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Abstract

The statistical wave field theory mathematically establishes the statistical laws of the solutions to the wave equation in a bounded domain, which hold at high frequency. It provides the closed-form expressions of the power distribution and the correlations of the wave field jointly over time, frequency and space, in terms of the geometry and the specific admittance of the boundary surface. In this paper, the mathematical foundations of the theory are reworked at the fundamental level of the eigenfunctions of the Laplace operator. We thus introduce a random wave model of the complex Robin Laplacian, which encompasses and generalizes previous works dedicated to the real Robin Laplacian, and which accounts for the absorption of energy at the boundary surface. This model will allow us to establish the first and second order statistics of the room impulse response, and to finally achieve the unification of geometrical acoustics, which is generally used to predict the first echoes that appear in early reverberation, and of probabilistic models, which are generally used to predict the statistical properties of late reverberation.

Keywords: Statistical physics, wave equation, Helmholtz equation, reverberation

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1. Introduction

The statistical wave field theory was originally introduced in Badeau (2024). Its stated goal is to mathematically establish the statistical laws of the solutions to the wave equation in a connected and bounded domain. These laws hold at high frequency, provided that the geometrical shape of the domain satisfies mild assumptions related to chaos. In particular, this theory provides two different kinds of predictions, which are expressed in terms of the geometry and the specific admittance at the boundary surface:

- the isotropic reverberation time in diffuse rooms (Badeau, 2024), and the directional reverberation time in anisotropic wave fields (Badeau, 2026b);
- more generally, the power distribution and the correlations of the wave field jointly over time, frequency and space (Badeau, 2024, 2026b).

The main results obtained so far with this theory were summarized in Badeau (2026c). In particular, the prediction of the reverberation time in diffuse rooms has been verified by experiment, with a remarkable accuracy (Prinn and Badeau, 2026).

The main mathematical tool of the statistical wave field theory is the "B-function", which is a random field that represents the distribution of image sources over space at fixed wave number k. This random field is real-valued when there is no energy absorption at the domain's boundary, and it is complex-valued when there is energy absorption. From the statistics of this B-function, the statistics of the Green's function of the Helmholtz equation are derived, from which the statistics of the room response to a punctual source are finally deduced. In all the previous papers that addressed various aspects of the theory (Badeau, 2024, 2025b,c, 2026a,b), this theory was presented as being based on the three following mathematical assumptions:

- Assumption 1: the source's position is a random variable uniformly distributed in the domain's volume V;
- Assumption 2: the frequency f (or equivalently the wave number k) is large;
- Assumption 3: the mean and pseudo-covariances of the B-function are stationary.

This particular ordering of the three assumptions was chosen in order to closely follow the original mathematical developments in Badeau (2024). However, the most important assumption is the second one. Indeed, at high frequency, it is well known that wave propagation can be approximated in the classical limit by considering the trajectory of rays that undergo successive specular reflections on the domain's boundary, similarly to optical rays (Kuttruff, 2014, Chap. 4). The ray trajectory can then be interpreted as a dynamical billiard that, depending on the boundary geometry, may follow different statistical properties (Tabachnikov, 1995). For instance, a classical billiard is *ergodic* when over time t, the position $\boldsymbol{x}(t) \in V$ and the unit direction vector $\boldsymbol{v}(t) \in \mathcal{S}(0,1)$ of the ray are jointly uniformly distributed in the *phase space* $V \times \mathcal{S}(0,1)$, where the *configuration space* (or *position space*) V is the bounded volume of the billiard, and the *momentum space* $\mathcal{S}(0,1)$ is the unit sphere. In other respects, *mixing* billiards, in addition to being ergodic, are such that two different rays that are arbitrarily close in the phase space at $t = 0$ diverge completely after an asymptotically long elapsed time. Under this assumption and when there is no energy absorption at the boundary, the wave field is asymptotically *diffuse* (Polack, 1992), which means that all its statistics are invariant over space under any translation (it is *stationary*), and any rotation (it is *isotropic*). When on the contrary there is energy absorption at the boundary, then the sound power decreases exponentially over time, at a rate that is uniform in the room and depends on the frequency. The *reverberation time*, often denoted T_{60}, is then defined as the time it takes for the sound pressure level to reduce by 60 dB. In this case, the power distribution of the wave field is stationary neither in the time domain (since the power varies exponentially), nor in the space domain (where the power variations are more complex).

The second most important assumption of the theory is the third one. This assumption is explicitly related to the asymptotic stationarity of the wave field in mixing rooms, when there is no energy absorption. Indeed, in this case the wave field statistics are independent of the receiver's position, the B-function is real-valued, and its first and second order moments are stationary. When on the contrary there is energy absorption at the boundary, then the power distribution depends on the receiver's position, the B-function is complex-valued, and its autocovariance function is *not* stationary. Nevertheless, what was not immediately obvious, but turned out to be true, is that in this case its complex pseudo-covariance function is asymptotically stationary. The prediction of the reverberation time presented in Badeau (2024) is based on this property.

Finally, the first assumption of the theory is related to the third one, through the acoustic reciprocity principle: if the wave field statistics are independent of the receiver's position, then they should also be independent of the *source*'s position, so the source's position can be modeled as a random variable uniformly distributed in V. However, as previously mentioned, the wave field is truly stationary only when there is no energy absorption at the boundary. In this case, the theory's predictions regarding the power distribution over time, frequency and space presented in Badeau (2024, 2025b, 2026a) are asymptotically accurate. When on the contrary there is energy absorption, then the wave field is no longer stationary, so the first assumption is no longer fully justified. Yet, because the temporal decrease of the sound power is uniform in the room, this assumption permitted us to accurately predict the reverberation time (Badeau, 2024, 2025b, 2026b). However, in this case the predictions of the power distribution over space presented in Badeau (2024, 2025b, 2026b) do not enjoy the same remarkable accuracy as that of the reverberation time: they correspond to an average, over all possible source positions in V, of the true power distribution, which actually depends on the source position.

This observation was the impetus for the research presented in this article. Our main goal was to rule out Assumption 1, which was responsible for the reduced accuracy of the predicted power distribution in the case of energy absorption. To do so, the mathematical foundations of the theory have been reworked at a more fundamental level than that of the B-function. Indeed, our new assumptions focus on the statistical properties of the eigenspectrum and eigenfunctions of the complex Robin Laplacian, from which the B-function is defined. To help us with this task, the statistical properties of the eigenfunctions of the *real* Robin Laplacian have already been well investigated in the literature. In this introduction, we do not focus on the existing works, which will be presented in details in Sec. 3. Here, let us just mention that our main contributions are the unification of these existing works, and more importantly their generalization to the *complex* Robin Laplacian. This generalization will be achieved through the powerful approach of the statistical wave field theory, which was originally developed in our previous works. This approach relies on the general framework of *semi-ergodic billiards* introduced in Badeau (2026a), which encompasses both particular cases of mixing billiards addressed in Badeau (2024, 2025b), and of *special polyhedra* investigated in Badeau (2025c).

These new mathematical foundations of the statistical wave field theory will turn out to be extremely powerful: in our next article, not only will we retrieve all the previous predictions of the theory, and improve the accuracy of the predicted power distribution in the case of energy absorption, but we will also end up with an unexpected and remarkable result. Indeed, this new approach will allow us to achieve the unification of geometrical acoustics (Kuttruff, 2014, Chap. 4), which is generally used to predict the first echoes that appear in early reverberation, and of probabilistic models (Kuttruff, 2014, Chap. 5), which are generally used to predict the statistical properties of late reverberation. We will also show that Assumption 3 becomes a mathematical result, proved from the more fundamendal assumptions on the eigenspectrum and eigenfunctions of the Robin Laplacian.

This paper is organized as follows. In the next section, we introduce some mathematical notations that will be used in the rest of the paper. Then in Sec. 2, we summarize a few fundamental notions regarding wave propagation that are needed to develop the statistical wave field theory. In Sec. 3, we present an overview of the proposed contribution, in relation to several related works from the existing literature. In Sec. 4, we introduce our general *random wave model* of the complex Robin Laplacian. Then, in Sec. 5 we summarize the main contributions of this paper, and present a few perspectives for future work. Finally, in Appendix A, we calculate the phase and magnitude distortions of the eigenfunctions that are involved in the random wave model introduced in Sec. 4.

Mathematical notations

- $\triangleq$: equal by definition to
- $\mathbb{N}$: set of whole numbers
- $\mathbb{R}$, $\mathbb{C}$: sets of real and complex numbers, respectively
- $\imath = \sqrt{-1}$: imaginary unit
- $\mathbb{R}_+$: set of nonnegative real numbers
- $\boldsymbol{x}$ (bold font), z (regular): vector and scalar, respectively
- $A \backslash B$: relative complement (set difference) of set B in set A
- $A \subseteq B$: A is a subset of B, possibly equal to B
- $|V|$: Lebesgue measure (volume) of a subset V of $\mathbb{R}^3$
- $\lambda = \frac{1}{|V|}$: mean density of image sources over space
- ∂V: boundary of a subset V of $\mathbb{R}^3$
- $\boldsymbol{n}(\boldsymbol{x})$ where $\boldsymbol{x} \in \partial V$: outward normal to the boundary surface of subset V
- $S(A)$: surface area of a 2-dimensional sub-manifold A of $\mathbb{R}^3$
- $\|.\|_2$: Euclidean/Hermitian norm of a vector or a function
- $\overline{z}$: complex conjugate of $z \in \mathbb{C}$
- $\mathrm{Re}(z)$ (resp. $\mathrm{Im}(z)$): real (resp. imaginary) part of a complex number $z \in \mathbb{C}$
- $\boldsymbol{x}^\top$: transpose of vector $\boldsymbol{x}$
- $\boldsymbol{I}$: identity matrix
- $\mathcal{S}(0,k)$: sphere centered at the origin and of radius k: $\mathcal{S}(0,k) = \{\boldsymbol{k} \in \mathbb{R}^3; \|\boldsymbol{k}\|_2 = k\}$
- $L^1(\mathbb{R})$: Lebesgue space of measurable functions f, such that $\|f\|_1 = \int_{\mathbb{R}} |f(\tau)| d\tau < +\infty$
- $L^2(V)$ where V is a Borel subset of $\mathbb{R}^3$: Hilbert space of measurable functions f supported in V, such that $\|f\|_2 = \sqrt{\int_V |f(\boldsymbol{x})|^2 d\boldsymbol{x}} < +\infty$
- δ: Dirac delta function
- $H(t)$: Heaviside function: $H(t) = 1\ \forall t > 0$ and $H(t) = 0\ \forall t < 0$
- $\mathrm{sign}(t)$: sign function: $\mathrm{sign}(t) = 1\ \forall t > 0$ and $\mathrm{sign}(t) = -1\ \forall t < 0$
- $\Delta\phi(\boldsymbol{x})$: Laplacian of function $\phi(\boldsymbol{x})$
- $Y_{l,m}(\boldsymbol{u})$ for $l \in \mathbb{N}$, $m \in \{-l, \dots, l\}$, and $\boldsymbol{u} \in \mathcal{S}(0,1)$: real spherical harmonic of degree l and order m;
- 1D direct and inverse Fourier transforms of a function $\psi : \mathbb{R} \to \mathbb{C}$:

$$\begin{aligned} \widehat{\psi}(f) &= \int_{t\in\mathbb{R}} \psi(t) e^{-2\imath\pi f t} dt \\ \text{and} \quad \psi(t) &= \int_{f\in\mathbb{R}} \widehat{\psi}(f) e^{+2\imath\pi f t} df \end{aligned}$$

