it is smooth, the less it is mixing. Nevertheless, there exist very smooth and simple geometric shapes such as the Bunimovich stadium (Bunimovich, 1979) that are mixing, though the mixing rate is slower in this case.

The statistical wave field theory is closely related to the Sturm-Liouville theory. Indeed, the solutions to the wave equation in a bounded domain are characterized by the Helmholtz equation that, along with its boundary conditions, forms a particular Sturm-Liouville problem (Al-Gwaiz, 2008). The Sturm-Liouville theory shows that this problem admits a discrete set of solutions, called *normal modes* (Kuttruff, 2014, Chap. 3). In several dimensions of space, the density of discrete modes increases with the frequency, in a way that has been investigated mathematically for the first time by Weyl (1911). Since then, a rich literature has been devoted to the study of asymptotic expansions of the modal density as a function of frequency f when $f \to +\infty$, in various space dimensions and various boundary conditions (Arendt *et al.*, 2009). The case of a three-dimensional (3D) space and of Robin's boundary condition, which is of special interest to us, was addressed by the physicists Balian and Bloch (1970).

Up to the first order of the asymptotic expansion, wave propagation can be approximated by considering the trajectory of rays traveling in straight lines and undergoing successive specular reflections on the boundary surface, as in geometric acoustics (Kuttruff, 2014, Chap. 4) and optics (Greivenkamp, 2004). The ray trajectory can then be interpreted as a dynamical billiard that, depending on the boundary geometry, may satisfy different statistical properties, such as ergodicity. In particular, *mixing* billiards, in addition to being ergodic, are such that after an asymptotically long elapsed time, the orientation of the ray at any receiver's position is statistically independent of the source's position and orientation. Under this assumption, the resulting wave field is both homogeneous and isotropic (thus the classical definition of a *diffuse field* can be considered as equivalent to the mixing property), and the reverberation time is constant w.r.t. the space position and the orientation in the room (Polack, 1992). To sum up, wave propagation can be described as a mixing dynamical billiard in a rectangular region of the time-frequency plane that we depicted in Badeau (2019, Fig. 1): in the frequency domain, at high frequency, so that the conditions of geometric acoustics and optics are met, and in the time domain, after the *mixing time* as defined by Polack (1992), so that the mixing conditions of a dynamical billiard are met.

Based on the mixing assumption, we proved in Badeau (2024) that, if there is no energy absorption at the boundary surface, then the wave field is asymptotically *wide sense stationary* (WSS). Moreover, the asymptotic expansion of the modal density directly provides us with a closed-form expression of the *power spectrum* of the WSS wave field: indeed, because all normal modes are uncorrelated and carry on average the same quantity of power when the source position is random, the power spectrum is proportional to the modal density. If on the contrary there is energy absorption, then the wave field is non-stationary, and the theory proves that its statistics are actually related to the analytic continuation of the modal density to the domain of complex frequencies.

In Badeau (2024), we have investigated the first and second order statistics of the wave field that result from the asymptotic expansion at high frequency of the modal density, up to the first order *surface term*. So the theory predictions hold under the same high frequency approximation as in geometric acoustics and optics. In this paper, we investigate the impact of a curved boundary surface on the wave field statistics, by exploiting the second order *curvature term* of the asymptotic expansion, calculated by Balian and Bloch (1970). The second order asymptotic expansion is based on the additional assumption that the boundary is twice continuously differentiable, so edges are excluded in Balian and Bloch (1970), and they are also excluded in the present paper (the effect of edges will be investigated in future publications, see Sec. 9).

This paper is structured accordingly to Badeau (2024), in order to permit an easy comparison of the mathematical developments in the two documents. Note that Badeau (2024) includes a comprehensive literature review, as well as detailed information regarding the various assumptions, concepts and implications of the statistical wave field theory, which will not be repeated here, in order to avoid unneeded redundancies. Nevertheless, the present paper is written so that it can be read independently from Badeau (2024). In Sec. 2, we introduce some acronyms and mathematical notations that will be used in the rest of the paper. Then in Sec. 3, we summarize a few fundamental notions regarding wave propagation that are needed to develop the statistical wave field theory. In Sec. 4, we list the three mathematical assumptions on which the statistical wave field theory relies, and we briefly present the Wigner time-frequency distribution that we will use to characterize the second-order properties of non-stationary random processes. Then in Sec. 5 we present the *special* theory dedicated to Neumann's boundary condition, and in Sec. 6 we present the *general* theory dedicated to Robin's boundary condition. The main results are summarized in Secs. 5.4 and 6.6. In Sec. 7, we investigate numerically the impact of the surface term and that of the curvature term on the reverberation time and on the frequency distortion. Then in Sec. 8, we discuss how in practice the statistical wave field theory can be applied to room acoustics. Finally, in Sec. 9 we summarize the main contributions of this paper, and we propose a few perspectives for future work. Another important contribution of this paper is the proof presented in Appendix A that under mild assumptions, the set of complex eigenfunctions of the Robin Laplacian forms a pseudo-orthonormal basis of the Hilbert space of square-integrable functions.

2. Acronyms and mathematical notations

Acronyms:

ACF auto-covariance function

PCF pseudo-covariance function

WSS wide sense stationary

Mathematical notations:

- $\triangleq$: equal by definition to

- $\mathbb{N}$: set of whole numbers
- $\mathbb{R}$, $\mathbb{C}$: sets of real and complex numbers, respectively
- $\imath = \sqrt{-1}$: imaginary unit
- $\mathbb{R}_+$: set of nonnegative real numbers
- $\boldsymbol{x}$ (bold font), z (regular): vector and scalar, respectively
- $A \backslash B$: relative complement (set difference) of set B in set A
- $A \subseteq B$: A is a subset of B, possibly equal to B
- $\mathring{V}$: interior of a subset V of $\mathbb{R}^3$
- $\overline{V}$: closure of a subset V of $\mathbb{R}^3$
- $|V|$: Lebesgue measure (volume) of a subset V of $\mathbb{R}^3$
- $\lambda = \frac{1}{|V|}$: mean density of sources over space
- $\partial V = \overline{V} \backslash \mathring{V}$: boundary of a subset V of $\mathbb{R}^3$
- $\boldsymbol{n}(\boldsymbol{x})$ where $\boldsymbol{x} \in \partial V$: outward unit normal vector to the boundary surface of subset V
- $S(A)$: surface area of a bidimensional sub-manifold A of $\mathbb{R}^3$
- $\|.\|_2$: Euclidean/Hermitian norm of a vector or a function
- $\overline{z}$: complex conjugate of $z \in \mathbb{C}$
- $\mathrm{Re}(z)$ (respectively, $\mathrm{Im}(z)$): real (respectively, imaginary) part of a complex number $z \in \mathbb{C}$
- $\boldsymbol{x}^\top$: transpose of vector $\boldsymbol{x}$
- $\mathcal{S}(\mathbf{0}, k)$: sphere centered at the origin and of radius k: $\mathcal{S}(\mathbf{0}, k) = \{\boldsymbol{k} \in \mathbb{R}^3; \|\boldsymbol{k}\|_2 = k\}$
- δ_{n_1,n_2}: Kronecker delta: $\delta_{n_1,n_2} = 1$ if $n_1 = n_2$, $\delta_{n_1,n_2} = 0$ otherwise
- $\delta(.)$: Dirac delta function
- $H(t)$: Heaviside function: $H(t) = 1\ \forall t > 0$ and $H(t) = 0\ \forall t < 0$
- $\mathrm{sinc}(x) = \frac{\sin(x)}{x}$: cardinal sine function
- $\Delta\phi(\boldsymbol{x})$: Laplacian of function $\phi(\boldsymbol{x})$

- 1D direct and inverse Fourier transforms of a function $\psi : \mathbb{R} \to \mathbb{C}$:

$$\widehat{\psi}(f) = \int_{t\in\mathbb{R}} \psi(t) e^{-2\imath\pi f t} dt \text{ and } \psi(t) = \int_{f\in\mathbb{R}} \widehat{\psi}(f) e^{+2\imath\pi f t} df \tag{1}$$

- 3D direct and inverse Fourier transform of a function $\psi : \mathbb{R}^3 \to \mathbb{C}$:

$$\widehat{\psi}(\boldsymbol{k}) = \int_{\boldsymbol{x}\in\mathbb{R}^3} \psi(\boldsymbol{x}) e^{-2\imath\pi \boldsymbol{k}^\top \boldsymbol{x}} d\boldsymbol{x} \text{ and } \psi(\boldsymbol{x}) = \int_{\boldsymbol{k}\in\mathbb{R}^3} \widehat{\psi}(\boldsymbol{k}) e^{+2\imath\pi \boldsymbol{k}^\top \boldsymbol{x}} d\boldsymbol{k} \tag{2}$$

- $\mathbb{E}[X]$: expected value of a random variable X
- Covariance of two complex random variables X and Y:

$$\text{cov}[X, Y] = \mathbb{E}[(X - \mathbb{E}[X])\overline{(Y - \mathbb{E}[Y])}]$$

3. Fundamentals of waves revisited

In this section, we summarize a few fundamental notions regarding wave propagation that are needed in the rest of the paper. Most of these notions are well-known and are described for instance in Morse and Ingard (1968). However, a few concepts presented here are not standard, such as the *source response* in Sec. 3.1 and the *B-function* in Sec. 3.2. These and other notions were already presented in more detail in Badeau (2024, Sec. III), except for a thorough investigation of the validity of Eq. (11) in Sec. 3.3, and for the asymptotic expansion of the modal density in Sec. 3.4, which now includes the curvature term.

3.1. Main definitions

In a simply connected open domain $V \subseteq \mathbb{R}^3$, the homogeneous wave equation states that

$$\Delta p(\boldsymbol{x}, t) - \frac{1}{c^2} \frac{\partial^2 p(\boldsymbol{x}, t)}{\partial t^2} = 0, \tag{3}$$

where $p(\boldsymbol{x}, t)$ is the wave amplitude at position $\boldsymbol{x} \in V$ and time $t \in \mathbb{R}$, Δ is the Laplacian, and c is the propagation speed of the wave. By applying the one-dimensional (1D) Fourier transform [Eq. (1)] w.r.t. time to Eq. (3), we get the Helmholtz equation:

$$\Delta \phi(\boldsymbol{x}) + 4\pi^2 k^2 \phi(\boldsymbol{x}) = 0, \tag{4}$$

where the scalar $k = \frac{f}{c}$ is the *wave number* and f denotes the frequency[1].

Given a punctual source position $\boldsymbol{x}_0 \in V$ and a space position $\boldsymbol{x} \in V$, we define the *source response* p as the unique causal solution to the following inhomogeneous wave equation:

$$\Delta p(\boldsymbol{x}, \boldsymbol{x}_0, t) - \frac{1}{c^2} \frac{\partial^2 p(\boldsymbol{x}, \boldsymbol{x}_0, t)}{\partial t^2} = -\delta(\boldsymbol{x} - \boldsymbol{x}_0)\dot{\delta}(t). \tag{5}$$

In the same way, given a punctual source position $\boldsymbol{x}_0 \in V$ and a space position $\boldsymbol{x} \in V$, a Green's function G of the Helmholtz equation (4) is a particular solution to the following inhomogeneous Helmholtz equation:

$$\Delta G(\boldsymbol{x}, \boldsymbol{x}_0, k) + 4\pi^2 k^2 G(\boldsymbol{x}, \boldsymbol{x}_0, k) = -\delta(\boldsymbol{x} - \boldsymbol{x}_0). \tag{6}$$

[1] Note the unusual presence of the term $4\pi^2$ in Eq. (4), which induces a normalization of the wave number different from what is usually found in the literature. This convention is related to our definition of the Fourier transform in Eq. (1) as a function of the *frequency*, instead of the *pulsation* or *angular frequency*.

3.2. B-function

In the case of a simply connected domain $V \subset \mathbb{R}^3$ with boundaries, any Green's function $G(\boldsymbol{x}, \boldsymbol{x}_0, k)$ can generally be analytically continued on a mathematical vicinity $\mathcal{D}$ of V, which depends on the geometry and the specific admittance of the boundary surface. In some cases, this extension holds in the full space $\mathcal{D} = \mathbb{R}^3$. The B-function on $\mathcal{D} \subseteq \mathbb{R}^3$ is then defined as:

$$B(\boldsymbol{y}, \boldsymbol{x}_0, k) = -\left(\Delta G(\boldsymbol{y}, \boldsymbol{x}_0, k) + 4\pi^2 k^2 G(\boldsymbol{y}, \boldsymbol{x}_0, k)\right). \tag{7}$$

By definition of the Green's function G in Eq. (6), the restriction of the B-function to V is $\delta(\boldsymbol{y} - \boldsymbol{x}_0)$. Reciprocally, when $\mathcal{D} = \mathbb{R}^3$, a particular Green's function G is obtained as:

$$G(\boldsymbol{x}, \boldsymbol{x}_0, k) = \int_{\boldsymbol{y} \in \mathbb{R}^3} G_0(\boldsymbol{x} - \boldsymbol{y}, k)\, B(\boldsymbol{y}, \boldsymbol{x}_0, k) d\boldsymbol{y} \tag{8}$$

where G_0 is a free-field Green's function. Equation (8) permits us to interpret the B-function as a spatial distribution of image sources in the free field, which collectively generate inside V the same response as that of the single original source within the bounded domain V. This characterization of the wave field in terms of image sources is usual for instance in the particular case of the rectangular cuboid (Allen and Berkley, 1979).

3.3. Robin's boundary condition

We now consider a simply connected domain $V \not\subseteq \mathbb{R}^3$, whose boundary ∂V is a Lipschitz continuous bidimensional (2D) manifold (i.e. ∂V is locally the graph of a Lipschitz function). The boundary ∂V is characterized by the *specific admittance* $\widehat{\beta}(\boldsymbol{x}, k) \in \mathbb{C}$, which is an essentially bounded function of the position $\boldsymbol{x} \in \partial V$. Then the boundary condition of the Helmholtz equation is written

$$\forall \boldsymbol{x} \in \partial V, \; \frac{\partial \varphi(\boldsymbol{x}, k)}{\partial \boldsymbol{n}(\boldsymbol{x})} + 2\imath\pi k \widehat{\beta}(\boldsymbol{x}, k) \varphi(\boldsymbol{x}, k) = 0, \tag{9}$$

where $\frac{\partial}{\partial \boldsymbol{n}(\boldsymbol{x})}$ denotes partial differentiation in the direction of the outward unit normal vector $\boldsymbol{n}(\boldsymbol{x})$ to the boundary surface at $\boldsymbol{x} \in \partial V$. In the case of *non-rigid* surfaces, which absorb a part of the energy of the incident wave, $\widehat{\beta}(\boldsymbol{x}, k)$ is complex and the real part of $\widehat{\beta}(\boldsymbol{x}, k)$ is positive. Also note that when $k = 0$, Eq. (9) reduces to Neumann's boundary condition.

Since the boundary condition explicitly depends on the wave number k, the solutions to the homogeneous Helmholtz equation also depend on k, thus Eq. (4) has to be rewritten

$$\Delta \varphi(\boldsymbol{x}, k) + 4\pi^2 \kappa(k)^2 \varphi(\boldsymbol{x}, k) = 0, \tag{10}$$

where the wave number is now denoted $\kappa(k) \in \mathbb{C}$.

Now, let us assume in addition that V is a bounded domain. Then the set of eigenvalues $\kappa_n(k)$ and eigenfunctions $\varphi_n(\boldsymbol{x}, k)$ that are solutions to Eqs. (9) and (10) is discrete and indexed by $n \in \mathbb{N}$. Moreover, when the specific admittance $\widehat{\beta}(\boldsymbol{x}, k)$ is purely imaginary, all the eigenvalues $\kappa_n(k)$ and eigenfunctions $\varphi_n(\boldsymbol{x}, k)$ are real-valued, the Robin Laplacian operator is real symmetric, and the set $\{\varphi_n(.,k)\}_{n\in\mathbb{N}}$ forms a Hilbert basis of $L^2(V)$, which is reflected by Eq. (11). When on the contrary $\mathrm{Re}(\widehat{\beta}(\boldsymbol{x},k)) > 0$, then there is energy absorption at the boundary, so both $\kappa_n(k)$ and $\varphi_n(\boldsymbol{x},k)$ are complex and $\mathrm{Im}(\kappa_n(k)) > 0$ (which implies an exponential decay over time). In this case, the Robin Laplacian operator is not Hermitian, and the set $\{\varphi_n(.,k)\}_{n\in\mathbb{N}}$ no longer forms a Hilbert basis of $L^2(V)$, because it is not orthogonal w.r.t. the Hermitian inner product. Nevertheless, it has been recently proved[2] in Bögli *et al.* (2022, Theorem 5.7) that the set $\{\varphi_n(.,k)\}_{n\in\mathbb{N}}$ always forms an Abel basis of $L^2(V)$, a notion which involves a weaker form of convergence than the usual convergence in $L^2(V)$ [see Bögli *et al.* (2022, Definition 5.5)].

In this paper, we introduce a stronger result. Indeed, Proposition 1 in Appendix A proves that if the Robin Laplacian is diagonalizable and if the set $\{\varphi_n(\boldsymbol{x},k)\}_{n\in\mathbb{N}}$ forms a basis of $L^2(V)$ (which is always the case of the real Robin Laplacian), then without loss of generality, this set can be chosen so as to form a *pseudo-orthonormal* basis of $L^2(V)$, which means that

$$\forall \boldsymbol{x}, \boldsymbol{y} \in V,\ \sum_{n\in\mathbb{N}} \varphi_n(\boldsymbol{x},k)\varphi_n(\boldsymbol{y},k) = \delta(\boldsymbol{x}-\boldsymbol{y}). \tag{11}$$

In other words, the dual basis of $\{\varphi_n(\boldsymbol{x},k)\}_{n\in\mathbb{N}}$ w.r.t. the Hermitian inner product is the set $\{\overline{\varphi_n(\boldsymbol{x},k)}\}_{n\in\mathbb{N}}$. The idea of considering such biorthogonal sets of eigenfunctions in the general case of non-self-adjoint linear operators was originally developed in Morse and Feshbach (1953, Sec. 7.5), but it turned out that their mathematical developments based on supposedly convergent series expansions were flawed, since Kostenbauder *et al.* (1997) exhibited examples in physics of diverging series expansions involving non-self-adjoint linear operators. Here, the convergence of the series in Eq. (11) is guaranteed by Proposition 1. In addition, note that Eq. (11) is equivalent to $\forall \psi_1, \psi_2 \in L^2(V)$,

$$\sum_{n\in\mathbb{N}} \int_V \varphi_n(\boldsymbol{x},k)\psi_1(\boldsymbol{x})d\boldsymbol{x} \int_V \varphi_n(\boldsymbol{x},k)\psi_2(\boldsymbol{x})d\boldsymbol{x} = \int_V \psi_1(\boldsymbol{x})\psi_2(\boldsymbol{x})d\boldsymbol{x}. \tag{12}$$

If the set $\{\varphi_n(\boldsymbol{x},k)\}_{n\in\mathbb{N}}$ is not just a simple basis of $L^2(V)$, but in addition satisfies the frame condition of a *Riesz* basis [as defined in Bögli *et al.* (2022, Definition 5.3)], then the series in Eq. (12) converges absolutely, which is a stronger property than conditional convergence.

In other respects, given a punctual source position $\boldsymbol{x}_0 \in V$, a Green's function of the Helmholtz equation as defined in Eq. (6) is expressed as

$$G(\boldsymbol{x},\boldsymbol{x}_0,k) = \sum_{n\in\mathbb{N}} \frac{\varphi_n(\boldsymbol{x}_0,k)\,\varphi_n(\boldsymbol{x},k)}{4\pi^2(\kappa_n(k)^2 - k^2)}. \tag{13}$$

Indeed, Eq. (11) shows that function G in (13) is a solution to the inhomogeneous Helmholtz equation (6) in V, and it satisfies the boundary condition (9) on ∂V because all functions $\varphi_n(\boldsymbol{x},k)$ satisfy this condition.

[2]This property was proved in Bögli *et al.* (2022) when $\widehat{\beta}(\boldsymbol{x},k)$ is constant on ∂V, but the same proof actually applies when $\widehat{\beta}(\boldsymbol{x},k)$ is an essentially bounded function.

Then applying the residue theorem (Ahlfors, 1979) to the derivative of the inverse Fourier transform [Eq. (1)] of function $f \mapsto G(\boldsymbol{x}, \boldsymbol{x}_0, \frac{f}{c})$ leads to the following expression of the causal source response introduced in Eq. (5):

$$p(\boldsymbol{x}, \boldsymbol{x}_0, t) = H(t)\, q(\boldsymbol{x}, \boldsymbol{x}_0, t), \tag{14}$$

where

$$q(\boldsymbol{x}, \boldsymbol{x}_0, t) = c^2 \mathrm{Re}\left(\sum_{n \in \mathbb{N}} \varphi_n(\boldsymbol{x}_0, \tfrac{\nu_n}{c}) \varphi_n(\boldsymbol{x}, \tfrac{\nu_n}{c}) e^{2\imath\pi\nu_n t} \right), \tag{15}$$

$H(t)$ denotes the Heaviside function, which is such that $H(t) = 1$ $\forall t > 0$ and $H(t) = 0$ $\forall t < 0$, and $\forall n \in \mathbb{N}^*$, $\nu_n \in \mathbb{C}$ denotes the unique solution to the equation $\frac{f}{c} = \kappa_n(\frac{f}{c})$, which has both nonnegative real and imaginary parts.

Finally, every eigenfunction $\varphi_n(., k)$ is holomorphic in V, so it can generally be continued as an analytic function on a mathematical vicinity $\mathcal{D}$ of V, which is a solution to the Helmholtz equation (10) on $\mathcal{D}$. By substituting Eqs. (10) and (13) into Eq. (7), we get the closed-form expression of the B-function on $\mathcal{D}$:

$$B(\boldsymbol{y}, \boldsymbol{x}_0, k) = \sum_{n \in \mathbb{N}} \varphi_n(\boldsymbol{x}_0, k)\, \varphi_n(\boldsymbol{y}, k). \tag{16}$$

Note that Eq. (11) confirms that the restriction of the B-function to V is $\delta(\boldsymbol{y} - \boldsymbol{x}_0)$.

3.4. Asymptotic expansion of the modal density

We have seen in Sec. 3.3 that in a simply connected bounded domain, the set of normal modes of the Helmholtz equation is discrete and countable. Moreover, it is well-known that the density of modes is quadratically increasing with the wave number κ, so when the frequency is high enough, the spectrum is well described by a smooth density function $\rho(\kappa, k)$. When the boundary surface of the domain V is twice continuously differentiable and when $\widehat{\beta}(\boldsymbol{x}, k)$ is Lipschitz continuous with a small Lipschitz constant[3], Balian and Bloch (1970) have shown that function $\rho(\kappa, k)$ admits the following second order expansion[4] when $\kappa \to +\infty$:

$$\begin{aligned} \rho(\kappa, k) = \;& 4\pi |V| \kappa^2 + \kappa \int_{\boldsymbol{s} \in \partial V} \left(\tfrac{\pi}{2} - 2 \arctan\!\left(\tfrac{k}{\kappa}\, \imath\widehat{\beta}(\boldsymbol{s}, k)\right) \right) dS(\boldsymbol{s}) \\ & + \tfrac{1}{2\pi} \int_{\boldsymbol{s} \in \partial V} \left(\tfrac{1}{3} + \frac{1}{1 + \left(\frac{k}{\kappa}\, \imath\widehat{\beta}(\boldsymbol{s},k)\right)^2} - \frac{\arctan\left(\frac{k}{\kappa}\, \imath\widehat{\beta}(\boldsymbol{s},k)\right)}{\frac{k}{\kappa}\, \imath\widehat{\beta}(\boldsymbol{s},k)} \right) \left(\tfrac{1}{R_1(\boldsymbol{s})} + \tfrac{1}{R_2(\boldsymbol{s})} \right) dS(\boldsymbol{s}). \end{aligned} \tag{17}$$

This equation holds both in the case of Neumann's boundary condition, which corresponds to $k = 0$ as mentioned previously, and in the case of Robin's boundary condition if and

[3]See the discussion in Balian and Bloch (1970, Sec. V.A).

[4]In Eq. (17), we have modified the notation used in Balian and Bloch (1970) so as to adapt it to the notation used in this paper.

only if $\widehat{\beta}$ is purely imaginary (indeed, this assumption is necessary to guarantee that the eigenvalues $\kappa_n(k)$ are real, in order to be able to define a density function $\rho(\kappa, k)$ over $\mathbb{R}$).