- 3D direct and inverse Fourier transform of a function $\psi : \mathbb{R}^3 \to \mathbb{C}$:

$$\begin{aligned} \widehat{\psi}(\boldsymbol{k}) &= \int_{\boldsymbol{x}\in\mathbb{R}^3} \psi(\boldsymbol{x}) e^{-2\imath\pi \boldsymbol{k}^\top \boldsymbol{x}} d\boldsymbol{x} \\ \text{and} \quad \psi(\boldsymbol{x}) &= \int_{\boldsymbol{k}\in\mathbb{R}^3} \widehat{\psi}(\boldsymbol{k}) e^{+2\imath\pi \boldsymbol{k}^\top \boldsymbol{x}} d\boldsymbol{k} \end{aligned}$$

- $\mathbb{E}[X]$: expected value of a random variable X
- Covariance of two complex random variables X and Y:

$$\mathrm{cov}[X, Y] = \mathbb{E}[(X - \mathbb{E}[X])\overline{(Y - \mathbb{E}[Y])}]$$

- Pseudo-covariance of two complex random variables X and Y:

$$\mathrm{cov}[X, \overline{Y}] = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$$

- $\boldsymbol{k} \in \mathbb{R}^3$: wave vector related to Neumann's boundary condition
- $K = \|\boldsymbol{k}\|_2 \in \mathbb{R}$: wave number related to Neumann's boundary condition
- $k \in \mathbb{R}$: parameter of the Helmholtz equation under Robin's boundary condition characterized by the specific admittance $\widehat{\beta}(\boldsymbol{x}, k)$
- $\boldsymbol{\kappa} \in \mathbb{C}^3$: wave vector related to Robin's boundary condition
- $\kappa = \sqrt{\boldsymbol{\kappa}^\top \boldsymbol{\kappa}} \in \mathbb{C}$: wave number related to Robin's boundary condition
- $\mathcal{K}$: function that assigns $\kappa \in \mathbb{C}$ to $K \in \mathbb{R}$
- $\boldsymbol{\mathcal{K}}$: function that assigns $\boldsymbol{\kappa} \in \mathbb{C}^3$ to $\boldsymbol{k} \in \mathbb{R}^3$

2. Fundamentals of waves revisited

This section summarizes a few fundamental notions regarding wave propagation that are needed in the rest of the paper. Most of these notions are well-known and are described for instance in Morse and Ingard (1968). These and other notions were already presented in more details in Badeau (2024, Sec. III).

2.1. Main definitions

In a connected open domain $V \subseteq \mathbb{R}^3$, the homogeneous wave equation states that

$$\Delta p(\boldsymbol{x}, t) - \tfrac{1}{c^2} \tfrac{\partial^2 p(\boldsymbol{x}, t)}{\partial t^2} = 0, \tag{1}$$

where $p(\boldsymbol{x}, t)$ is the wave amplitude at position $\boldsymbol{x} \in V$ and time $t \in \mathbb{R}$, Δ is the Laplacian, and c is the propagation speed. Applying the Fourier transform w.r.t. time to Eq. (1) yields the Helmholtz equation:

$$\Delta \varphi(\boldsymbol{x}) + 4\pi^2 k^2 \varphi(\boldsymbol{x}) = 0, \tag{2}$$

where the scalar $k = \frac{f}{c}$ is the *wave number* and f denotes the frequency. Any solution φ to Eq. (2) is an eigenfunction of the Laplace operator $-\Delta$, of eigenvalue $4\pi^2 k^2$.

2.2. Boundary condition

Let us now consider a connected domain $V \subset \mathbb{R}^3$, whose boundary ∂V is a Lipschitz continuous bidimensional (2D) manifold (i.e., ∂V is locally the graph of a Lipschitz function). The boundary ∂V is characterized by the *specific admittance* $\widehat{\beta}(\boldsymbol{s}, k) \in \mathbb{C}$, which is an essentially bounded function of the position $\boldsymbol{s} \in \partial V$. Then, Robin's boundary condition of Eq. (2) states that

$$\forall \boldsymbol{s} \in \partial V, \ \tfrac{\partial \varphi(\boldsymbol{s}, k)}{\partial \boldsymbol{n}(\boldsymbol{s})} + 2\imath\pi k \widehat{\beta}(\boldsymbol{s}, k) \varphi(\boldsymbol{s}, k) = 0, \tag{3}$$

where $\frac{\partial}{\partial \boldsymbol{n}(\boldsymbol{s})}$ denotes partial differentiation in the direction of the outward normal $\boldsymbol{n}(\boldsymbol{s})$ to the boundary surface at $\boldsymbol{s}$. This boundary condition explicitly depends on k, so Eq. (2) has to be rewritten

$$\Delta \varphi(\boldsymbol{x}, k) + 4\pi^2 \kappa(k)^2 \varphi(\boldsymbol{x}, k) = 0, \tag{4}$$

where the wave number is now denoted $\kappa(k) \in \mathbb{C}$. In the case of *non-rigid* surfaces, which absorb a part of the energy of the incident wave, $\widehat{\beta}(\boldsymbol{s}, k) \in \mathbb{C}$ and $\mathrm{Re}(\widehat{\beta}(\boldsymbol{s}, k)) > 0$. Then the Robin Laplace operator $-\Delta$ is complex and not self-adjoint, and when the domain V is bounded, the set of (generalized) eigenvalues and eigenfunctions is discrete and complex, and it was proved in Badeau (2025a) that the (generalized) eigenfunctions can be chosen to form a pseudo-orthonormal *Abel basis with brackets* of the Hilbert space $L^2(V)$, a notion that is defined in Badeau (2025a, Sec. III A). However, when there is no energy absorption, $\widehat{\beta}(\boldsymbol{s}, k)$ is purely imaginary, the Robin Laplace operator is real and self-adjoint, the set of eigenvalues and eigenfunctions is discrete and real, and the eigenfunctions can be chosen to form an orthonormal basis of $L^2(V)$. In particular, when $\widehat{\beta}(\boldsymbol{s}, k) = 0$, Eq. (3) reduces to Neumann's boundary condition.

3. Overview of the contribution in relation to existing works

In this section, we review and comment the existing probabilistic models of the eigenfunctions $\{\varphi_n(\boldsymbol{x}, k)\}_{n\in\mathbb{N}}$ of the real Robin Laplacian that are related to the statistical wave field theory, in order to pave the way for the random wave model of the complex Robin Laplacian that will be introduced in Sec. 4. More precisely, we will investigate the statistical properties of certain functionals of the eigenfunctions related to mixing billiards, which are averaged over a small bandwidth. For instance, we will be interested in averages of the form

$$\mathcal{F}^{(1)}(\boldsymbol{x}, \kappa, k) = \frac{\sum\limits_{\{n\in\mathbb{N}:\, \kappa-\frac{\epsilon}{2}\leq\kappa_n(k)<\kappa+\frac{\epsilon}{2}\}} \varphi_n(\boldsymbol{x},k)}{\int_{\kappa-\frac{\epsilon}{2}}^{\kappa+\frac{\epsilon}{2}} \rho(\kappa',k)d\kappa'}, \tag{5}$$

and

$$\mathcal{F}^{(2)}(\boldsymbol{x}_1, \boldsymbol{x}_2, \kappa, k) = \frac{\sum\limits_{\{n\in\mathbb{N}:\, \kappa-\frac{\epsilon}{2}\leq\kappa_n(k)<\kappa+\frac{\epsilon}{2}\}} \varphi_n(\boldsymbol{x}_1,k)\,\varphi_n(\boldsymbol{x}_2,k)}{\int_{\kappa-\frac{\epsilon}{2}}^{\kappa+\frac{\epsilon}{2}} \rho(\kappa',k)d\kappa'}, \tag{6}$$

for a given bandwidth $\epsilon > 0$, where $\kappa_n(k)$ denotes the wave number of $\varphi_n(\boldsymbol{x}, k)$, which is such that $\Delta\varphi_n(\boldsymbol{x}, k) = -4\pi^2\kappa_n(k)^2\varphi_n(\boldsymbol{x}, k)$, and $\rho(\kappa, k)$ denotes the density of eigenmodes as a function of the wave number κ. When ϵ is fixed and $\kappa \to +\infty$, the number of terms involved in the averages in Eqs. (5) and (6) tends to infinity. These averages then become statistically insensitive to certain anomalies that may appear in some rare eigenfunctions, such as quantum scars (Lu *et al.*, 2025). The statistical properties of $\mathcal{F}^{(1)}(\boldsymbol{x}, \kappa, k)$ and $\mathcal{F}^{(2)}(\boldsymbol{x}_1, \boldsymbol{x}_2, \kappa, k)$ then become regular, and can be described by simple probabilistic models.