In the right member of Eq. (17), the dominant term $4\pi|V|\kappa^2$ is known as the *volume term.* The first order term, which involves the first integral over the boundary surface, is called the *surface term.* The second order term, which involves the second integral over the boundary surface, is called the *curvature term.* Indeed, this curvature term depends explicitly on the two *main curvature radii* $R_1(\boldsymbol{s})$ and $R_2(\boldsymbol{s})$ at any point $\boldsymbol{s}$ of the boundary surface ∂V. These main curvature radii are defined as follows (Kobayashi and Nomizu, 1996): since ∂V is twice continuously differentiable, then in a vicinity of any point $\boldsymbol{s} \in \partial V$, the 2D manifold ∂V can be locally parameterized as $x_n = -f(\boldsymbol{x}_T)$, where $f : \mathbb{R}^2 \to \mathbb{R}$ is a twice continuously differentiable function, $x_n = \boldsymbol{n}(\boldsymbol{s})^\top(\boldsymbol{x} - \boldsymbol{s}) \in \mathbb{R}$, and $\boldsymbol{x}_T \in \mathbb{R}^2$ is the orthogonal projection of vector $\boldsymbol{x} - \boldsymbol{s}$ onto the 2D subspace tangent to ∂V at $\boldsymbol{s}$. In particular, $f(\boldsymbol{s}) = 0$ and $\nabla f(\boldsymbol{s}) = 0$. Then the Hessian matrix of f at $\boldsymbol{s}$ is generally denoted $I\!I(\boldsymbol{s})$ and it defines the *second fundamental form* (or *shape tensor*) of ∂V at $\boldsymbol{s}$. Its two eigenvectors are called the *principal directions*, and its two eigenvalues are called the *principal curvatures.* Finally, the main curvature radii $R_1(\boldsymbol{s})$ and $R_2(\boldsymbol{s})$ are defined as the inverses of the two principal curvatures.

As explained in Balian and Bloch (1970), a few conditions are required for the asymptotic expansion (17) to hold true up to the second order curvature term: first, since we assumed that ∂V is twice continuously differentiable, the principal curvatures $\frac{1}{R_1(\boldsymbol{s})}$ and $\frac{1}{R_2(\boldsymbol{s})}$ are continuous functions on the compact set ∂V, so they are bounded, which implies $\inf_{\boldsymbol{s}\in\partial V} |R_1(\boldsymbol{s})| > 0$ and $\inf_{\boldsymbol{s}\in\partial V} |R_2(\boldsymbol{s})| > 0$. Second, the wave number κ has to be much larger than both $\sup_{\boldsymbol{s}\in\partial V} \frac{1}{|R_1(\boldsymbol{s})|}$ and $\sup_{\boldsymbol{s}\in\partial V} \frac{1}{|R_2(\boldsymbol{s})|}$. Actually, a third condition is hidden in Balian and Bloch (1970, p. 435): the wave number κ should never get close to $k\widehat{\beta}(\boldsymbol{s}, k)$. The authors did not highlight this condition, because in Eq. (17), κ is real, whereas $k\widehat{\beta}(\boldsymbol{s}, k)$ is purely imaginary. Nevertheless, their proof is based on an analytic continuation to the domain of complex wave numbers, in which case an imaginary κ should not get too close to $k\widehat{\beta}(\boldsymbol{s}, k)$. However, in the statistical wave field theory, we will also consider the analytic continuation of Eq. (17) to complex values of $\widehat{\beta}(\boldsymbol{s}, k)$ that are not purely imaginary, so the condition $\kappa \neq k\widehat{\beta}(\boldsymbol{s}, k)$ should be understood in the broadest sense, i.e. for complex values of both κ and $\widehat{\beta}(\boldsymbol{s}, k)$. Indeed, we can note that this assumption is required in Eq. (17), in order to prevent the denominator $1 + \left(\frac{k}{\kappa}\imath\widehat{\beta}(\boldsymbol{s}, k)\right)^2$ from getting close to zero.

4. Fundamentals of the statistical wave field theory

4.1. Mathematical assumptions

The statistical wave field theory relies on three mathematical assumptions (Badeau, 2024, Sec. IV).

The first assumption states that the punctual source's position $\boldsymbol{x}_0$ is a random variable uniformly distributed in V. This assumption is related to the mixing property, and it turns the source response $p(\boldsymbol{x}, \boldsymbol{x}_0, t)$ introduced in Eq. (5) into a random process, which from now on will be simply denoted $p(\boldsymbol{x}, t)$.

The second assumption states that the frequency f (or equivalently the wave number k) is large, which permits us to apply *spectral smoothing*, i.e. to approximate the discrete modal distribution by a smooth modal density, and to consider asymptotic expansions of various functions of k, including the modal density.

Last, the third assumption, like the first one, is related to the mixing property. It states that the mean and (pseudo-)covariances of the B-function are stationary and isotropic, so that the statistics are independent from the receiver's position and orientation. More precisely, in the case of Neumann's boundary condition, the B-function is a real WSS random process, which means that both its mean $\mu_B = \mathbb{E}[B(\boldsymbol{y}, 0)]$ and its covariances $\text{cov}[B(\boldsymbol{y} + \boldsymbol{z}, 0), B(\boldsymbol{y}, 0)]$ are well-defined and do not depend on $\boldsymbol{y}$. Its stationary first and second order statistics are then characterized by the mean $\mu_B = \lambda \triangleq \frac{1}{|V|}$, the *auto-covariance function* (ACF) $\Gamma_B(\boldsymbol{z}) \triangleq \text{cov}[B(\boldsymbol{y} + \boldsymbol{z}, 0), B(\boldsymbol{y}, 0)]$, and the isotropic power spectrum

$$\widehat{\Gamma}_B(k) \triangleq \int_{\boldsymbol{k} \in \mathcal{S}(0,k)} \widehat{\Gamma}_B(\boldsymbol{k}) \, dS(\boldsymbol{k}).$$

In the case of Robin's boundary condition, the B-function is a complex pseudo-stationary random process, which means that both its mean $\mu_B = \mathbb{E}[B(\boldsymbol{y}, k)]$ and its pseudo-covariances $\text{cov}[B(\boldsymbol{y}+\boldsymbol{z}, k), \overline{B(\boldsymbol{y}, k)}]$ are well-defined and do not depend on $\boldsymbol{y}$. Its stationary first and second order statistics are then characterized by the same mean $\mu_B = \lambda$, the *pseudo-covariance function* (PCF) $J_B(\boldsymbol{z}, k) \triangleq \text{cov}[B(\boldsymbol{y} + \boldsymbol{z}, k), \overline{B(\boldsymbol{y}, k)}]$, and the isotropic *pseudo spectrum*

$$\widehat{J}_B(\kappa, k) \triangleq \int_{\boldsymbol{\kappa} \in \mathcal{S}(0,\kappa)} \widehat{J}_B(\boldsymbol{\kappa}, k) \, dS(\boldsymbol{\kappa}).$$

4.2. Relationship between the modal density and the power and pseudo spectra

In the case of Neumann's boundary condition, we established at the beginning of Badeau (2024, Sec. V) the following relationship between the modal density $\rho(k, 0)$ and the power spectrum $\widehat{\Gamma}_B(k)$:

$$\widehat{\Gamma}_B(k) = \lambda^2 \, \rho(k, 0), \tag{18}$$

which holds asymptotically, when $k \to +\infty$. Equation (18) was proved independently of the asymptotic expansion in Eq. (17), therefore it holds at all orders of this asymptotic expansion. Then in the case of Robin's boundary condition, we proved in Badeau (2024, Sec. VI.A.3) a similar relationship between the modal density $\rho(\kappa, k)$ and the pseudo spectrum $\widehat{J}_B(\kappa, k)$:

$$\widehat{J}_B(\kappa, k) = \lambda^2 \, \rho(\kappa, k), \tag{19}$$

which also holds asymptotically, when $\kappa \to +\infty$, and where $\rho(\kappa, k)$ denotes the analytic continuation of the asymptotic expansion of the modal density in Eq. (17), from a purely imaginary to a complex-valued specific admittance $\widehat{\beta}(\boldsymbol{s}, k)$. Equation (19) was thus proved for any value of $\widehat{\beta}(\boldsymbol{s},k) \in \mathbb{C}$, but this proof was based on the truncation to the first order of the asymptotic expansion in Eq. (17). In this paper, we will admit that Eq. (19) actually holds at all orders of this asymptotic expansion, in the same way as Eq. (18). Nevertheless, we can *prove* this property when the specific admittance $\widehat{\beta}(\boldsymbol{s},k)$ is purely imaginary (so that all eigenfunctions $\varphi_n(\boldsymbol{x},k)$ and eigenvalues $\kappa_n(k)$ that are solutions to the Helmholtz equation (10) are real-valued). To do so, let us adapt the same line of reasoning as in Badeau (2024, Sec. V), to the general case of Robin's boundary condition. According to the third assumption introduced in Sec. 4.1, the B-function is a pseudo-stationary random process defined on $\mathbb{R}^3$. Its second order statistics are thus characterized by its pseudo spectrum $\widehat{J}_B(\boldsymbol{\kappa},k)$, which is defined on the wave vector space $\mathbb{R}^3$. Then Eq. (16) shows that $B(\boldsymbol{y},\boldsymbol{x}_0,k) = \sum_{n\in\mathbb{N}} B_n(\boldsymbol{y},\boldsymbol{x}_0,k)$ with $B_n(\boldsymbol{y},\boldsymbol{x}_0,k) = \varphi_n(\boldsymbol{x}_0,k)\,\varphi_n(\boldsymbol{y},k)$. Since every eigenfunction φ_n is a solution to the Helmholtz equation (10) with the eigenvalue $\kappa_n(k)$, then so is the PCF $J_n(\boldsymbol{z},k)$ of the pseudo-stationary random process B_n. Therefore its 3D-Fourier transform [Eq. (2)], i.e. the measure $\widehat{J}_n(\boldsymbol{\kappa},k)$, is supported in $\mathcal{S}(0,\kappa_n(k))$. Indeed, since $J_n(\boldsymbol{z},k) = \int_{\boldsymbol{\kappa}\in\mathbb{R}^3} e^{2\imath\pi\boldsymbol{\kappa}^\top\boldsymbol{z}} d\widehat{J}_n(\boldsymbol{\kappa},k)$, the Helmholtz equation (10) applied to $J_n(\boldsymbol{z},k)$ yields $4\pi^2(\kappa_n(k)^2 - \|\boldsymbol{\kappa}\|^2)\widehat{J}_n(\boldsymbol{\kappa},k) = 0$, therefore either $\boldsymbol{\kappa} \in \mathcal{S}(0,\kappa_n(k))$ or $\widehat{J}_n(\boldsymbol{\kappa},k) = 0$. In other respects, Eq. (11) shows that $\int_{\boldsymbol{x}\in V} \varphi_n(\boldsymbol{x},k)^2 d\boldsymbol{x} = 1$. Therefore the pseudo-energy of each mode B_n, averaged over both the source position $\boldsymbol{x}_0$ and the receiver position $\boldsymbol{y}$, is $\lambda^2 \int_{\boldsymbol{x}_0\in V}\int_{\boldsymbol{y}\in V} B_n(\boldsymbol{y},\boldsymbol{x}_0)^2 d\boldsymbol{y}d\boldsymbol{x}_0 = \lambda^2$. We thus conclude that $\int_{\mathcal{S}(0,\kappa_n(k))} d\widehat{J}_n(\boldsymbol{\kappa},k) = \lambda^2$. Since the density of modes is given by $\rho(\kappa,k)$, we finally obtain Eq. (19) by smoothing the discrete pseudo spectrum over κ.

4.3. Wigner distribution

In the statistical wave field theory, the power distribution of the wave field is characterized by the Wigner time-frequency distribution of the random process $q(\boldsymbol{x},\boldsymbol{x}_0,t)$ introduced in Eqs. (14) and (15), which from now on will be simply denoted $q(\boldsymbol{x},t)$, since the source position $\boldsymbol{x}_0$ is random. More precisely, let

$$\Gamma_q(\boldsymbol{x}_1,\boldsymbol{x}_2,t_1,t_2) = \mathrm{cov}[q(\boldsymbol{x}_1,t_1),q(\boldsymbol{x}_2,t_2)] \tag{20}$$

denote the ACF of the non-stationary random process $q(\boldsymbol{x},t)$. Then its (cross-)Wigner distribution W_q is defined as follows (Cohen, 1989): $\forall f,t\in\mathbb{R}$,

$$W_q(\boldsymbol{x}_1,\boldsymbol{x}_2,f,t) = \int_{\mathbb{R}} \Gamma_q(\boldsymbol{x}_1,\boldsymbol{x}_2,t+\tfrac{\tau}{2},t-\tfrac{\tau}{2})e^{-2\imath\pi f\tau}d\tau. \tag{21}$$

5. Special theory (Neumann's boundary condition)

5.1. Asymptotic expansion of the power spectrum

By substituting Eq. (17) into Eq. (18), we get

$$\widehat{\Gamma}_B(k) = 4\pi\lambda\left(k^2 + \tfrac{\lambda S(\partial V)}{8}k + \tfrac{\lambda}{24\pi^2}\int_{\boldsymbol{s}\in\partial V}\left(\tfrac{1}{R_1(\boldsymbol{s})} + \tfrac{1}{R_2(\boldsymbol{s})}\right)dS(\boldsymbol{s})\right). \tag{22}$$

More precisely, the spectral representation [see Theorem 8.4.IV in Daley and Vere-Jones (2003, Chap. 8)] of the WSS random process $B(\boldsymbol{y},0)$ can be written in the same way as in Badeau (2024, Sec. V.A.3):

$$B(\boldsymbol{y},0) = \lambda + \int_{\boldsymbol{k}\in\mathbb{R}^3}\int_{\boldsymbol{s}\in\overline{V}} e^{2\imath\pi\boldsymbol{k}^\top(\boldsymbol{y}-\boldsymbol{s})} d\widehat{\xi}^0(\boldsymbol{k},\boldsymbol{s}), \tag{23}$$

where $\widehat{\xi}^0(\boldsymbol{k},\boldsymbol{s})$ is a centered complex random measure with uncorrelated increments on $\mathbb{R}^3\times\overline{V}$, which is Hermitian symmetric w.r.t. $\boldsymbol{k}$, such that for any Borel sets $\mathcal{K}\subset\mathbb{R}^3$ and $\mathcal{V}\subset\overline{V}$,

$$\mathbb{E}\left[\left(\widehat{\xi}^0(\mathcal{K},\mathcal{V})\right)^2\right] = 0 \text{ and } \mathbb{E}\left[\left|\widehat{\xi}^0(\mathcal{K},\mathcal{V})\right|^2\right] = \int_{k\in\mathbb{R}_+} \tfrac{S(\mathcal{K}\cap\mathcal{S}(0,k))}{S(\mathcal{S}(0,k))}\,\widehat{\Lambda}^0(k,\mathcal{V})\,dk \tag{24}$$

with $\widehat{\Lambda}^0(k,\mathcal{V})$ the nonnegative spectral measure on $\mathbb{R}_+\times\overline{V}$ defined as

$$\widehat{\Lambda}^0(k,\mathcal{V}) = 4\pi\lambda^2\left(|\mathcal{V}|k^2 + \tfrac{S(\mathcal{V}\cap\partial V)}{8}k + \tfrac{1}{24\pi^2}\int_{\boldsymbol{s}\in\mathcal{V}\cap\partial V}\left(\tfrac{1}{R_1(\boldsymbol{s})}+\tfrac{1}{R_2(\boldsymbol{s})}\right)dS(\boldsymbol{s})\right), \tag{25}$$

which is such that $\widehat{\Gamma}_B(k) = \widehat{\Lambda}^0(k,\overline{V})$.

5.2. *Green's function*

In the same way as in Badeau (2024, Sec. V.B), Eqs. (7) and (23) lead to the following spectral representation of the Green's function:

$$G(\boldsymbol{x},k) = \mu_G(k) + \int_{\boldsymbol{k}\in\mathbb{R}^3}\int_{\boldsymbol{s}\in\overline{V}} \frac{e^{2\imath\pi\boldsymbol{k}^\top(\boldsymbol{x}-\boldsymbol{s})}}{4\pi^2(\|\boldsymbol{k}\|_2^2-k^2)} d\widehat{\xi}^0(\boldsymbol{k},\boldsymbol{s}) \tag{26}$$

with

$$\mu_G(k) = -\frac{\lambda}{4\pi^2k^2}. \tag{27}$$

5.3. *Source response*

In the same way as in Badeau (2024, Sec. V.C), applying the residue theorem to the derivative of the inverse Fourier transform [Eq. (1)] of function $f\mapsto G(\boldsymbol{x},\frac{f}{c})$ leads to the following expression of the source response:

$$p(\boldsymbol{x},t) = H(t)\,q(\boldsymbol{x},t), \tag{28}$$

where the random process $q(\boldsymbol{x},t)$ admits the following spectral representation:

$$q(\boldsymbol{x},t) = c^2\left(\lambda + \int_{\boldsymbol{k}\in\mathbb{R}^3}\cos(2\pi c\|\boldsymbol{k}\|_2 t)\int_{\boldsymbol{s}\in\overline{V}} e^{2\imath\pi\boldsymbol{k}^\top(\boldsymbol{x}-\boldsymbol{s})} d\widehat{\xi}^0(\boldsymbol{k},\boldsymbol{s})\right). \tag{29}$$

5.4. *Wigner distribution*

In the same way as in Badeau (2024, Sec. V.D), substituting Eqs. (29) and (20) into Eq. (21) leads to the following asymptotic expansion of the Wigner distribution of the random process q, which holds when $f \to +\infty$:

$$W_q(\boldsymbol{x}_1, \boldsymbol{x}_2, f, t) = \tfrac{c^3}{4}\,\mathrm{sinc}(2\pi \tfrac{f}{c} \|\boldsymbol{x}_1 - \boldsymbol{x}_2\|_2)\,\widehat{\Gamma}_B(\tfrac{f}{c}). \tag{30}$$

Then by substituting Eq. (22) into Eq. (30), we get

$$W_q(\boldsymbol{x}_1, \boldsymbol{x}_2, f, t) = W_q(f)\,\gamma(\boldsymbol{x}_1, \boldsymbol{x}_2, f), \tag{31}$$

where

$$W_q(f) \triangleq W_q(\boldsymbol{x}, \boldsymbol{x}, f, t) = \pi\lambda c\left(f^2 + \tfrac{\lambda c S(\partial V)}{8} f + \tfrac{\lambda c^2}{24\pi^2}\int_{\boldsymbol{s}\in\partial V}\left(\tfrac{1}{R_1(\boldsymbol{s})} + \tfrac{1}{R_2(\boldsymbol{s})}\right) dS(\boldsymbol{s})\right) \tag{32}$$

is the stationary Wigner distribution at any point $\boldsymbol{x} \in V$, and

$$\gamma(\boldsymbol{x}_1, \boldsymbol{x}_2, f) = \mathrm{sinc}\left(\frac{2\pi f \|\boldsymbol{x}_1 - \boldsymbol{x}_2\|_2}{c}\right) \tag{33}$$

corresponds to the usual expression of the spectral correlation in a diffuse acoustic field, as established by Cook *et al.* (1955).

6. General theory (Robin's boundary condition)

6.1. *Asymptotic expansion of the pseudo spectrum*

By substituting Eq. (17) into Eq. (19) and by using the well-known identity $\arctan(x) = \frac{1}{2\imath}\ln\left(\frac{1+\imath x}{1-\imath x}\right)$, we get

$$\begin{aligned}\widehat{J}_B(\kappa, k) = \;& 4\pi\lambda\Big(\kappa^2 + \tfrac{\lambda S(\partial V)}{8}\kappa + \tfrac{\lambda}{8\pi}\textstyle\int_{\boldsymbol{s}\in\partial V} 2\imath\kappa \ln\left(\tfrac{\kappa - k\widehat{\beta}(\boldsymbol{s},k)}{\kappa + k\widehat{\beta}(\boldsymbol{s},k)}\right)\\ &+\tfrac{1}{\pi}\left(\tfrac{1}{3} + \tfrac{\kappa^2}{\kappa^2 - k^2\,\widehat{\beta}(\boldsymbol{s},k)^2} + \tfrac{\kappa}{2k\,\widehat{\beta}(\boldsymbol{s},k)}\ln\left(\tfrac{\kappa - k\widehat{\beta}(\boldsymbol{s},k)}{\kappa + k\widehat{\beta}(\boldsymbol{s},k)}\right)\right)\left(\tfrac{1}{R_1(\boldsymbol{s})} + \tfrac{1}{R_2(\boldsymbol{s})}\right) dS(\boldsymbol{s})\Big).\end{aligned} \tag{34}$$

6.2. *Wave numbers distortion*

We will now show that the power spectrum $\widehat{\Gamma}_B(K)$ in Eq. (22) and the pseudo spectrum $\widehat{J}_B(\kappa, k)$ in Eq. (34) are related through the equation

$$\widehat{\Gamma}_B(K) = \widehat{J}_B(\mathcal{K}(K, k), k)\,\tfrac{d\mathcal{K}(K,k)}{dK}. \tag{35}$$

In Eq. (35), function $K \mapsto \mathcal{K}(K, k)$ is such that $\mathcal{K}(K, 0) = K$ and $\mathcal{K}(0, k) = 0$, and it can be interpreted as a distortion of the wave number K when the specific admittance $\widehat{\beta}$ is non-zero.

Indeed, with the change of variable $\kappa = \mathcal{K}(K, k)$ in the right member of Eq. (35), integrating Eq. (35) w.r.t. K yields

$$\int_0^K \widehat{\Gamma}_B(\kappa) d\kappa = \int_0^{\mathcal{K}(K,k)} \widehat{J}_B(\kappa, k)\, d\kappa. \tag{36}$$

Then substituting Eqs. (22) and (34) into Eq. (36) yields

$$\begin{array}{rl} & \int_0^K \left(\kappa^2 + \frac{\lambda S(\partial V)}{8}\kappa + \frac{\lambda}{24\pi^2}\int_{\boldsymbol{s}\in\partial V}\left(\frac{1}{R_1(\boldsymbol{s})}+\frac{1}{R_2(\boldsymbol{s})}\right)dS(\boldsymbol{s})\right)d\kappa \\ = & \int_0^{\mathcal{K}(K,k)} \left(\kappa^2 + \frac{\lambda S(\partial V)}{8}\kappa + \frac{\lambda}{8\pi}\int_{\boldsymbol{s}\in\partial V} 2\imath\kappa \ln\left(\frac{\kappa - k\widehat{\beta}(\boldsymbol{s},k)}{\kappa + k\widehat{\beta}(\boldsymbol{s},k)}\right)\right. \\ & \left. +\frac{1}{\pi}\left(\frac{1}{3} + \frac{\kappa^2}{\kappa^2 - k^2\,\widehat{\beta}(\boldsymbol{s},k)^2} + \frac{\kappa}{2k\,\widehat{\beta}(\boldsymbol{s},k)}\ln\left(\frac{\kappa - k\widehat{\beta}(\boldsymbol{s},k)}{\kappa + k\widehat{\beta}(\boldsymbol{s},k)}\right)\right)\left(\frac{1}{R_1(\boldsymbol{s})}+\frac{1}{R_2(\boldsymbol{s})}\right)dS(\boldsymbol{s})\right)d\kappa, \end{array}$$

which can be rewritten

$$\begin{array}{l} \frac{K^3}{3} + \frac{\lambda S(\partial V)}{8}\frac{K^2}{2} + \frac{\lambda}{24\pi^2}K\int_{\boldsymbol{s}\in\partial V}\frac{1}{R_1(\boldsymbol{s})}+\frac{1}{R_2(\boldsymbol{s})}dS(\boldsymbol{s}) = \frac{\mathcal{K}(K,k)^3}{3} + \frac{\lambda S(\partial V)}{8}\frac{\mathcal{K}(K,k)^2}{2} \\ +\frac{\lambda}{8\pi}\int_{\boldsymbol{s}\in\partial V}\imath\left(\mathcal{K}(K,k)^2\ln\left(\frac{\mathcal{K}(K,k)-k\widehat{\beta}(\boldsymbol{s},k)}{\mathcal{K}(K,k)+k\widehat{\beta}(\boldsymbol{s},k)}\right) - k^2\widehat{\beta}(\boldsymbol{s},k)^2\ln\left(\frac{k\widehat{\beta}(\boldsymbol{s},k)-\mathcal{K}(K,k)}{k\widehat{\beta}(\boldsymbol{s},k)+\mathcal{K}(K,k)}\right) - 2k\widehat{\beta}(\boldsymbol{s},k)\mathcal{K}(K,k)\right) \\ +\frac{1}{4\pi k\,\widehat{\beta}(\boldsymbol{s},k)}\left(\mathcal{K}(K,k)^2\ln\left(\frac{\mathcal{K}(K,k)-k\widehat{\beta}(\boldsymbol{s},k)}{\mathcal{K}(K,k)+k\widehat{\beta}(\boldsymbol{s},k)}\right) + k^2\widehat{\beta}(\boldsymbol{s},k)^2\ln\left(\frac{k\widehat{\beta}(\boldsymbol{s},k)-\mathcal{K}(K,k)}{k\widehat{\beta}(\boldsymbol{s},k)+\mathcal{K}(K,k)}\right) + \frac{10k\,\widehat{\beta}(\boldsymbol{s},k)\mathcal{K}(K,k)}{3}\right) \\ \times\left(\frac{1}{R_1(\boldsymbol{s})}+\frac{1}{R_2(\boldsymbol{s})}\right)dS(\boldsymbol{s}). \end{array} \tag{37}$$