In the literature, various random wave models have thus been proposed, which generally represent the eigenfunctions $\varphi_n(\boldsymbol{x}, k)$ as centered and pairwise independent random fields (Urbina and Richter, 2006). In particular, the independence of the eigenfunctions is consistent with the quantum weak mixing theorem in phase space for classically weak mixing billiards (Zelditch, 2006) (see Badeau (2026a, Sec. I) for more details on this topic). From now on, we will thus write equations such as $\mathbb{E}[\varphi_n(\boldsymbol{x}, k)] = 0$, and

$$\forall n_1 \neq n_2,\ \mathrm{cov}[\varphi_{n_1}(\boldsymbol{x}_1, k_1)\, \varphi_{n_2}(\boldsymbol{x}_2, k_2)] = 0,$$

which might misleadingly suggest that we are focusing on individual eigenfunctions. The reader should keep in mind that these equations implicitly involve spectral averages, like in Eqs. (5) and (6). Nevertheless, this simplified probabilistic notation will prove to be very useful, and it will allow us to get accurate predictions of the wave field statistics.

3.1. Gaussian distribution

It was originally conjectured by Berry (1977) that the eigenfunctions of classically chaotic systems can be represented as a random superposition of plane waves with fixed wave number. Then, due to the central limit theorem, these eigenfunctions have the same statistics as a Gaussian random field (Urbina and Richter, 2007). This conjecture was originally formulated for the real eigenfunctions of the real Robin Laplacian, but in fact the same argument applies to the complex eigenfunctions of the complex Robin Laplacian. In Sec. 4, we will show that the Gaussian distribution of these complex eigenfunctions is characterized by the following covariances, for two possibly different values of k_1 and k_2,

$$\mathrm{cov}\left[\varphi_n(\boldsymbol{x}_1, k_1), \varphi_n(\boldsymbol{x}_2, k_2)\right] = \lambda\gamma(\boldsymbol{x}_1, \boldsymbol{x}_2, K_n, k_1, k_2), \tag{7}$$

where $\lambda \triangleq \frac{1}{|V|}$ is the inverse of the room's volume, γ is the *correlation function*, and K_n denotes the wave number related to Neumann's boundary condition. It is also characterized by the following pseudo-covariances, for $k_1 = k_2 = k$:

$$\mathrm{cov}\left[\varphi_n(\boldsymbol{x}_1, k), \overline{\varphi_n(\boldsymbol{x}_2, k)}\right] = \lambda C(\boldsymbol{x}_1, \boldsymbol{x}_2, \mathcal{K}(K_n, k), k), \tag{8}$$

where C is the *pseudo-correlation function*, and $\mathcal{K}(K_n, k)$ denotes the wave number related to Robin's boundary condition, which will be expressed in Eq. (15). In the particular case of the real Robin Laplacian, for $k_1 = k_2 = k$, we get $\gamma(\boldsymbol{x}_1, \boldsymbol{x}_2, K, k, k) = C(\boldsymbol{x}_1, \boldsymbol{x}_2, \mathcal{K}(K, k), k) \in \mathbb{R}$.

3.2. Universal correlation function

Let us focus again on the real Robin Laplacian in an ergodic billiard. The correlation function predicted by Berry (1977) is given by the well-known spherical Bessel function:

$$C(\boldsymbol{x}_1, \boldsymbol{x}_2, \kappa, k) = \mathrm{sinc}\left(2\pi\kappa\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|_2\right), \tag{9}$$

where notation C was introduced in Eq. (8). Note that this expression is actually independent of k. This function is often referred to as the *universal correlation function* (Urbina and Richter, 2007), because it applies to any ergodic billiard, independently of its particular geometrical shape.

Note that if $C(\boldsymbol{x}_1, \boldsymbol{x}_2, \kappa, k)$ is defined as in Eq. (9), then for $\boldsymbol{x}_1 = \boldsymbol{x}_2 = \boldsymbol{x}$, $C(\boldsymbol{x}, \boldsymbol{x}, \kappa, k) = 1$, therefore this function does correspond to the mathematical definition of a *correlation* function in probability theory. Equation (9) models the eigenfunctions as isotropically distributed, stationary random fields, which is consistent with the quantum ergodicity theorem in phase space for classically ergodic billiards (Zelditch and Zworski, 1996), and with the eigenstate stacking theorem (Lu *et al.*, 2025) (see Badeau (2026a, Sec. I) for more details on this topic). However, even though Eq. (9) does hold when the two points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are close enough, it was proved inaccurate when the distance between $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ is not much smaller than their distances to the boundary (see Sec. 3.4).

In this paper, we will show that in the general case of the complex Robin Laplacian, the analytic continuation of function $C(\boldsymbol{x}_1, \boldsymbol{x}_2, \kappa, k)$ defined in Eq. (9) to complex values of the wave number κ still satisfies Eq. (8), which describes the pseudo-covariances of the eigenfunctions. In particular, for $\boldsymbol{x}_1 = \boldsymbol{x}_2 = \boldsymbol{x}$, Eqs. (8) and (9) yield $\mathbb{E}[\varphi_n(\boldsymbol{x}, k)^2] = \lambda$, which shows that the random wave model respects the (pseudo)-normality of the eigenfunctions:

$$\mathbb{E}\left[\int_{\boldsymbol{x}\in V} \varphi_n(\boldsymbol{x}, k)^2 d\boldsymbol{x}\right] = 1. \tag{10}$$

3.3. Normalization constraint

By definition, the eigenfunctions of the real Robin Laplacian are normalized, in the sense that $\int_{\boldsymbol{x}\in V} \varphi_n(\boldsymbol{x}, k)^2 d\boldsymbol{x} = 1$. It was first noted by Gornyi and Mirlin (2002) that this normalization constraint was unfortunately incompatible with the Gaussian assumption introduced in Sec. 3.1. Indeed, when $\varphi_n(\boldsymbol{x}, k)$ is Gaussian, then $\mathbb{E}\left[\int_{\boldsymbol{x}\in V} \varphi_n(\boldsymbol{x}, k)^2 d\boldsymbol{x}\right] = 1$ as shown in Eq. (10); however it can be easily verified that $\mathrm{var}\left[\int_{\boldsymbol{x}\in V} \varphi_n(\boldsymbol{x}, k)^2 d\boldsymbol{x}\right] > 0$, whereas this variance should be zero. To address this issue, the principle of maximum entropy, which aims to find an appropriate probability distribution for φ_n by maximizing the entropy of this distribution, subject to fixed mean, covariances, and the normalization constraint, results in projecting the Gaussian distribution onto the so-called *Gaussian Projected Ensemble* (Urbina and Richter, 2007).

This approach leaves the second order statistics (covariances, pseudo-covariances) of the eigenfunctions unchanged w.r.t. the usual Gaussian statistics introduced in Sec. 3.1. However, it slightly modifies the fourth order moments involving the eigenfunctions. For instance, Urbina and Richter (2007, Eq. (52)) show that[1]

$$\begin{aligned} & \mathbb{E}\left[\varphi_n(\boldsymbol{x}^r, k)^2 \varphi_n(\boldsymbol{x}^s, k)^2\right] \\ = \; & \lambda^2 \Big(1 + 2\left(C(\boldsymbol{x}^r, \boldsymbol{x}^s, \mathcal{K}(K_n, k), k)\right)^2 \\ & -2\lambda \textstyle\int_V \left(C(\boldsymbol{x}^r, \boldsymbol{x}, \mathcal{K}(K_n, k), k)\right)^2 d\boldsymbol{x} \\ & -2\lambda \textstyle\int_V \left(C(\boldsymbol{x}, \boldsymbol{x}^s, \mathcal{K}(K_n, k), k)\right)^2 d\boldsymbol{x} \\ & +2\lambda^2 \textstyle\iint_V \left(C(\boldsymbol{x}, \boldsymbol{x}', \mathcal{K}(K_n, k), k)\right)^2 d\boldsymbol{x} d\boldsymbol{x}' \Big). \end{aligned} \tag{11}$$

In the right member of Eq. (11), the dominant term λ^2 corresponds to the product $\mathbb{E}\left[\varphi_n(\boldsymbol{x}^r, k)^2\right] \times \mathbb{E}\left[\varphi_n(\boldsymbol{x}^s, k)^2\right]$. The following term, $2\lambda^2 \left(C(\boldsymbol{x}^r, \boldsymbol{x}^s, \mathcal{K}(K_n, k), k)\right)^2$, is related to the backscattering phenomenon described, e.g., in Weaver and Lobkis (2000): the intensity of the wave field is enhanced at the original source position (i.e., when $\boldsymbol{x}^r = \boldsymbol{x}^s$). Finally, the three remaining terms are related to the normalization constraint. Indeed, it can be easily verified that Eq. (11) implies that $\mathrm{var}\left[\int_{\boldsymbol{x}\in V} \varphi_n(\boldsymbol{x}, k)^2 d\boldsymbol{x}\right] = 0$. In other respects, Eq. (11) shows that

[1] The original formula in Urbina and Richter (2007, Eq. (52)) was established for a two-dimensional space. Here, Eq. (11) provides the equivalent formulation for a three-dimensional space.

the impact of the three last terms, which are all of second order (i.e., of order $O(\frac{1}{\mathcal{K}(K_n,k)^2})$), is always negligible w.r.t. the dominant term λ^2, contrary to the enhanced backscattering term $2\lambda^2 \left(C(\boldsymbol{x}^r, \boldsymbol{x}^s, \mathcal{K}(K_n, k), k)\right)^2$, which tends to $2\lambda^2$ when $\boldsymbol{x}^r \to \boldsymbol{x}^s$. Consequently, the effect of the normalization constraint is always negligible at high frequency. We can reasonably assume that the same remark also holds in the general case of the complex Robin Laplacian. Consequently, we will stick in this paper to the Gaussian assumption introduced in Sec. 3.1.