Equation (37) defines function $\mathcal{K}(K,k)$ implicitly, but it is hardly exploitable. Instead, we will prove that asymptotically (i.e. when $K \to +\infty$), $\mathcal{K}(K,k)$ admits the following asymptotic expansion:

$$\mathcal{K}(K,k) = K + \frac{\lambda}{16\pi}\left(\imath\left(1 - \frac{\lambda S(\partial V)}{8\mathcal{K}(K,k)}\right)\epsilon_1(K,k) - \frac{\lambda\,\epsilon_1(K,k)^2}{16\pi\mathcal{K}(K,k)} + \frac{\epsilon_2(K,k)}{k}\right) \tag{38}$$

with

$$\epsilon_1(K,k) = 2\int_{\boldsymbol{s}\in\partial V}\left(\ln\left(\frac{\mathcal{K}(K,k)+k\widehat{\beta}(\boldsymbol{s},k)}{\mathcal{K}(K,k)-k\widehat{\beta}(\boldsymbol{s},k)}\right) - \left(\frac{k\widehat{\beta}(\boldsymbol{s},k)}{\mathcal{K}(K,k)}\right)^2\ln\left(\frac{k\widehat{\beta}(\boldsymbol{s},k)+\mathcal{K}(K,k)}{k\widehat{\beta}(\boldsymbol{s},k)-\mathcal{K}(K,k)}\right) + \frac{2k\widehat{\beta}(\boldsymbol{s},k)}{\mathcal{K}(K,k)}\right)dS(\boldsymbol{s}) \tag{39}$$

and

$$\begin{array}{rl} \epsilon_2(K,k) = & \int_{\boldsymbol{s}\in\partial V}\frac{1}{2\pi\,\widehat{\beta}(\boldsymbol{s},k)}\left(\ln\left(\frac{\mathcal{K}(K,k)+k\widehat{\beta}(\boldsymbol{s},k)}{\mathcal{K}(K,k)-k\widehat{\beta}(\boldsymbol{s},k)}\right) + \left(\frac{k\,\widehat{\beta}(\boldsymbol{s},k)}{\mathcal{K}(K,k)}\right)^2\ln\left(\frac{k\widehat{\beta}(\boldsymbol{s},k)+\mathcal{K}(K,k)}{k\widehat{\beta}(\boldsymbol{s},k)-\mathcal{K}(K,k)}\right) - \frac{2k\,\widehat{\beta}(\boldsymbol{s},k)}{\mathcal{K}(K,k)}\right) \\ & \times\left(\frac{1}{R_1(\boldsymbol{s})}+\frac{1}{R_2(\boldsymbol{s})}\right)dS(\boldsymbol{s}). \end{array} \tag{40}$$

Indeed, let $\epsilon(K,k) = \mathcal{K}(K,k) - K$, and suppose that $\mathcal{K}(K,k)$ is large w.r.t. both $\epsilon(K,k)$ and $\lambda S(\partial V)$. Then by substituting $K = \mathcal{K}(K,k) - \epsilon(K,k)$ in the left member of Eq. (37) and by then dividing both members by $\mathcal{K}(K,k)^2$, we get

$$\begin{array}{l} \epsilon(K,k) = \frac{\epsilon(K,k)^2}{\mathcal{K}(K,k)} - \frac{\lambda S(\partial V)}{8}\frac{\epsilon(K,k)}{\mathcal{K}(K,k)} \\ +\frac{\lambda}{8\pi}\int_{\boldsymbol{s}\in\partial V}\imath\left(\ln\left(\frac{\mathcal{K}(K,k)+k\widehat{\beta}(\boldsymbol{s},k)}{\mathcal{K}(K,k)-k\widehat{\beta}(\boldsymbol{s},k)}\right) - \left(\frac{k\widehat{\beta}(\boldsymbol{s},k)}{\mathcal{K}(K,k)}\right)^2\ln\left(\frac{k\widehat{\beta}(\boldsymbol{s},k)+\mathcal{K}(K,k)}{k\widehat{\beta}(\boldsymbol{s},k)-\mathcal{K}(K,k)}\right) + \frac{2k\widehat{\beta}(\boldsymbol{s},k)}{\mathcal{K}(K,k)}\right) \\ +\frac{1}{4\pi k\,\widehat{\beta}(\boldsymbol{s},k)}\left(\ln\left(\frac{\mathcal{K}(K,k)+k\widehat{\beta}(\boldsymbol{s},k)}{\mathcal{K}(K,k)-k\widehat{\beta}(\boldsymbol{s},k)}\right) + \left(\frac{k\widehat{\beta}(\boldsymbol{s},k)}{\mathcal{K}(K,k)}\right)^2\ln\left(\frac{k\widehat{\beta}(\boldsymbol{s},k)+\mathcal{K}(K,k)}{k\widehat{\beta}(\boldsymbol{s},k)-\mathcal{K}(K,k)}\right) - \frac{2k\widehat{\beta}(\boldsymbol{s},k)}{\mathcal{K}(K,k)}\right)\left(\frac{1}{R_1(\boldsymbol{s})}+\frac{1}{R_2(\boldsymbol{s})}\right)dS(\boldsymbol{s}). \end{array}$$

Hence asymptotically,

$$\epsilon(K,k) = \frac{\lambda}{16\pi}\left(\imath\left(1 - \frac{\lambda S(\partial V)}{8\mathcal{K}(K,k)}\right)\epsilon_1(K,k) - \frac{\lambda\,\epsilon_1(K,k)^2}{16\pi\mathcal{K}(K,k)} + \frac{\epsilon_2(K,k)}{k}\right),$$

which finally proves Eq. (38).

6.3. Green's function

In the same way as in Badeau (2024, Sec. VI.C), Eq. (35) permits us to calculate the isotropic PCF $J_G(\boldsymbol{z}, k)$ of the Green's function, from which the following spectral representation of the random process $G(\boldsymbol{x}, k)$ can be deduced:

$$G(\boldsymbol{x}, k) = \mu_G(k) + \int_{\boldsymbol{k}\in\mathbb{R}^3} \int_{\boldsymbol{s}\in\overline{V}} \frac{e^{2\imath\pi \frac{\mathcal{K}(\|\boldsymbol{k}\|_2, k)}{\|\boldsymbol{k}\|_2} \boldsymbol{k}^\top(\boldsymbol{x}-\boldsymbol{s})}}{4\pi^2(\mathcal{K}(\|\boldsymbol{k}\|_2, k)^2 - k^2)} d\widehat{\xi}^0(\boldsymbol{k}, \boldsymbol{s}), \tag{41}$$

where $\widehat{\xi}^0$ denotes the same complex random measure as in Eq. (26), and with the same mean as in Eq. (27): $\mu_G(k) = -\frac{\lambda}{4\pi^2 k^2}$.

6.4. Source response

In the same way as in Badeau (2024, Sec. VI.D), by applying the residue theorem to the derivative of the inverse Fourier transform [Eq. (1)] of function $f \mapsto G(\boldsymbol{x}, \frac{f}{c})$, we get Eq. (28), with the following spectral representation of the random process q:

$$q(\boldsymbol{x}, t) = c^2 \left(\lambda + \mathrm{Re}\left(\int_{\boldsymbol{k}\in\mathbb{R}^3} \int_{\boldsymbol{s}\in\overline{V}} e^{2\imath\pi\left(\frac{\kappa(\|\boldsymbol{k}\|_2)}{\|\boldsymbol{k}\|_2}\boldsymbol{k}^\top(\boldsymbol{x}-\boldsymbol{s}) + c\,\kappa(\|\boldsymbol{k}\|_2)t\right)} d\widehat{\xi}^0(\boldsymbol{k}, \boldsymbol{s})\right)\right), \tag{42}$$

where $\kappa(k) \in \mathbb{C}$ denotes the unique solution to the equation $\kappa(k) = \mathcal{K}(k, \kappa(k))$ that has both nonnegative real and imaginary parts.

6.5. Simplification of the wave numbers distortion

By substituting $K \leftarrow k$ and $k \leftarrow \kappa(k)$ into Eq. (38), we get the asymptotic expansion

$$\kappa(k) = k + \tfrac{\lambda}{16\pi}\left(\imath\epsilon_1(\kappa(k)) + \tfrac{1}{\kappa(k)}\left(\epsilon_2(\kappa(k)) - \tfrac{\lambda\,\epsilon_1(\kappa(k))}{8}\left(\tfrac{\epsilon_1(\kappa(k))}{2\pi} + \imath S(\partial V)\right)\right)\right) \tag{43}$$

with

$$\epsilon_1(\kappa) = 2\int_{\boldsymbol{s}\in\partial V} \left(\ln\left(\tfrac{1+\widehat{\beta}(\boldsymbol{s},\kappa)}{1-\widehat{\beta}(\boldsymbol{s},\kappa)}\right) - \widehat{\beta}(\boldsymbol{s},\kappa)^2 \ln\left(\tfrac{\widehat{\beta}(\boldsymbol{s},\kappa)+1}{\widehat{\beta}(\boldsymbol{s},\kappa)-1}\right) + 2\widehat{\beta}(\boldsymbol{s},\kappa)\right) dS(\boldsymbol{s}), \tag{44}$$

and

$$\epsilon_2(\kappa) = \int_{\boldsymbol{s}\in\partial V} \tfrac{1}{2\pi\widehat{\beta}(\boldsymbol{s},\kappa)} \left(\ln\left(\tfrac{1+\widehat{\beta}(\boldsymbol{s},\kappa)}{1-\widehat{\beta}(\boldsymbol{s},\kappa)}\right) + \widehat{\beta}(\boldsymbol{s},\kappa)^2 \ln\left(\tfrac{\widehat{\beta}(\boldsymbol{s},\kappa)+1}{\widehat{\beta}(\boldsymbol{s},\kappa)-1}\right) - 2\widehat{\beta}(\boldsymbol{s},\kappa)\right)\left(\tfrac{1}{R_1(\boldsymbol{s})} + \tfrac{1}{R_2(\boldsymbol{s})}\right) dS(\boldsymbol{s}). \tag{45}$$

Equation (43) provides an implicit expression of function $\kappa(k)$. Note that functions $\epsilon_1(\kappa)$ and $\mathrm{Im}(\epsilon_2(\kappa))$ are bounded, as illustrated in Figs. 1(a), 1(b) and 1(d) in Sec. 7. If we further assume that $\widehat{\beta}$ does not get close to 1, then function $\mathrm{Re}(\epsilon_2(\kappa))$ is also bounded, as illustrated in Fig. 1(c) in Sec. 7. Thus if $\lim_{k\to+\infty} \frac{d\widehat{\beta}(\boldsymbol{s},k)}{dk} = 0$, then we can conclude that $\widehat{\beta}(\boldsymbol{s}, \kappa(k)) \underset{k\to+\infty}{\sim} \widehat{\beta}(\boldsymbol{s}, k)$. Equation (43) can then be simplified into an explicit expression:

$$\kappa(k) = k + \tfrac{\lambda}{16\pi}\left(\imath\epsilon_1(k) + \tfrac{1}{k}\left(\epsilon_2(k) - \tfrac{\lambda\epsilon_1(k)}{8}\left(\tfrac{\epsilon_1(k)}{2\pi} + \imath S(\partial V)\right)\right)\right). \tag{46}$$

Note that the condition $\widehat{\beta} \neq 1$ is related to the condition $\kappa \neq k\widehat{\beta}(\boldsymbol{s}, k)$ that we mentioned at the end of Sec. 3.4. Indeed, with the same substitution $k \leftarrow \kappa(k)$ that we used here, the latter condition yields $\widehat{\beta}(\boldsymbol{s}, \kappa(k)) \neq 1$.

6.6. Wigner distribution

In the same way as in Badeau (2024, Sec. VI.F), substituting Eqs. (42) and (20) into Eq. (21) leads to the following asymptotic expansion of the Wigner distribution of the random process q, which holds when $f \to +\infty$:

$$W_q(\boldsymbol{x}_1, \boldsymbol{x}_2, f, t) = \tfrac{c^3}{4} e^{-4\pi c \mathrm{Im}(\kappa(\frac{f}{c}))t} \int_{\boldsymbol{s}\in\overline{V}} \gamma\left(\boldsymbol{x}_1 - \boldsymbol{s}, \boldsymbol{x}_2 - \boldsymbol{s}, \tfrac{f}{c} + \imath \mathrm{Im}(\kappa(\tfrac{f}{c}))\right) d\widehat{\Lambda}^1(\tfrac{f}{c}, \boldsymbol{s}), \tag{47}$$

where the second order asymptotic expansion of function $\kappa(k)$ was given in Eq. (46).

In Eq. (47), function γ is defined as $\forall \boldsymbol{y}_1, \boldsymbol{y}_2 \in \mathbb{R}^3$, $\forall \kappa \in \mathbb{C}$,

$$\gamma(\boldsymbol{y}_1, \boldsymbol{y}_2, \kappa) = \mathrm{sinc}\left(2\pi\sqrt{\left(\kappa \boldsymbol{y}_1 - \overline{\kappa}\boldsymbol{y}_2\right)^\top \left(\kappa \boldsymbol{y}_1 - \overline{\kappa}\boldsymbol{y}_2\right)}\right) \tag{48}$$

where sinc(.) denotes the analytic continuation of the cardinal sine function on $\mathbb{C}$, and $\sqrt{(.)}$ can denote any of the two complex square roots of opposite sign, since function sinc(.) in Eq. (48) is even. In addition, the distorted spectral measure $\widehat{\Lambda}^1(k, \mathcal{V})$ in Eq. (47) is expressed as

$$\widehat{\Lambda}^1(k, \mathcal{V}) = \frac{\widehat{\Lambda}^0((\mathrm{Re}\kappa)^{-1}(k), \mathcal{V})}{(\mathrm{Re}\kappa)'((\mathrm{Re}\kappa)^{-1}(k))}, \tag{49}$$

where the second order asymptotic expansion of $\widehat{\Lambda}^0(., \mathcal{V})$ was given in Eq. (25).

Note that Eq. (47) can be rewritten

$$W_q(\boldsymbol{x}_1, \boldsymbol{x}_2, f, t) = W_q(\boldsymbol{x}_1, \boldsymbol{x}_2, f)\, e^{-2\alpha(\frac{f}{c})t}, \tag{50}$$

where

$$W_q(\boldsymbol{x}_1, \boldsymbol{x}_2, f) \triangleq W_q(\boldsymbol{x}_1, \boldsymbol{x}_2, f, 0) = \tfrac{c^3}{4} \int_{\boldsymbol{s}\in\overline{V}} \gamma\left(\boldsymbol{x}_1 - \boldsymbol{s}, \boldsymbol{x}_2 - \boldsymbol{s}, \tfrac{f}{c} + \imath \mathrm{Im}(\kappa(\tfrac{f}{c}))\right) d\widehat{\Lambda}^1(\tfrac{f}{c}, \boldsymbol{s}), \tag{51}$$

and Eq. (46) yields the first order asymptotic expansion of the spectral attenuation:

$$\alpha(k) \triangleq 2\pi c\, \mathrm{Im}(\kappa(k)) = \tfrac{\lambda c}{8}\left(\mathrm{Re}\left(\epsilon_1(k)\right) + \tfrac{1}{k}\left(\mathrm{Im}\left(\epsilon_2(k) - \tfrac{\lambda\, \epsilon_1(k)^2}{16\pi}\right) - \tfrac{\lambda S(\partial V)}{8}\,\mathrm{Re}\left(\epsilon_1(k)\right)\right)\right).$$

We note that the Wigner distribution $W_q(\boldsymbol{x}_1, \boldsymbol{x}_2, f, t)$ in Eq. (50) has the same factorized form as the Polack time-frequency distribution (Polack, 1988) (which was originally known only for $\boldsymbol{x}_1 = \boldsymbol{x}_2$). Moreover, we get the closed-form expression of the reverberation time in mixing rooms:

$$T_{60}(f) \triangleq \tfrac{3\ln(10)}{\alpha(\frac{f}{c})} = \tfrac{24\ln(10)}{c}\frac{|V|}{\mathrm{Re}\left(\epsilon_1\left(\frac{f}{c}\right)\right) + \frac{c}{f}\left(\mathrm{Im}\left(\epsilon_2\left(\frac{f}{c}\right) - \frac{\lambda\epsilon_1\left(\frac{f}{c}\right)^2}{16\pi}\right) - \frac{\lambda S(\partial V)}{8}\mathrm{Re}\left(\epsilon_1\left(\frac{f}{c}\right)\right)\right)} \tag{52}$$

where functions $\epsilon_1(.)$ and $\epsilon_2(.)$ were defined in Eqs. (44) and (45). If the asymptotic expansion of function $\kappa(k)$ in Eq. (46) is truncated to the first order, then we retrieve the same closed-form expression of $T_{60}(f)$ as in Badeau (2024, Sec. VI.F). Moreover, we have proved in Badeau (2024, Sec. VI.F) that this expression can be rewritten in the same form as Eyring's formula (Eyring, 1930). Therefore Eq. (52) refines Eyring's formula by introducing the first order term in the denominator, which decreases as $\frac{1}{f}$.

Finally, when $\boldsymbol{x}_1 = \boldsymbol{x}_2$, by substituting Eq. (48) into Eq. (51), we get the simplified expression of the power distribution over space at frequency f:

$$W_q(\boldsymbol{x}, \boldsymbol{x}, f) = \tfrac{c^3}{4} \int_{\boldsymbol{s} \in \overline{V}} \operatorname{sinhc}\left(4\pi \operatorname{Im}(\kappa(\tfrac{f}{c})) \|\boldsymbol{x} - \boldsymbol{s}\|_2\right) d\widehat{\Lambda}^1(\tfrac{f}{c}, \boldsymbol{s}),$$

where sinhc denotes the hyperbolic cardinal sine function: $\operatorname{sinhc}(u) = \frac{\sinh(u)}{u}$.

7. Numerical analysis

In this section, we aim to investigate numerically how the specific admittance and the curvature radii of the boundary surface impact the reverberation time. In order to simplify the problem, we will assume that function $\widehat{\beta}(\boldsymbol{s}, \kappa)$ is equal to a constant $\widehat{\beta} \in \mathbb{C}$ at high frequency. Then $\epsilon_1(\kappa)$ in Eq. (44) is also equal to a constant $\epsilon_1 = \zeta_1 \, S(\partial V)$, where

$$\zeta_1 = 2\left(\ln\left(\tfrac{1+\widehat{\beta}}{1-\widehat{\beta}}\right) - \widehat{\beta}^2 \ln\left(\tfrac{\widehat{\beta}+1}{\widehat{\beta}-1}\right) + 2\widehat{\beta}\right).$$

7.1. Surface term

Let us first investigate numerically the surface term. If we neglect the second order terms, Eq. (52) is reduced to

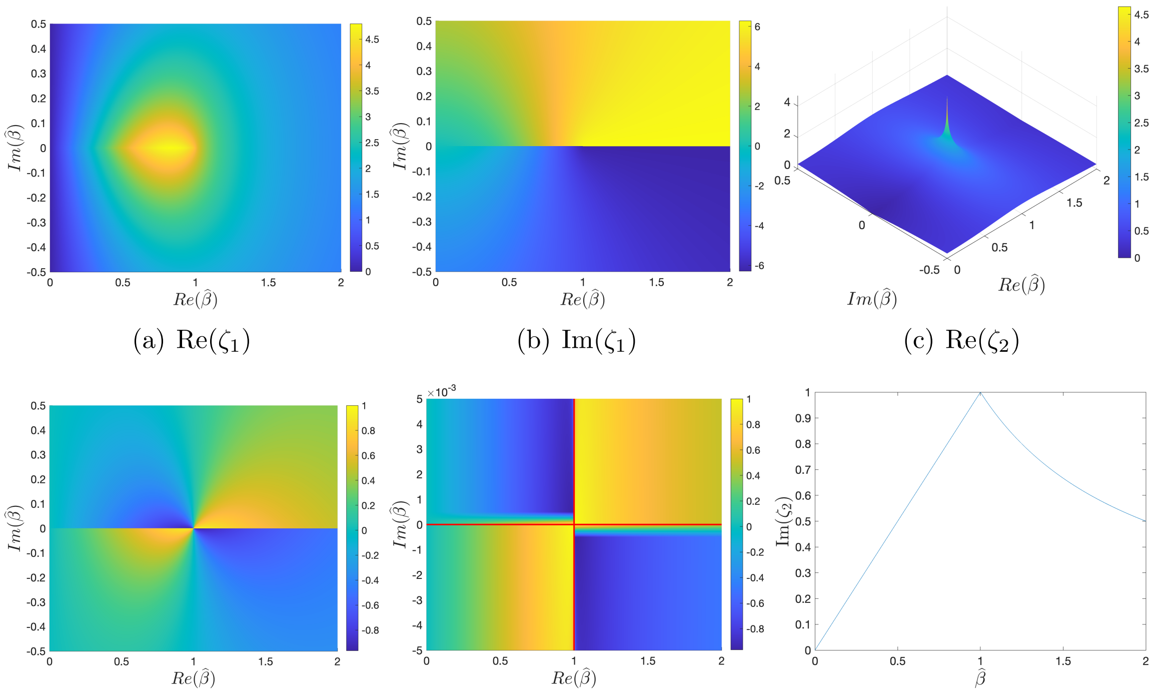
$$T_{60}(f) = \tfrac{24\ln(10)}{\operatorname{Re}(\zeta_1)} \tfrac{|V|}{cS(\partial V)}.$$

Figure 1(a) represents the term $\operatorname{Re}(\zeta_1)$ that appears in the denominator of this last expression, as a function of $\widehat{\beta} \in \mathbb{C}$. As expected, it is always nonnegative, and we also note that it reaches its maximum at a value $\widehat{\beta}_{\max} \in]0, 1[$. By zeroing the derivative of $\operatorname{Re}(\zeta_1)$ w.r.t. $\widehat{\beta}$, we get $\widehat{\beta}_{\max} = \tanh(\frac{1}{\widehat{\beta}_{\max}})$, which can be computed numerically by means of a fixed-point iteration. We thus get $\widehat{\beta}_{\max} \approx 0.8336$, as can be seen in Fig. 1(a). For this particular value of $\widehat{\beta}$, we get $\max\left(\operatorname{Re}(\zeta_1)\right) \approx 4.7987$, which can also be seen in Fig. 1(a).

We can now make the following very important remark: *At high frequency in mixing rooms, the statistical wave field theory predicts that the reverberation time is lower bounded, independently of the specific admittance (and even when $\widehat{\beta}$ depends on $\boldsymbol{s} \in \partial V$ and κ), by a constant that depends on the room geometry only through its volume and surface:*

$$T_{60}(f) \geq \tfrac{24\ln(10)}{4.7987} \tfrac{|V|}{c\,S(\partial V)} \approx \tfrac{11.5160\,|V|}{c\,S(\partial V)}. \tag{53}$$

At first sight, this remark may seem surprising, because we claimed in Sec. 6.6 that the closed-form expression of $T_{60}(f)$ in Badeau (2024, Sec. VI.F) can be rewritten in the same form as Eyring's formula, and Eyring's formula is known to allow the reverberation time to be zero when the average absorption coefficient is 1 on a portion of the boundary of



(a) $\mathrm{Re}(\zeta_1)$ (b) $\mathrm{Im}(\zeta_1)$ (c) $\mathrm{Re}(\zeta_2)$

(d) $\mathrm{Im}(\zeta_2)$ (e) $\mathrm{Im}(\zeta_2)$ (zoom in around $\mathrm{Im}(\widehat{\beta}) = 0$) (f) $\mathrm{Im}(\zeta_2)$ for $\widehat{\beta} \in \mathbb{R}$

Figure 1: Numerical analysis of the surface and curvature terms

positive measure (Eyring, 1930). This apparent contradiction is resolved by noting that in our expression of $T_{60}(f)$, the absorption coefficient is averaged over all possible directions of incidence [see Eq. (125) in Badeau (2024)]; however, the angle-dependent absorption can only reach the maximal value of 1 at a single angle of incidence [see Eq. (28) in Badeau (2024)], therefore the average absorption coefficient is upper bounded by a value that is strictly lower than 1, everywhere on the boundary ∂V.