3.4. Boundary constraint

In Sec. 3.2, we mentioned that the universal correlation function was proved inaccurate when the positions $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ get close to the boundary surface. To address this issue, a general, system-dependent, non-stationary correlation function that is accurate even near the boundary surface was introduced in Hortikar and Srednicki (1998, Eq. (17)). This *multiple-path correlation function* is defined as a sum over all possible classical trajectories from $\boldsymbol{x}_1$ to $\boldsymbol{x}_2$ in the billiard, including the direct path:

$$C(\boldsymbol{x}_1, \boldsymbol{x}_2, \kappa, k) = \sum_p \epsilon^{r_p} \operatorname{sinc}(2\pi\kappa L_p(\boldsymbol{x}_1, \boldsymbol{x}_2)), \tag{12}$$

where:

- $\epsilon = 1$ for Neuman's boundary condition ($\widehat{\beta} = 0$), and $\epsilon = -1$ for Dirichlet's boundary condition ($\mathrm{Im}(\widehat{\beta}) \to -\infty$);
- r_p is the number of reflections along the p-th trajectory;
- L_p is the length of the p-th trajectory; for the direct path ($p = 0$), $L_0(\boldsymbol{x}_1, \boldsymbol{x}_2) = \|\boldsymbol{x}_1 - \boldsymbol{x}_2\|_2$, which corresponds to the universal correlation function introduced in Sec. 3.2.

In Eq. (12), the longer the trajectory, the lower the corresponding term, since its magnitude is proportional to $\frac{1}{L_p}$. In practice, the sum is finite because the discrete spectrum is smoothed over a bandwidth $\epsilon > 0$, as explained at the beginning of Sec. 3, thus all trajectories of length $L_p \gg \frac{1}{\epsilon}$ are smoothed out. For instance, with a bandwidth $\Delta f = 100$ Hz and $c = 340$ ms^{-1}, we get $\frac{1}{\epsilon} = \frac{c}{\Delta f} = 3{,}4$ m. Finally, note that Eq. (12) also holds when the direct path between $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ is blocked by an obstacle. In such a case, the contribution of the universal correlation function in Eq. (12) is zero.

4. Random wave model of the eigenfunctions of the complex Robin Laplacian

To summarize the discussion in Sec. 3, we consider in this article a Gaussian model of the eigenfunctions, involving either the universal correlation function introduced in Sec. 3.2, or the more accurate multiple-path correlation function introduced in Sec. 3.4. This section aims to unify and generalize these two correlation functions to the general case of the complex Robin Laplacian, and to derive the closed-form expressions of the covariances and pseudo-covariances of the eigenfunctions that have been introduced in Eqs. (7) and (8). The general spectral representation of the Robin eigenfunctions is first introduced in Sec. 4.1. This representation involves a *phase distortion*, which can be computed numerically (Sec. 4.2). It also involves a ramdom measure on the momentum space $\mathcal{S}(0, 1)$. Two different probability distributions of this random measure are presented, leading either to the universal (pseudo-)correlation functions (Sec. 4.3), or to the multiple-path (pseudo-)correlation functions (Sec. 4.4).

4.1. Spectral representation of the Robin eigenfunctions

Let us first focus on the real Robin boundary condition ($\widehat{\beta} \in \imath\mathbb{R}$), which includes the Neumann ($\widehat{\beta} = 0$) and Dirichlet ($\mathrm{Im}(\widehat{\beta}) \to -\infty$) boundary conditions as particular cases. In this case, the eigenfunctions can be represented as $\forall n > 0$,

$$\varphi_n(\boldsymbol{x}, k) = \int_{\boldsymbol{u}\in\mathcal{S}(0,1)} e^{\imath 2\pi\kappa_n(k)\boldsymbol{u}^\top \boldsymbol{x}}\, d\alpha_n(\boldsymbol{u}, k), \tag{13}$$

where $\kappa_n(k) \in \mathbb{R}$ is the wave number, and $\alpha_n(., k)$ is a Hermitian, centered, complex, Gaussian random measure on the momentum space $\mathcal{S}(0, 1)$. Note that if $k\widehat{\beta}(\boldsymbol{s}, k) = 0$, then $\kappa_0(k) = 0$ and $\varphi_0(\boldsymbol{x}, k) = \sqrt{\lambda}$, otherwise φ_0 also satisfies Eq. (13). This exception will be ignored in the rest of this paper, because the statistical wave field theory focuses on high frequencies (i.e., $\kappa_n(k) \to +\infty$).

From Eq. (13), it can be readily verified that $\varphi_n(\boldsymbol{x}, k) \in \mathbb{R}$, that $\forall n > 0$, φ_n is a centered Gaussian random process (i.e., $\mathbb{E}[\varphi_n(\boldsymbol{x}, k)] = 0$), and that $\varphi_n(\boldsymbol{x}, k)$ is an eigenfunction of the Laplace operator $-\Delta$, of eigenvalue $\lambda_n(k) = 4\pi^2\kappa_n(k)^2$. In addition, we will assume that the random measures α_n for all n are jointly independent, which makes the eigenfunctions $\varphi_n(\boldsymbol{x}, k)$ statistically independent, as explained in Sec. 3.

The statistical wave field theory permits us to establish the relationship between the parameters of the various spectral representations of the eigenfunctions, for different values of the Robin parameter $\widehat{\beta}$:

$$\left\{\begin{array}{rcl} \kappa_n(k) &=& \mathcal{K}(K_n, k) \\ d\alpha_n(\boldsymbol{u}, k) &=& M(\mathcal{K}(K_n, k), k)e^{\imath\theta(\boldsymbol{u}, K_n, k)} da_n(\boldsymbol{u}), \end{array}\right. \tag{14}$$

where:

- K_n and a_n denote the wave number and random measure associated with Neumann's boundary condition ($\widehat{\beta} = 0$).
- $\mathcal{K}(K, k)$ is the *wave number distortion*, whose first order asymptotic expansion was expressed in (Badeau, 2024, Eq. (102)):

$$\begin{array}{l}\mathcal{K}(K, k) = K + \frac{\imath\lambda}{8\pi}\int_{\boldsymbol{s}\in\partial V}\left(\ln\left(\frac{\mathcal{K}(K,k)+k\widehat{\beta}(\boldsymbol{s},k)}{\mathcal{K}(K,k)-k\widehat{\beta}(\boldsymbol{s},k)}\right)\right. \\ \left. - \left(\frac{k\widehat{\beta}(\boldsymbol{s},k)}{\mathcal{K}(K,k)}\right)^2 \ln\left(\frac{k\widehat{\beta}(\boldsymbol{s},k)+\mathcal{K}(K,k)}{k\widehat{\beta}(\boldsymbol{s},k)-\mathcal{K}(K,k)}\right) + \frac{2k\widehat{\beta}(\boldsymbol{s},k)}{\mathcal{K}(K,k)}\right) dS(\boldsymbol{s}).\end{array} \tag{15}$$

- $M(\kappa, k)$ is the *magnitude distortion*. In Appendix A, it is shown that $\lim_{\kappa\to+\infty} M(\kappa, k) = 1$: at high frequency, the magnitude distortion is always negligible.
- $\theta(\boldsymbol{u}, K, k)$ is the *phase distortion*. In Appendix A, it is shown that function $\boldsymbol{u} \mapsto \theta(\boldsymbol{u}, K, k)$ is the unique odd solution to the following equation: $\forall K > 0$, $\forall \boldsymbol{u} \in \mathcal{S}(0, 1)$,

$$\begin{array}{rl} & \lambda\int_{\boldsymbol{s}\in\partial V}\theta(\boldsymbol{u}, K, k) - \theta(\boldsymbol{R}(\boldsymbol{s})\boldsymbol{u}, K, k)dS(\boldsymbol{s}) \\ = & -2\pi(\mathcal{K}(K, k) - K)\boldsymbol{u}^\top \\ & \times\left(2\lambda\int_{\boldsymbol{s}\in\partial V}\left(\boldsymbol{n}(\boldsymbol{s})\boldsymbol{n}(\boldsymbol{s})^\top\boldsymbol{s}\right)dS(\boldsymbol{s})\right) \\ & +\imath\lambda\int_{\boldsymbol{s}\in\partial V}\ln\left(\frac{\mathcal{K}(K,k)\,\boldsymbol{u}^\top\boldsymbol{n}(\boldsymbol{s})+k\widehat{\beta}(\boldsymbol{s},k)}{\mathcal{K}(K,k)\,\boldsymbol{u}^\top\boldsymbol{n}(\boldsymbol{s})-k\widehat{\beta}(\boldsymbol{s},k)}\right)dS(\boldsymbol{s}), \end{array} \tag{16}$$

where $\boldsymbol{n}(\boldsymbol{s})$ is the outward normal to the boundary surface at $\boldsymbol{s}$, and $\boldsymbol{R}(\boldsymbol{s})$ denotes the reflection matrix w.r.t. the 2D subspace parallel to the plane tangent to ∂V at $\boldsymbol{s}$: $\boldsymbol{R}(\boldsymbol{s}) = \boldsymbol{I} - 2\,\boldsymbol{n}(\boldsymbol{s})\,\boldsymbol{n}(\boldsymbol{s})^\top$, where $\boldsymbol{I}$ is the identity matrix.

Then, substituting Eq. (14) into Eq. (13) yields

$$\varphi_n(\boldsymbol{x}, k) = \int_{\boldsymbol{u}\in\mathcal{S}(0,1)} e^{\imath\left(2\pi\mathcal{K}(K_n,k)\boldsymbol{u}^\top\boldsymbol{x}+\theta(\boldsymbol{u},K_n,k)\right)} da_n(\boldsymbol{u}). \tag{17}$$

Finally, an important property of this spectral representation of the eigenfunctions is that, in fact, Eqs. (13) to (17) *continue to hold, even in the general case of the complex Robin's boundary condition* ($\widehat{\beta} \in \mathbb{C}$), by simply considering the analytic continuation of functions $\mathcal{K}(K, k)$, $M(\kappa, k)$ and $\theta(\boldsymbol{u}, K, k)$ to the complex domain. The differences with the real case are that $\mathcal{K}(K, k)$, $M(\kappa, k)$, $\theta(\boldsymbol{u}, K, k)$, and $\varphi_n(\boldsymbol{x}, k)$ are complex-valued, and the complex random measure α_n is no longer Hermitian.