In other respects, note that this remarkable result has been established by assuming that the room has *locally reacting* boundary surfaces, which means that $\widehat{\beta}$ depends only on the position $\boldsymbol{s} \in \partial V$ and on the wave number κ [as we assumed both in this paper and in Badeau (2024)], but not on the angle of sound incidence. In other words, $\widehat{\beta}$ does not depend on the *orientation* of the wave vector $\boldsymbol{\kappa} \in \mathbb{R}^3$, but only on its norm. Consequently, this inequality might be violated when the room surfaces are not locally reacting. Kuttruff (2014, Chap. 2) explained that in practice, surfaces with local reaction are rather the exception than the rule, and gave several examples of non-locally reacting surfaces. In future work, it will thus be interesting to address the general case of a specific admittance that explicitly depends on the wave vector $\boldsymbol{\kappa}$.

Finally, if we neglect the second order terms, Eq. (46) is reduced to $\kappa(k) = k + \frac{\imath \zeta_1}{16\pi} \frac{S(\partial V)}{|V|}$, so that the frequency distortion is

$$\nu(f) \triangleq cRe(\kappa(\tfrac{f}{c})) = f - \tfrac{\mathrm{Im}(\zeta_1)}{16\pi}\tfrac{cS(\partial V)}{|V|},$$

where $f = ck$. Fig. 1(b) represents the term $\mathrm{Im}(\zeta_1)$. We can observe that, as expected, it is bounded. Moreover, we retrieve a property that was described in Badeau (2024, Sec. III.E.1): if $\mathrm{Im}(\widehat{\beta}) < 0$, then frequency $\nu(f)$ is larger than f, else if $\mathrm{Im}(\widehat{\beta}) > 0$, then $\nu(f)$ is lower than f.

7.2. Curvature term

Let us now consider the curvature term. In addition to the previous simplification, we also assume that function $\frac{1}{R_1(\boldsymbol{s})} + \frac{1}{R_2(\boldsymbol{s})}$ is equal to a constant $\frac{2}{R} \in \mathbb{R}$ at high frequency, so that $\epsilon_2(\kappa)$ in Eq. (45) is also equal to a constant $\epsilon_2 = \frac{\zeta_2\, S(\partial V)}{R}$, where

$$\zeta_2 = \tfrac{1}{\pi\widehat{\beta}}\left(\ln\left(\tfrac{1+\widehat{\beta}}{1-\widehat{\beta}}\right) + \widehat{\beta}^2\ln\left(\tfrac{\widehat{\beta}+1}{\widehat{\beta}-1}\right) - 2\widehat{\beta}\right).$$

Fig. 1(c) represents the real part of ζ_2. We can observe that it presents a singularity at $\widehat{\beta} = 1$, which explains why in Sec. 6.5 we had to assume that $\widehat{\beta}$ never gets close to 1, in order to guarantee that the term $\epsilon_2(\kappa)$ stays bounded. Moreover, we also remark that $\mathrm{Re}(\zeta_2)$ is nonnegative, which shows that convex boundaries always tend to increase the frequency $\nu(f)$, whereas concave boundaries always tend to decrease it, as can be noticed from Eq. (46).

In other respects, Eq. (52) becomes

$$T_{60}(f) = \frac{24\ln(10)}{\mathrm{Re}(\zeta_1)+\frac{c\mathrm{Im}(\zeta_2)}{fR}-\frac{\lambda S(\partial V)c}{8f}\left(\frac{\mathrm{Im}\left(\zeta_1^2\right)}{2\pi}+\mathrm{Re}(\zeta_1)\right)}\frac{|V|}{cS(\partial V)}. \tag{54}$$

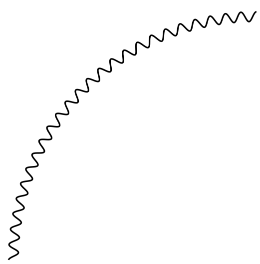
In Eq. (54), the correction to the reverberation time that is related to the curvature is the term $\frac{c\mathrm{Im}(\zeta_2)}{fR}$ that appears in the denominator. In Sec. 3.4, we have listed the conditions that are required for the asymptotic expansion (17) to hold true up to the second order curvature term; these conditions imply $\frac{c}{fR} \ll 1$. In addition, we have represented $\mathrm{Im}(\zeta_2)$ in Fig. 1(d), as a function of $\widehat{\beta} \in \mathbb{C}$. It can be noted that it is bounded, therefore the correction term $\frac{c\mathrm{Im}(\zeta_2)}{fR}$ in the denominator in Eq. (54) is much lower than $\mathrm{Re}\,(\zeta_1)$, which corresponds to the surface term. However, this correction term competes with the other second order term $\frac{\lambda S(\partial V)c}{8f}\left(\frac{\mathrm{Im}\left(\zeta_1^2\right)}{2\pi} + \mathrm{Re}\,(\zeta_1)\right)$, which naturally appeared when pursuing the asymptotic expansion of the wave numbers distortion up to order 2 (see Sec. 6.2). In addition, we observe in Fig. 1(d) that the sign of $\mathrm{Im}(\zeta_2)$ depends on both the real and imaginary parts of $\widehat{\beta}$. Therefore the sign of the correction term $\frac{c\mathrm{Im}(\zeta_2)}{fR}$ in Eq. (54) depends not only on the sign of the curvature radius R, but also on the value of $\widehat{\beta} \in \mathbb{C}$.

Fig. 1(e) represents a zoom in Fig. 1(d) for small values of $\mathrm{Im}(\widehat{\beta})$ (the axes $\mathrm{Im}(\widehat{\beta}) = 0$ and $\mathrm{Re}(\widehat{\beta}) = 1$ are represented as red lines). In particular, it can be noticed that the abscissa axis $\mathrm{Im}(\widehat{\beta}) = 0$ corresponds to a ridge of $\mathrm{Im}\,(\zeta_2)$, both for $\mathrm{Re}(\widehat{\beta}) < 1$ and $\mathrm{Re}(\widehat{\beta}) > 1$. We have thus represented the variations of $\mathrm{Im}\,(\zeta_2)$ when $\widehat{\beta}$ is real in Fig. 1(f). We observe that in this case $\mathrm{Im}\,(\zeta_2) \geq 0$, thus we can see from Eq. (54) that a convex boundary reduces the reverberation time, whereas a concave boundary increases it.

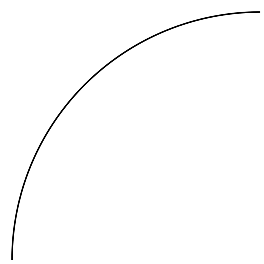
8. Application of the theory to room acoustics

In Sec. 3.4, we listed the conditions for the asymptotic expansion of the modal density to hold, up to the second order curvature term. In particular, these assumptions imply that the wavelength is much lower than the curvature radii of the boundary surface. In room acoustics however, the wavelength is generally larger than the fine details of the room surfaces, so strictly speaking, this condition is generally not met. Thus one may wonder how in practice the statistical wave field theory can be applied to room acoustics. The answer is quite simple: the room geometry must be simplified, so as to stay macroscopic w.r.t. the wavelength, as generally assumed in geometric acoustics (Kuttruff, 2014, Chap. 4).

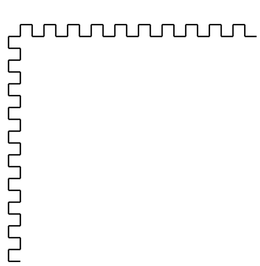
In Fig. 2, we illustrated two examples of such simplifications. First, Fig. 2(a) represents a boundary characterized by curvatures at two different scales: one that is smaller than the wavelength, and one that is larger than the wavelength. Figure 2(b) represents the geometric simplification that is needed to apply the statistical wave field theory: only the large curvature radius is kept, the small oscillations are smoothed out. Even though we have not studied here the second order edge term (which is left for a future publication, see Sec. 9), we also illustrated in Fig. 2(c) a boundary with edges at two different scales: one that is smaller than the wavelength, and one that is larger than the wavelength. Figure 2(d) represents the geometric simplification that is needed to apply the statistical wave field theory: only the edge with long sides is kept, the ones with short sides are smoothed out.



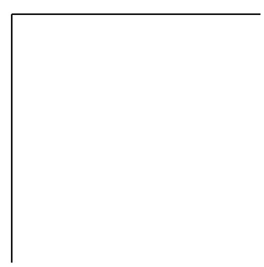
(a) Detailed curved boundary



(b) Smoothed curved boundary



(c) Detailed boundary with edges



(d) Smoothed boundary with edges

Figure 2: Geometric simplifications in room acoustics

In other respects, we have reminded in the introduction that irregular (rough) surfaces produce wave scattering, so one might be afraid that smoothing out the fine details of the room surfaces would result in canceling out, or at least reducing, diffusion. Fortunately, this is not the case: since we assumed right from the beginning, both in Badeau (2024) and in this paper, that the room shape is mixing, diffusion is innately taken into account in the framework of the theory by sticking to this mixing assumption.

9. Conclusion

In this paper, we have refined the results of the statistical wave field theory presented in Badeau (2024), by investigating the impact of a curved boundary surface on the wave field statistics, in the case of mixing rooms. To do so, we have proved that under mild assumptions the set of complex eigenfunctions of the Robin Laplacian forms a pseudo-orthonormal basis of $L^2(V)$, and we have exploited the asymptotic expansion of the modal density up to the second order curvature term, which was given in Balian and Bloch (1970). Compared to Badeau (2024), these improved results rely on the same physical assumptions, plus a few additional ones: the boundary surface is twice continuously differentiable, the specific admittance is Lipschitz continuous with a small Lipschitz constant, the wave number is much greater than the inverse of the curvature radii, and the specific admittance never gets close to 1.

In particular, we were able to provide an improved Eyring-like formula of the reverberation time in mixing rooms, which holds at lower frequency by accounting for reflections on curved surfaces, as explained in Sec. 6.6. To the best of our knowledge, this is the first time that this kind of correction is brought to Eyring's formula. Of course, the improved formula of the reverberation time still needs to be tested by experiments in future work.

In other respects, our numerical analysis of the impact of the surface and curvature terms on the reverberation time has permitted us to draw two important conclusions. First, at high frequency in mixing rooms, the reverberation time is lower bounded, independently of the value of the specific admittance, by a constant that depends on the room geometry only through its volume and surface. Second, the impact of the curvature on the reverberation time depends jointly on the curvature radii and on the value of the specific admittance. Finally, we discussed the simplifications that are required in practice to apply the statistical wave field theory to room acoustics.

In future work, in order to address geometric shapes involving a piecewise twice continuously differentiable boundary including edges and vertices, which is e.g. the case of polyhedral surfaces, we will need to explicitly account for edge diffraction. In particular, we will show that vertices actually generate negligible terms in the asymptotic expansion, whereas edges generate a second order *edge term*, which will be expressed in closed-form. Equipped with the two second order *curvature* and *edge* terms, the improved predictions of the statistical wave field theory will then hold at lower frequencies for a large variety of geometric shapes, including both curved surfaces and edges/vertices.

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their very insightful questions.

Author Declarations

Conflict of Interest: The author of this paper has no conflict of interest to disclose.

Data Availability

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Appendix A. Proof of equation (11)

The following proposition formulates sufficient conditions for Eq. (11) to hold.

Proposition 1. *Let V be a simply connected bounded domain of $\mathbb{R}^3$, whose boundary ∂V is a Lipschitz continuous 2D manifold. We assume that the specific admittance $\widehat{\beta}(\boldsymbol{x}, k) \in \mathbb{C}$ is an essentially bounded function of the position $\boldsymbol{x} \in \partial V$. Then the set of eigenvalues $\kappa_n(k)$ and (generalized) eigenfunctions $\varphi_n(\boldsymbol{x}, k)$ that are solutions to Eqs.* (9) *and* (10) *is discrete and indexed by $n \in \mathbb{N}$.*

In addition, let us assume that the Robin Laplacian is diagonalizable, and that the set $\{\varphi_n(\boldsymbol{x}, k)\}_{n\in\mathbb{N}}$ forms a basis of $L^2(V)$. Then without loss of generality, the set $\{\varphi_n(\boldsymbol{x}, k)\}_{n\in\mathbb{N}}$ can be chosen so as to form a pseudo-orthonormal *basis of $L^2(V)$, which means that Eq.* (11) *holds.*

Before proving Proposition 1, let us comment the two assumptions made in the second part of this proposition. First, we assumed that *the Robin Laplacian is diagonalizable*. In Bögli *et al.* (2022, p. 12), a counterexample of non-diagonalizable Robin Laplacian is provided in 1D, where two different Neumann eigenvalues are mapped to the same Robin eigenvalue, creating a non-trivial eigennilpotent. However, such a case is completely singular, in the sense that it can only happen for a very special combination of both the geometry of ∂V and the function $\widehat{\beta}(\boldsymbol{x}, k)$ defined on ∂V. In other words, the first assumption holds almost surely in general.