4.2. Numerical computation of the phase distortion function

In Appendix A.2, it is shown that Eq. (16) can be solved in practice by decomposing function θ as a series of spherical harmonics. More precisely, for all $l \in \mathbb{N}$, let $\boldsymbol{Y}_l$ denote the $(2l+1)$-dimensional column vector of coefficients $Y_{l,m}$, where $\forall m \in \{-l, \dots, l\}$, $Y_{l,m}$ denotes the real spherical harmonic of degree l and order m (Courant and Hilbert, 1989). Function $\theta(\boldsymbol{u}, K, k)$ can be then decomposed as the following series:

$$\theta(\boldsymbol{u}, K, k) = \sum_{l\in 2\mathbb{N}+1} \boldsymbol{\theta}_l(K, k)^\top \boldsymbol{Y}_l\,(\boldsymbol{u})\,. \tag{18}$$

In Eq. (18), the $(2l + 1)$-dimensional column vector $\boldsymbol{\theta}_l(K, k)$ is the unique solution of the $(2l + 1) \times (2l + 1)$ linear system

$$\boldsymbol{A}_l\,\boldsymbol{\theta}_l(K, k) = \boldsymbol{b}_l(K, k), \tag{19}$$

where matrix $\boldsymbol{A}_l$ was defined in Badeau (2026a, Eq. (16)):

$$\begin{aligned}\boldsymbol{A}_l = \;& \lambda \int_{\boldsymbol{s}\in\partial V} \Big(\boldsymbol{I} - \Big(\int_{\boldsymbol{u}\in\mathcal{S}(0,1)} \boldsymbol{Y}_l(\boldsymbol{u}) \\ & \times \boldsymbol{Y}_l\left(\boldsymbol{u} - 2\boldsymbol{n}(\boldsymbol{s})(\boldsymbol{n}(\boldsymbol{s})^\top\boldsymbol{u})\right)^\top dS(\boldsymbol{u})\Big)\Big) dS(\boldsymbol{s}),\end{aligned} \tag{20}$$

and vector $\boldsymbol{b}_l(K, k)$ is defined as

$$\begin{aligned}\boldsymbol{b}_l(K, k) = \;& -\Big(2\pi\left(\mathcal{K}(K, k) - K\right) \int_{\boldsymbol{u}\in\mathcal{S}(0,1)} \boldsymbol{Y}_l(\boldsymbol{u})\boldsymbol{u}^\top dS(\boldsymbol{u})\Big) \\ & \times \left(2\lambda \int_{\boldsymbol{s}\in\partial V} \boldsymbol{n}(\boldsymbol{s}) \left(\boldsymbol{n}(\boldsymbol{s})^\top\boldsymbol{s}\right)\, dS(\boldsymbol{s})\right) \\ & + \int_{\boldsymbol{u}\in\mathcal{S}(0,1)} \boldsymbol{Y}_l(\boldsymbol{u})\Big(\imath\lambda \\ & \times \int_{\boldsymbol{s}\in\partial V} \ln\left(\frac{\mathcal{K}(K,k)\boldsymbol{u}^\top\boldsymbol{n}(\boldsymbol{s})+k\widehat{\beta}(\boldsymbol{s},k)}{\mathcal{K}(K,k)\boldsymbol{u}^\top\boldsymbol{n}(\boldsymbol{s})-k\widehat{\beta}(\boldsymbol{s},k)}\right) dS(\boldsymbol{s})\Big) dS(\boldsymbol{u}).\end{aligned} \tag{21}$$

Note that since $\boldsymbol{u} \mapsto \theta(\boldsymbol{u}, K, k)$ is an odd function, only the terms of odd degree l in the spherical harmonic expansion in Eq. (18) need to be computed. Moreover, in the case of ergodic billiards, all matrices $\boldsymbol{A}_l$ for $l > 0$ are positive definite (Badeau, 2026a, Remark 2), so the linear system of equations in Eq. (19) can be easily solved numerically.

4.3. Universal (pseudo-)correlation functions

In this section, we characterize the probability distribution of the random measure a_n in Eq. (17) that corresponds to the universal correlation function introduced in Sec. 3.2, and we establish Eqs. (7) and (8), which hold in the general case of the complex Robin Laplacian. So, let us assume that the Hermitian random measure a_n is defined as

$$da_n(\boldsymbol{u}) = \tfrac{db_n(\boldsymbol{u})+\overline{db_n(-\boldsymbol{u})}}{\sqrt{2}}, \tag{22}$$

where the random measure b_n is centered, complex, circular (i.e., its pseudo-covariance function is zero), Gaussian, and such that

$$\mathbb{E}[db_n(\boldsymbol{u}_1)\overline{db_n(\boldsymbol{u}_2)}] = \tfrac{\lambda}{4\pi}\delta(\boldsymbol{u}_2, \boldsymbol{u}_1)dS(\boldsymbol{u}_1)dS(\boldsymbol{u}_2). \tag{23}$$

Then, substituting Eqs. (22) and (23) into Eq. (17) yields

$$\begin{aligned}
& \operatorname{cov}\left[\varphi_n(\boldsymbol{x}_1, k), \overline{\varphi_n(\boldsymbol{x}_2, k)}\right] \\
= \;& \mathbb{E}\Big[\textstyle\int_{\boldsymbol{u}_1\in\mathcal{S}(0,1)} e^{\imath\left(2\pi\mathcal{K}(K_n,k)\boldsymbol{u}_1^\top\boldsymbol{x}_1+\theta(\boldsymbol{u}_1,K_n,k)\right)}\, da_n(\boldsymbol{u}_1) \\
& \times \textstyle\int_{\boldsymbol{u}_2\in\mathcal{S}(0,1)} e^{\imath\left(2\pi\mathcal{K}(K_n,k)\boldsymbol{u}_2^\top\boldsymbol{x}_2+\theta(\boldsymbol{u}_2,K_n,k)\right)}\, da_n(\boldsymbol{u}_2)\Big] \\
= \;& \tfrac{\lambda}{4\pi}\textstyle\int_{\boldsymbol{u}\in\mathcal{S}(0,1)} e^{\imath\left(2\pi\mathcal{K}(K_n,k)\boldsymbol{u}^\top\boldsymbol{x}_1+\theta(\boldsymbol{u},K_n,k)\right)} \\
& \times e^{-\imath\left(2\pi\mathcal{K}(K_n,k)\boldsymbol{u}^\top\boldsymbol{x}_2+\theta(\boldsymbol{u},K_n,k)\right)} dS(\boldsymbol{u}) \\
= \;& \tfrac{\lambda}{4\pi}\textstyle\int_{\boldsymbol{u}\in\mathcal{S}(0,1)} e^{\imath\left(2\pi\mathcal{K}(K_n,k)\boldsymbol{u}^\top(\boldsymbol{x}_1-\boldsymbol{x}_2)\right)} dS(\boldsymbol{u}) \\
= \;& \lambda\,\mathrm{sinc}\left(2\pi\mathcal{K}(K_n,k)\|\boldsymbol{x}_1-\boldsymbol{x}_2\|_2\right) \\
= \;& \lambda\, C(\boldsymbol{x}_1, \boldsymbol{x}_2, \mathcal{K}(K_n,k), k),
\end{aligned}$$

which is the same as Eq. (8), where $C(\boldsymbol{x}_1, \boldsymbol{x}_2, \kappa, k)$ is the universal correlation function defined in Eq. (9), which from now on will be referred to as the *universal pseudo-correlation function* in the general complex case. In addition, Eqs. (22), (23) and (17) also yield

$$\begin{aligned}
& \operatorname{cov}\left[\varphi_n(\boldsymbol{x}_1, k_1), \varphi_n(\boldsymbol{x}_2, k_2)\right] \\
= \;& \mathbb{E}\Big[\textstyle\int_{\boldsymbol{u}_1\in\mathcal{S}(0,1)} e^{\imath\left(2\pi\mathcal{K}(K_n,k_1)\boldsymbol{u}_1^\top\boldsymbol{x}_1+\theta(\boldsymbol{u}_1,K_n,k_1)\right)} da_n(\boldsymbol{u}_1) \\
& \textstyle\int_{\boldsymbol{u}_2\in\mathcal{S}(0,1)} e^{-\imath\left(2\pi\overline{\mathcal{K}(K_n,k_2)}\boldsymbol{u}_2^\top\boldsymbol{x}_2+\overline{\theta(\boldsymbol{u}_2,K_n,k_2)}\right)} \overline{da_n(\boldsymbol{u}_2)}\Big] \\
= \;& \tfrac{\lambda}{4\pi}\textstyle\int_{\boldsymbol{u}\in\mathcal{S}(0,1)} e^{\imath\left(2\pi\mathcal{K}(K_n,k_1)\boldsymbol{u}^\top\boldsymbol{x}_1+\theta(\boldsymbol{u},K_n,k_1)\right)} \\
& \times e^{-\imath\left(2\pi\overline{\mathcal{K}(K_n,k_2)}\boldsymbol{u}^\top\boldsymbol{x}_2+\overline{\theta(\boldsymbol{u},K_n,k_2)}\right)} dS(\boldsymbol{u}) \\
= \;& \lambda\,\gamma(\boldsymbol{x}_1, \boldsymbol{x}_2, K_n, k_1, k_2),
\end{aligned}$$

which is the same as Eq. (7), where the *universal correlation function* γ is defined as

$$\begin{aligned}
\gamma(\boldsymbol{x}_1, \boldsymbol{x}_2, K, k_1, k_2) \triangleq \;& \tfrac{1}{4\pi}\textstyle\int_{\boldsymbol{u}\in\mathcal{S}(0,1)} e^{\imath 2\pi\boldsymbol{u}^\top\left(\mathcal{K}(K,k_1)\boldsymbol{x}_1-\overline{\mathcal{K}(K,k_2)}\boldsymbol{x}_2\right)} \\
& \times e^{\imath\left(\theta(\boldsymbol{u},K,k_1)-\overline{\theta(\boldsymbol{u},K,k_2)}\right)} dS(\boldsymbol{u}).
\end{aligned} \tag{24}$$

Remark 1. *Note that the imaginary part of function $\theta(\boldsymbol{u}, K, k)$ defined in Eq. (16) depends on k and $\mathcal{K}(K,k)$, but not directly on K. Therefore, when $k_1 = k_2 = k$, $\gamma(\boldsymbol{x}_1, \boldsymbol{x}_2, K, k, k)$ in Eq. (24) depends on K only through $\mathcal{K}(K,k)$.*

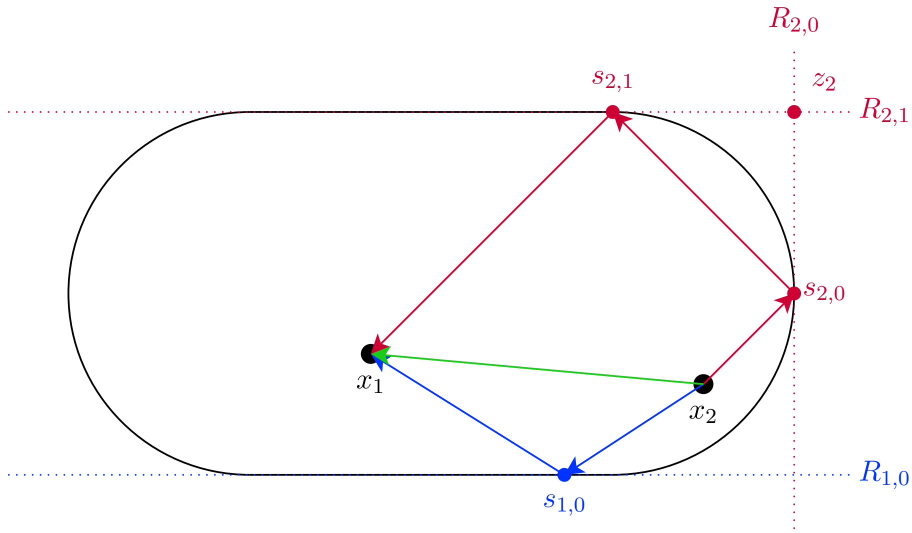

Figure 1: Examples of ray trajectories in a classical billiard (Bunimovich's stadium). The billiard's boundary is represented by the black closed line. The trajectories are represented by piecewise linear, colored lines.

4.4. Multiple-path (pseudo-)correlation functions

In this section, we characterize the probability distribution of the random measure a_n in Eq. (17) that corresponds to the multiple-path correlation function introduced in Sec. 3.4, and we establish Eqs. (7) and (8), which hold in the general case of the complex Robin Laplacian.

Suppose that from position $\boldsymbol{x}_2 \in V$ to position $\boldsymbol{x}_1 \in V$, the p-th trajectory in the classical billiard undergoes r_p reflections on the boundary ∂V. For $q \in \{0 \dots r_p - 1\}$, the q-th reflection occurs at a reflection point $\boldsymbol{s}_{p,q} \in \partial V$, and is represented by a reflection matrix $\boldsymbol{R}_{p,q} \in \mathbb{R}^3$ w.r.t. the 2D subspace parallel to the plane tangent to ∂V at $\boldsymbol{s}_{p,q} \in \partial V$.

Fig. 1 illustrates three examples of trajectories in a classical billiard (whose boundary is represented by the black closed line): the green one represents the direct path ($p = 0$), without any reflection ($r_0 = 0$); the blue one represents a first trajectory ($p = 1$) with one reflection ($r_1 = 1$); the red one represents a second trajectory ($p = 2$) with two reflections ($r_2 = 2$). At every reflection point, the plane tangent to ∂V is represented by a dotted line of the same color as the trajectory.

Every reflection along trajectory p is mathematically described by a similarity transformation $\psi_{p,q}$, which assigns to any point $\boldsymbol{x} \in \mathbb{R}^3$ the point $\psi_{p,q}(\boldsymbol{x}) = \boldsymbol{s}_{p,q} + \boldsymbol{R}_{p,q}(\boldsymbol{x} - \boldsymbol{s}_{p,q})$. The composition of all successive reflections along the trajectory is also a similarity transformation:

$$\psi_p \triangleq \psi_{p,r_p-1} \circ \dots \circ \psi_{p,0}(\boldsymbol{x}) = \boldsymbol{Q}_p \boldsymbol{x} + \boldsymbol{y}_p, \tag{25}$$

where $\boldsymbol{Q}_p$ is an orthogonal matrix defined as

$$\boldsymbol{Q}_p = \boldsymbol{R}_{p,r_p-1} \dots \boldsymbol{R}_{p,0} \tag{26}$$

if $r_p > 0$, or $\boldsymbol{Q}_p = \boldsymbol{I}$ if $r_p = 0$, and

$$\begin{aligned} \boldsymbol{y}_p = &\ \left(\boldsymbol{I} - \boldsymbol{R}_{p,r_p-1}\right) \boldsymbol{s}_{p,r_p-1} \\ &+ \textstyle\sum_{q=0}^{r_p-2} \boldsymbol{R}_{p,r_p-1} \dots \boldsymbol{R}_{p,q+1} \left(\boldsymbol{I} - \boldsymbol{R}_{p,q}\right) \boldsymbol{s}_{p,q} \\ = &\ \textstyle\sum_{q=0}^{r_p-1} \boldsymbol{R}_{p,r_p-1} \dots \boldsymbol{R}_{p,q+1} \left(\boldsymbol{I} - \boldsymbol{R}_{p,q}\right) \boldsymbol{s}_{p,q}, \end{aligned} \tag{27}$$

with the convention $\boldsymbol{R}_{p,r_p-1} \dots \boldsymbol{R}_{p,q+1} = \boldsymbol{I}$ for $q = r_p - 1$. Moreover, any point $\boldsymbol{z}_p \in \mathbb{R}^3$ is an invariant point of ψ_p in Eq. (25) if and only if it is a solution to the equation

$$\left(\boldsymbol{I} - \boldsymbol{Q}_p\right) \boldsymbol{z}_p = \boldsymbol{y}_p. \tag{28}$$

Finally, note that the total length of trajectory p is

$$L_p(\boldsymbol{x}_1, \boldsymbol{x}_2) = \left\|\boldsymbol{x}_1 - \psi_p(\boldsymbol{x}_2)\right\|_2, \tag{29}$$

where $\psi_p(\boldsymbol{x}_2)$ is the *image* of $\boldsymbol{x}_2$ through the composition of all reflections ψ_p.

In Fig. 1, the direct path ($p = 0$) is such that $\boldsymbol{Q}_0 = \boldsymbol{I}$ is the identity matrix and $\boldsymbol{y}_0 = \boldsymbol{0}$: any point $\boldsymbol{z}_0$ in $\mathbb{R}^3$ is an invariant point of ψ_0. The first trajectory ($p = 1$) is such that $\boldsymbol{Q}_1 = \boldsymbol{R}_{1,0}$ and $\boldsymbol{y}_1 = (\boldsymbol{I} - \boldsymbol{Q}_1)\boldsymbol{s}_{1,0}$: any point in the plane tangent to ∂V at $\boldsymbol{s}_{1,0}$ (blue dotted line) is an invariant point of ψ_1, including $\boldsymbol{z}_1 = \boldsymbol{s}_{1,0}$. The second trajectory ($p = 2$) is such that $\boldsymbol{Q}_2 = \boldsymbol{R}_{2,1}\boldsymbol{R}_{2,0} = -\boldsymbol{I}$; any point $\boldsymbol{z}_2$ (red dot) at the intersection of the two planes tangent to ∂V at $\boldsymbol{s}_{2,0}$ and $\boldsymbol{s}_{2,1}$ (red dotted lines) is an invariant point of ψ_2.

Based on these definitions, we can now introduce the random measure a_n that corresponds to the multiple-path correlation function defined in Eq. (12), in the Neumann case ($\epsilon = 1$). This Hermitian random measure a_n is defined as

$$da_n(\boldsymbol{u}) = \sum_p e^{-\imath 2\pi K_n \boldsymbol{u}^\top \boldsymbol{z}_p} \frac{db_{n,p}(\boldsymbol{u}) + \overline{db_{n,p}(-\boldsymbol{u})}}{\sqrt{2}}, \tag{30}$$

where the random measures $b_{n,p}$ are independent, centered, complex, circular, Gaussian, and such that

$$\mathbb{E}[db_{n,p}(\boldsymbol{u}_1)\overline{db_{n,p}(\boldsymbol{u}_2)}] = \tfrac{\lambda}{4\pi}\delta(\boldsymbol{u}_2, \boldsymbol{Q}_p^\top \boldsymbol{u}_1) dS(\boldsymbol{u}_1) dS(\boldsymbol{u}_2). \tag{31}$$

Then, substituting Eqs. (30) and (31) into Eq. (17) yields

$$\begin{aligned} &\operatorname{cov}\left[\varphi_n(\boldsymbol{x}_1, k), \overline{\varphi_n(\boldsymbol{x}_2, k)}\right] \\ = &\ \mathbb{E}\Big[\textstyle\int_{\boldsymbol{u}_1 \in \mathcal{S}(0,1)} e^{\imath\left(2\pi\mathcal{K}(K_n,k)\boldsymbol{u}_1^\top \boldsymbol{x}_1 + \theta(\boldsymbol{u}_1, K_n, k)\right)} \\ &\textstyle\sum_{p_1} e^{-\imath 2\pi K_n \boldsymbol{u}_1^\top \boldsymbol{z}_{p_1}} \frac{db_{n,p_1}(\boldsymbol{u}_1) + \overline{db_{n,p_1}(-\boldsymbol{u}_1)}}{\sqrt{2}} \\ &\times \textstyle\int_{\boldsymbol{u}_2 \in \mathcal{S}(0,1)} e^{\imath\left(2\pi\mathcal{K}(K_n,k)\boldsymbol{u}_2^\top \boldsymbol{x}_2 + \theta(\boldsymbol{u}_2, K_n, k)\right)} \\ &\textstyle\sum_{p_2} e^{-\imath 2\pi K_n \boldsymbol{u}_2^\top \boldsymbol{z}_{p_2}} \frac{db_{n,p_2}(\boldsymbol{u}_2) + \overline{db_{n,p_2}(-\boldsymbol{u}_2)}}{\sqrt{2}} \Big] \\ = &\ \tfrac{\lambda}{8\pi} \textstyle\sum_p \\ &\textstyle\int_{\boldsymbol{u} \in \mathcal{S}(0,1)} \Big(e^{\imath\left(2\pi\mathcal{K}(K_n,k)\boldsymbol{u}^\top \boldsymbol{x}_1 + \theta(\boldsymbol{u}, K_n, k) - 2\pi K_n \boldsymbol{u}^\top \boldsymbol{z}_p\right)} \\ &\times e^{\imath\left(-2\pi\mathcal{K}(K_n,k)\boldsymbol{u}^\top \boldsymbol{Q}_p \boldsymbol{x}_2 - \theta(\boldsymbol{Q}_p^\top \boldsymbol{u}, K_n, k) + 2\pi K_n \boldsymbol{u}^\top \boldsymbol{Q}_p \boldsymbol{z}_p\right)} \\ &+ e^{\imath\left(-2\pi\mathcal{K}(K_n,k)\boldsymbol{u}^\top \boldsymbol{Q}_p \boldsymbol{x}_1 - \theta(\boldsymbol{Q}_p^\top \boldsymbol{u}, K_n, k) + 2\pi K_n \boldsymbol{u}^\top \boldsymbol{Q}_p \boldsymbol{z}_p\right)} \\ &\times e^{\imath\left(2\pi\mathcal{K}(K_n,k)\boldsymbol{u}^\top \boldsymbol{x}_2 + \theta(\boldsymbol{u}, K_n, k) - 2\pi K_n \boldsymbol{u}^\top \boldsymbol{z}_p\right)} dS(\boldsymbol{u}) \\ = &\ \lambda\, C(\boldsymbol{x}_1, \boldsymbol{x}_2, \mathcal{K}(K_n, k), k), \end{aligned}$$