Second, we assumed that *the set $\{\varphi_n(\boldsymbol{x}, k)\}_{n\in\mathbb{N}}$ forms a basis of $L^2(V)$*. A weaker property is proved in Bögli *et al.* (2022, Theorem 5.7): for any space dimension, this set forms an *Abel* basis of $L^2(V)$, which is actually *not* a basis of $L^2(V)$ in the usual sense [see Bögli *et al.* (2022, Definition 5.5)]. On the contrary, in dimension 1, it forms a *Riesz* basis of $L^2(V)$, which is a stronger property [see Bögli *et al.* (2022, Definition 5.3)]. The authors leave open the question of whether this set continues to be a Riesz basis in any dimension (Bögli *et al.*, 2022, p. 17). Indeed, no counterexample which would violate the Riesz basis property is known to the best of our knowledge. This question is actually related to another one: is it possible for an eigenfunction φ_n to be such that $\int_V \varphi_n(\boldsymbol{x}, k)^2 d\boldsymbol{x} = 0$? The authors also leave this question as an open problem (Bögli *et al.*, 2022, p. 13), but Proposition 1 shows that this is not possible when the set $\{\varphi_n(\boldsymbol{x}, k)\}_{n\in\mathbb{N}}$ is a basis of $L^2(V)$. Anyway, even if this set were only an Abel basis, $\int_V \varphi_n(\boldsymbol{x},k)^2 d\boldsymbol{x} = 0$ might only occur for very special values of $\widehat{\beta}(\boldsymbol{x},k)$, because the squared eigenfunctions are meromorphic (i.e. holomorphic except for a set of isolated points) functions of $\widehat{\beta}(\boldsymbol{x},k)$, as proved in Bögli *et al.* (2022, Theorem 1.1). In particular, this cannot happen when $\widehat{\beta}(\boldsymbol{x},k)$ is in a mathematical neighborhood of zero, since $\int_V \varphi_n(\boldsymbol{x},k)^2 d\boldsymbol{x}$ depends continuously on $\widehat{\beta}(\boldsymbol{x},k)$, and when $\widehat{\beta}(\boldsymbol{x},k) = 0$ (Neumann's boundary condition), $\int_V \varphi_n(\boldsymbol{x},k)^2 d\boldsymbol{x} = 1$. In the same way, since the set $\{\varphi_n(\boldsymbol{x},k)\}_{n\in\mathbb{N}}$ depends continuously on $\widehat{\beta}(\boldsymbol{x},k)$, and when $\widehat{\beta}(\boldsymbol{x},k) = 0$, it is a Hilbert basis of $L^2(V)$, it seems very reasonable to conjecture that it still behaves nicely, i.e. it is still a Riesz basis as Bögli *et al.* (2022) suggest, at least when $\widehat{\beta}(\boldsymbol{x},k)$ stays in a mathematical neighborhood of zero. So the second assumption should hold in practice under mild conditions.

Let us now prove Proposition 1:

Proof of Proposition 1. First, $\forall n_1, n_2 \in \mathbb{N}$, since φ_{n_1} is an eigenfunction of the Robin Laplacian, we have

$$\begin{aligned}
& 4\pi^2 \kappa_{n_1}(k)^2 \int_V \varphi_{n_1}(\boldsymbol{x},k)\varphi_{n_2}(\boldsymbol{x},k)d\boldsymbol{x} \\
= & \int_V \Delta\varphi_{n_1}(\boldsymbol{x},k)\varphi_{n_2}(\boldsymbol{x},k)d\boldsymbol{x} \\
= & \int_{\partial V} \frac{\partial\varphi_{n_1}(\boldsymbol{x},k)}{\partial \boldsymbol{n}(\boldsymbol{x})}\varphi_{n_2}(\boldsymbol{x},k)dS(\boldsymbol{x}) - \int_V \left(\nabla\varphi_{n_1}(\boldsymbol{x},k)\right)^\top \left(\nabla\varphi_{n_2}(\boldsymbol{x},k)\right) d\boldsymbol{x} \\
= & -2\imath\pi k \int_{\partial V} \widehat{\beta}(\boldsymbol{x},k)\varphi_{n_1}(\boldsymbol{x},k)\varphi_{n_2}(\boldsymbol{x},k)dS(\boldsymbol{x}) - \int_V \left(\nabla\varphi_{n_1}(\boldsymbol{x},k)\right)^\top \left(\nabla\varphi_{n_2}(\boldsymbol{x},k)\right) d\boldsymbol{x}.
\end{aligned}$$

In the same way, we also have

$$\begin{aligned}
& 4\pi^2 \kappa_{n_2}(k)^2 \int_V \varphi_{n_1}(\boldsymbol{x},k)\varphi_{n_2}(\boldsymbol{x},k)d\boldsymbol{x} \\
= & -2\imath\pi k \int_{\partial V} \widehat{\beta}(\boldsymbol{x},k)\varphi_{n_1}(\boldsymbol{x},k)\varphi_{n_2}(\boldsymbol{x},k)dS(\boldsymbol{x}) - \int_V \left(\nabla\varphi_{n_1}(\boldsymbol{x},k)\right)^\top \left(\nabla\varphi_{n_2}(\boldsymbol{x},k)\right) d\boldsymbol{x}.
\end{aligned}$$

By subtracting the two equalities, we get

$$\left(\kappa_{n_1}(k)^2 - \kappa_{n_2}(k)^2\right) \int_V \varphi_{n_1}(\boldsymbol{x},k)\varphi_{n_2}(\boldsymbol{x},k)d\boldsymbol{x} = 0.$$

Therefore, either $\kappa_{n_1}(k)^2 = \kappa_{n_2}(k)^2$, which means that the two eigenfunctions $\varphi_{n_1}(\boldsymbol{x},k)$ and $\varphi_{n_2}(\boldsymbol{x},k)$ are in the same eigenspace of the Robin Laplacian, or $\int_V \varphi_{n_1}(\boldsymbol{x},k)\varphi_{n_2}(\boldsymbol{x},k)d\boldsymbol{x} = 0$, in which case we will say that the two eigenfunctions $\varphi_{n_1}(\boldsymbol{x},k)$ and $\varphi_{n_2}(\boldsymbol{x},k)$ are *pseudo-orthogonal*.

Let us now prove that $\forall n \in \mathbb{N}$, $\int_V \varphi_n(\boldsymbol{x},k)^2 d\boldsymbol{x} \neq 0$. First, if $\kappa_n(k)$ is a simple eigenvalue, then we have already proved that $\varphi_n(\boldsymbol{x},k)$ is pseudo-orthogonal to all the other eigenfunctions $\{\varphi_m(\boldsymbol{x},k)\}_{m\neq n}$. If we assume that in addition $\int_V \varphi_n(\boldsymbol{x},k)^2 d\boldsymbol{x} = 0$, then $\varphi_n(\boldsymbol{x},k)$ is pseudo-orthogonal to the whole set of eigenfunctions, which spans $L^2(V)$ since it forms a basis of $L^2(V)$. Therefore $\varphi_n(\boldsymbol{x},k)$ is pseudo-orthogonal to any function in $L^2(V)$. However, its conjugate $\overline{\varphi_n(\boldsymbol{x},k)}$ belongs to $L^2(V)$, thus $\varphi_n(\boldsymbol{x},k)$ is pseudo-orthogonal to $\overline{\varphi_n(\boldsymbol{x},k)}$, which means that $\int_V \varphi_n(\boldsymbol{x},k)\overline{\varphi_n(\boldsymbol{x},k)}d\boldsymbol{x} = \int_V \left|\varphi_n(\boldsymbol{x},k)\right|^2 d\boldsymbol{x} = 0$. Therefore $\varphi_n(\boldsymbol{x},k)$ is zero on V, which is in contradiction with the fact that it is an eigenfunction of the Robin Laplacian. We thus conclude that $\int_V \varphi_n(\boldsymbol{x},k)^2 d\boldsymbol{x} \neq 0$. So without loss of generality, we can assume that $\int_V \varphi_n(\boldsymbol{x},k)^2 d\boldsymbol{x} = 1$ (we then say that $\varphi_n(.,k)$ is *pseudo-unitary*).

Next, if $\kappa_n(k)$ is a multiple eigenvalue, let M be its multiplicity. Since we assumed that the Robin Laplacian is diagonalizable, there is a basis $\{\varphi_n(\boldsymbol{x},k),\dots,\varphi_{n+M-1}(\boldsymbol{x},k)\}$ of eigenfunctions of the corresponding eigenspace. Let us then define the $M\times M$ matrix $\boldsymbol{H}$ of entries $[\boldsymbol{H}]_{i,j}=\int_V \varphi_{n+i}(\boldsymbol{x},k)\varphi_{n+j}(\boldsymbol{x},k)d\boldsymbol{x}$ for $i,j\in\{0\dots M-1\}$. If we assume that $\boldsymbol{H}$ is singular, then the same line of reasoning as in the previous case shows that the matrix $\boldsymbol{G}$ of entries $[\boldsymbol{G}]_{i,j}=\int_V \varphi_{n+i}(\boldsymbol{x},k)\overline{\varphi_{n+j}(\boldsymbol{x},k)}d\boldsymbol{x}$ is also singular, which is in contradiction with the fact that the set $\{\varphi_n(\boldsymbol{x},k),\dots,\varphi_{n+M-1}(\boldsymbol{x},k)\}$ is linearly independent (since it is a subset of a basis of $L^2(V)$). We thus conclude that matrix $\boldsymbol{H}$ is non-singular, therefore we can apply a Gram–Schmidt-like process to make the set $\{\varphi_n(\boldsymbol{x},k),\dots,\varphi_{n+M-1}(\boldsymbol{x},k)\}$ *pseudo-orthonormal* (i.e. pseudo-orthogonal and such that $\int_V \varphi_m(\boldsymbol{x},k)^2 d\boldsymbol{x}=1\ \forall m\in\{n,\dots,n+M-1\}$). So without loss of generality, we can assume that the set $\{\varphi_n(\boldsymbol{x},k),\dots,\varphi_{n+M-1}(\boldsymbol{x},k)\}$ is pseudo-orthonormal.

To sum up, we have proved that, without loss of generality, the whole set of eigenfunctions $\{\varphi_n(\boldsymbol{x},k)\}_{n\in\mathbb{N}}$ is a pseudo-orthonormal basis of $L^2(V)$, which is equivalent to Eq. (11).

Let us now study the uniqueness of the decomposition in Eq. (11). First, if $\kappa_n(k)$ is a simple eigenvalue, then its two possible pseudo-unitary eigenfunctions are $\varphi_n(\boldsymbol{x},k)$ and $-\varphi_n(\boldsymbol{x},k)$, so the product $\varphi_n(\boldsymbol{x},k)\varphi_n(\boldsymbol{y},k)$ is unique. In the same way, if $\kappa_n(k)$ is a multiple eigenvalue, then any pseudo-orthonormal basis of eigenfunctions of the corresponding eigenspace will result in the same sum of products $\sum_{i=0}^{M}\varphi_{n+i}(\boldsymbol{x},k)\varphi_{n+i}(\boldsymbol{y},k)$. Finally, let us investigate the uniqueness of the pseudo-dual set of the pseudo-orthonormal basis $\{\varphi_n(\boldsymbol{x},k)\}_{n\in\mathbb{N}}$. Here, pseudo-duality is defined as follows: a set $\{\psi_n(\boldsymbol{x},k)\}_{n\in\mathbb{N}}$ is pseudo-dual to $\{\varphi_n(\boldsymbol{x},k)\}_{n\in\mathbb{N}}$ if and only if $\forall m,n\in\mathbb{N}$, $\int_{\boldsymbol{y}\in V}\psi_m(\boldsymbol{y},k)\varphi_n(\boldsymbol{y},k)d\boldsymbol{y}=\delta_{m,n}$. Note that Eq. (11) means that $\forall\psi\in L^2(V)$,

$$\psi(\boldsymbol{x})=\sum_{n\in\mathbb{N}}\varphi_n(\boldsymbol{x},k)\left(\int_{\boldsymbol{y}\in V}\psi(\boldsymbol{y})\varphi_n(\boldsymbol{y},k)d\boldsymbol{y}\right).$$

By applying this equation to every function ψ_n, we get $\psi_n(\boldsymbol{x},k)=\varphi_n(\boldsymbol{x},k)$, which proves the uniqueness of the pseudo-dual set: the pseudo-orthonormal basis $\{\varphi_n(\boldsymbol{x},k)\}_{n\in\mathbb{N}}$ is self-pseudo-dual. □

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