which is the same expression as in Eq. (8), where $C(\boldsymbol{x}_1, \boldsymbol{x}_2, \mathcal{K}(K_n, k), k)$ is the *multiple-path pseudo-correlation function*, which is expressed as

$$
\begin{array}{l}
C(\boldsymbol{x}_1, \boldsymbol{x}_2, \mathcal{K}(K,k), k) = \sum_p \frac{1}{4\pi} \int_{\boldsymbol{u}\in\mathcal{S}(0,1)} \\
\frac{e^{\imath 2\pi\mathcal{K}(K,k)\boldsymbol{u}^\top(\boldsymbol{x}_1 - \boldsymbol{Q}_p\boldsymbol{x}_2)} + e^{\imath 2\pi\mathcal{K}(K,k)\boldsymbol{u}^\top(\boldsymbol{x}_2 - \boldsymbol{Q}_p\boldsymbol{x}_1)}}{2} \\
\times e^{\imath\left(\theta(\boldsymbol{u},K,k) - \theta(\boldsymbol{Q}_p^\top\boldsymbol{u},K,k) - 2\pi K\boldsymbol{u}^\top\boldsymbol{y}_p\right)} dS(\boldsymbol{u}),
\end{array}
\tag{32}
$$

where we have substituted Eq. (28). In the right member of Eq. (32), Eq. (26) yields $\forall p \in \mathbb{N}$,

$$
\begin{array}{rl}
& \theta(\boldsymbol{u}, K, k) - \theta(\boldsymbol{Q}_p^\top\boldsymbol{u}, K, k) \\
= & \theta(\boldsymbol{u}, K, k) - \theta(\boldsymbol{R}_{p,0}\ldots\boldsymbol{R}_{p,r_p-1}\boldsymbol{u}, K, k) \\
= & \theta(\boldsymbol{u}, K, k) - \theta(\boldsymbol{R}_{p,r_p-1}\boldsymbol{u}, K, k) \\
& + \sum\limits_{q=0}^{r_p-2} \theta(\boldsymbol{R}_{p,q+1}\ldots\boldsymbol{R}_{p,r_p-1}\boldsymbol{u}, K, k) \\
& -\theta(\boldsymbol{R}_{p,q}\ldots\boldsymbol{R}_{p,r_p-1}\boldsymbol{u}, K, k).
\end{array}
$$

Then, Eq. (A.4) in Appendix A, applied to $\phi = \theta$, $\boldsymbol{s} = \boldsymbol{s}_{p,q}$ and $\boldsymbol{R}(\boldsymbol{s}) = \boldsymbol{R}_{p,q}$, yields

$$
\begin{array}{rl}
& \theta(\boldsymbol{u}, K, k) - \theta(\boldsymbol{Q}_p^\top\boldsymbol{u}, K, k) \\
= & -2\pi\left(\mathcal{K}(K,k) - K\right)\boldsymbol{u}^\top(\boldsymbol{I} - \boldsymbol{R}_{p,r_p-1})\boldsymbol{s}_{p,r_p-1} \\
& +\imath\ln\left(\frac{\mathcal{K}(K,k)\boldsymbol{u}^\top\boldsymbol{n}(\boldsymbol{s}_{p,r_p-1}) + k\widehat{\beta}(\boldsymbol{s}_{p,r_p-1},k)}{\mathcal{K}(K,k)\boldsymbol{u}^\top\boldsymbol{n}(\boldsymbol{s}_{p,r_p-1}) - k\widehat{\beta}(\boldsymbol{s}_{p,r_p-1},k)}\right) \\
& + \sum\limits_{q=0}^{r_p-2} -2\pi\left(\mathcal{K}(K,k) - K\right)\boldsymbol{u}^\top \\
& \times\boldsymbol{R}_{p,r_p-1}\ldots\boldsymbol{R}_{p,q+1}(\boldsymbol{I} - \boldsymbol{R}_{p,q})\boldsymbol{s}_{p,q} \\
& +\imath\ln\left(\frac{\mathcal{K}(K,k)\boldsymbol{u}^\top\boldsymbol{R}_{p,r_p-1}\ldots\boldsymbol{R}_{p,q+1}\boldsymbol{n}(\boldsymbol{s}_{p,q}) + k\widehat{\beta}(\boldsymbol{s}_{p,q},k)}{\mathcal{K}(K,k)\boldsymbol{u}^\top\boldsymbol{R}_{p,r_p-1}\ldots\boldsymbol{R}_{p,q+1}\boldsymbol{n}(\boldsymbol{s}_{p,q}) - k\widehat{\beta}(\boldsymbol{s}_{p,q},k)}\right) \\
= & -2\pi\left(\mathcal{K}(K,k) - K\right)\boldsymbol{u}^\top\boldsymbol{y}_p + \imath\sum\limits_{q=0}^{r_p-1} \\
& \ln\left(\frac{\mathcal{K}(K,k)\boldsymbol{u}^\top\boldsymbol{R}_{p,r_p-1}\ldots\boldsymbol{R}_{p,q+1}\boldsymbol{n}(\boldsymbol{s}_{p,q}) + k\widehat{\beta}(\boldsymbol{s}_{p,q},k)}{\mathcal{K}(K,k)\boldsymbol{u}^\top\boldsymbol{R}_{p,r_p-1}\ldots\boldsymbol{R}_{p,q+1}\boldsymbol{n}(\boldsymbol{s}_{p,q}) - k\widehat{\beta}(\boldsymbol{s}_{p,q},k)}\right),
\end{array}
\tag{33}
$$

where we have substituted Eq. (27) in the last equality. By substituting Eq. (33) into Eq. (32), the multiple-path pseudo-correlation function $C(\boldsymbol{x}_1, \boldsymbol{x}_2, \mathcal{K}(K,k), k)$ can be rewritten as a function of (κ, k), instead of a function of $(\mathcal{K}(K,k), k)$:

$$
\begin{array}{l}
C(\boldsymbol{x}_1, \boldsymbol{x}_2, \kappa, k) = \sum\limits_p \frac{1}{4\pi} \int_{\boldsymbol{u}\in\mathcal{S}(0,1)} \\
\left(\prod\limits_{q=0}^{r_p-1} \frac{\kappa\boldsymbol{u}^\top\boldsymbol{R}_{p,r_p-1}\ldots\boldsymbol{R}_{p,q+1}\boldsymbol{n}(\boldsymbol{s}_{p,q}) - k\widehat{\beta}(\boldsymbol{s}_{p,q},k)}{\kappa\boldsymbol{u}^\top\boldsymbol{R}_{p,r_p-1}\ldots\boldsymbol{R}_{p,q+1}\boldsymbol{n}(\boldsymbol{s}_{p,q}) + k\widehat{\beta}(\boldsymbol{s}_{p,q},k)}\right) \\
\times\frac{e^{\imath 2\pi\kappa\boldsymbol{u}^\top(\boldsymbol{x}_1 - \boldsymbol{Q}_p\boldsymbol{x}_2 - \boldsymbol{y}_p)} + e^{\imath 2\pi\kappa\boldsymbol{u}^\top(\boldsymbol{x}_2 - \boldsymbol{Q}_p\boldsymbol{x}_1 - \boldsymbol{y}_p)}}{2} dS(\boldsymbol{u}).
\end{array}
\tag{34}
$$

In the Neumann case ($\epsilon = 1$) and in the Dirichlet case ($\epsilon = -1$), Eq. (34) is simplified as

$$\begin{aligned}
& C(\boldsymbol{x}_1, \boldsymbol{x}_2, \kappa, k) = \sum_p \epsilon^{r_p} \tfrac{1}{4\pi} \int_{\boldsymbol{u} \in \mathcal{S}(0,1)} \\
& \tfrac{e^{\imath 2\pi\kappa \boldsymbol{u}^\top(\boldsymbol{x}_1 - (\boldsymbol{Q}_p \boldsymbol{x}_2 + \boldsymbol{y}_p))} + e^{\imath 2\pi\kappa \boldsymbol{u}^\top(\boldsymbol{x}_2 - (\boldsymbol{Q}_p \boldsymbol{x}_1 + \boldsymbol{y}_p))}}{2} dS(\boldsymbol{u}) \\
& = \sum_p \epsilon^{r_p} \tfrac{\operatorname{sinc}\left(2\pi\kappa \|\boldsymbol{x}_1 - \psi_p(\boldsymbol{x}_2)\|_2\right) + \operatorname{sinc}\left(2\pi\kappa \|\boldsymbol{x}_2 - \psi_p(\boldsymbol{x}_1)\|_2\right)}{2} \\
& = \sum_p \epsilon^{r_p} \operatorname{sinc}(2\pi\kappa L_p(\boldsymbol{x}_1, \boldsymbol{x}_2)),
\end{aligned} \tag{35}$$

where we have substituted Eq. (25), and L_p was defined in Eq. (29). We thus retrieve Eq. (12) of the multiple-path correlation function. In the same way, Eqs. (30), (31) and (17) yield

$$\begin{aligned}
& \operatorname{cov}\left[\varphi_n(\boldsymbol{x}_1, k_1), \varphi_n(\boldsymbol{x}_2, k_2)\right] \\
= \; & \mathbb{E}\Big[\int_{\boldsymbol{u}_1 \in \mathcal{S}(0,1)} e^{\imath\left(2\pi\mathcal{K}(K_n,k_1)\boldsymbol{u}_1^\top \boldsymbol{x}_1 + \theta(\boldsymbol{u}_1, K_n, k_1)\right)} \\
& \textstyle\sum_{p_1} e^{-\imath 2\pi K_n \boldsymbol{u}_1^\top \boldsymbol{z}_{p_1}} \frac{db_{n,p_1}(\boldsymbol{u}_1) + \overline{db_{n,p_1}(-\boldsymbol{u}_1)}}{\sqrt{2}} \\
& \times \int_{\boldsymbol{u}_2 \in \mathcal{S}(0,1)} e^{-\imath\left(2\pi\overline{\mathcal{K}(K_n,k_2)}\boldsymbol{u}_2^\top \boldsymbol{x}_2 + \overline{\theta(\boldsymbol{u}_2, K_n, k_2)}\right)} \\
& \textstyle\sum_{p_2} e^{+\imath 2\pi K_n \boldsymbol{u}_2^\top \boldsymbol{z}_{p_2}} \frac{\overline{db_{n,p_2}(\boldsymbol{u}_2)} + db_{n,p_2}(-\boldsymbol{u}_2)}{\sqrt{2}} \Big] \\
= \; & \tfrac{\lambda}{8\pi} \textstyle\sum_p \\
& \int_{\boldsymbol{u} \in \mathcal{S}(0,1)} e^{\imath\left(2\pi\mathcal{K}(K_n,k_1)\boldsymbol{u}^\top \boldsymbol{x}_1 + \theta(\boldsymbol{u}, K_n, k_1) - 2\pi K_n \boldsymbol{u}^\top \boldsymbol{z}_p\right)} \\
& \times e^{\imath\left(-2\pi\overline{\mathcal{K}(K_n,k_2)}\boldsymbol{u}^\top \boldsymbol{Q}_p \boldsymbol{x}_2 - \overline{\theta(\boldsymbol{Q}_p^\top \boldsymbol{u}, K_n, k_2)} + 2\pi K_n \boldsymbol{u}^\top \boldsymbol{Q}_p \boldsymbol{z}_p\right)} \\
& + e^{\imath\left(2\pi\mathcal{K}(K_n,k_1)\boldsymbol{u}^\top \boldsymbol{Q}_p \boldsymbol{x}_1 + \theta(\boldsymbol{Q}_p^\top \boldsymbol{u}, K_n, k_1) - 2\pi K_n \boldsymbol{u}^\top \boldsymbol{Q}_p \boldsymbol{z}_p\right)} \\
& \times e^{\imath\left(-2\pi\overline{\mathcal{K}(K_n,k_2)}\boldsymbol{u}^\top \boldsymbol{x}_2 - \overline{\theta(\boldsymbol{u}, K_n, k_2)} + 2\pi K_n \boldsymbol{u}^\top \boldsymbol{z}_p\right)} dS(\boldsymbol{u}) \\
= \; & \lambda\, \gamma(\boldsymbol{x}_1, \boldsymbol{x}_2, K_n, k_1, k_2),
\end{aligned}$$

which is the same expression as in Eq. (7), where the *multiple-path correlation function* γ is defined as

$$\gamma(\boldsymbol{x}_1, \boldsymbol{x}_2, K, k_1, k_2) = \tfrac{\mathring{\gamma}(\boldsymbol{x}_1, \boldsymbol{x}_2, K, k_1, k_2) + \overline{\mathring{\gamma}(\boldsymbol{x}_2, \boldsymbol{x}_1, K, k_2, k_1)}}{2}, \tag{36}$$

with

$$\begin{aligned}
\mathring{\gamma}(\boldsymbol{x}_1, \boldsymbol{x}_2, K, k_1, k_2) = \; & \sum_p \tfrac{1}{4\pi} \int_{\boldsymbol{u} \in \mathcal{S}(0,1)} e^{\imath 2\pi \boldsymbol{u}^\top\left(\mathcal{K}(K,k_1)\boldsymbol{x}_1 - \overline{\mathcal{K}(K,k_2)}\boldsymbol{Q}_p \boldsymbol{x}_2 - K \boldsymbol{y}_p\right)} \\
& \times e^{\imath\left(\theta(\boldsymbol{u}, K, k_1) - \overline{\theta\left(\boldsymbol{Q}_p^\top \boldsymbol{u}, K, k_2\right)}\right)} dS(\boldsymbol{u}),
\end{aligned} \tag{37}$$

and where we have substituted Eq. (28).

Remark 2. *In the right member of Eq. (37), as we already noted in Remark 1, the imaginary part of function $\theta(\boldsymbol{u}, K, k)$ defined in Eq. (16) depends on k and $\mathcal{K}(K,k)$, but not directly on K. In the same way, when $k_1 = k_2 = k$, note that Eq. (33) implies*

$$\begin{aligned}
& -2\pi K \boldsymbol{u}^\top \boldsymbol{y}_p + \operatorname{Re}\left(\theta(\boldsymbol{u}, K, k) - \overline{\theta(\boldsymbol{Q}_p^\top \boldsymbol{u}, K, k)}\right) \\
= \; & \operatorname{Re}\Big(-2\pi\mathcal{K}(K,k)\boldsymbol{u}^\top \boldsymbol{y}_p + \imath \sum_{q=0}^{r_p - 1} \\
& \ln\left(\tfrac{\mathcal{K}(K,k)\boldsymbol{u}^\top \boldsymbol{R}_{p,r_p-1} \ldots \boldsymbol{R}_{p,q+1} \boldsymbol{n}(\boldsymbol{s}_{p,q}) + k\widehat{\beta}(\boldsymbol{s}_{p,q},k)}{\mathcal{K}(K,k)\boldsymbol{u}^\top \boldsymbol{R}_{p,r_p-1} \ldots \boldsymbol{R}_{p,q+1} \boldsymbol{n}(\boldsymbol{s}_{p,q}) - k\widehat{\beta}(\boldsymbol{s}_{p,q},k)} \right) \Big),
\end{aligned}$$

which also depends on k and $\mathcal{K}(K,k)$, but not directly on K. Therefore, $\mathring{\gamma}(\boldsymbol{x}_1, \boldsymbol{x}_2, K, k, k)$, as defined in Eq. (37)*, and thus $\gamma(\boldsymbol{x}_1, \boldsymbol{x}_2, K, k, k)$, as defined in Eq.* (36)*, depend on K only through $\mathcal{K}(K,k)$.*

5. Conclusion

In this paper, the mathematical foundations of the statistical wave field theory have been reworked at the fundamental level of the eigenfunctions of the complex Robin Laplacian. We first identified the relevant works regarding the asymptotic distribution of the eigenfunctions of the real Robin Laplacian (Sec. 3), which include the Gaussian random wave model and universal correlation function (Berry, 1977), and the normalization (Urbina and Richter, 2007) and boundary (Hortikar and Srednicki, 1998) constraints.

Based on these existing works, we then introduced our general random wave model of the eigenfunctions of the complex Robin Laplacian (Sec. 4). This contribution is twofold: first, it unifies the existing random wave models dedicated to the real Robin Laplacian, and second, it generalizes these random wave models to the *complex* Robin Laplacian, which makes it possible to account for the absorption of energy at the boundary surface. We have demonstrated the existence of phase and magnitude distortions between the Neumann and Robin eigenfunctions (Appendix A), in addition to the wave number and wave vector distortions that had already been highlighted in Badeau (2024) and Badeau (2026b), respectively. We showed that the magnitude distortion is always negligible, contrary to the phase distortion, which plays an important role. Moreover, the formula of the phase distortion possesses the amazing ability to account for any reflection of any order, even in the case of energy absorption, as shown in Sec 4.4. Finally, this phase distortion can be computed numerically thanks to its decomposition as a series of spherical harmonics (Sec. 4.2), which was obtained by following the same approach as in our previous works on semi-ergodic billiards (Badeau, 2026a,b).

The possible future extensions and applications of the work presented in this article are numerous. In our next article, we will calculate the first and second order statistics of the wave field that result from the random wave model of the complex Robin Laplacian introduced in Sec. 4. We will thus achieve a remarkable result: the unification of geometrical acoustics (Kuttruff, 2014, Chap. 4), which is generally used to predict the first echoes that appear in early reverberation, and of probabilistic models (Kuttruff, 2014, Chap. 5), which are generally used to predict the statistical properties of late reverberation. In other respects, the random wave model presented in Sec. 4 for the class of mixing billiards, which produce diffuse wave fields, can be extended to the general class of semi-mixing billiards introduced in Badeau (2026a), which produce anisotropic wave fields, based on the phase and magnitude distortions investigated in Appendix A.

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and on the boundary constraint (Hortikar and Srednicki, 1998), which are mentioned in his topical review paper (Tanner and Søndergaard, 2007).

Author Declarations

Conflict of Interest: The author of this paper has no conflict of interest to disclose.

Data Availability

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Appendix A. Phase and magnitude distortions

In this appendix, we derive Eq. (16), which characterizes the phase distortion that appears in the spectral representation of Eq. (17).

To do so, we extend the study to the larger class of semi-ergodic billiards introduced in Badeau (2026a, Sec. III). Indeed, this class of billiards encompasses both particular cases of mixing billiards, which are the focus of this paper, and of the special polyhedra, which were introduced in Badeau (2025c). The special polyhedra are especially interesting geometrical shapes, because they are *integrable* in the sense of Arnold's theorem (Arnold and Avez, 1989), which means that their eigenfunctions admit exact closed-form expressions, which can be expressed in the same form as Eq. (17).

Due to space constraints, it is not possible here to undertake a comprehensive study of the eigenfunctions of these special polyhedra. We will just state the main fact: these eigenfunctions can be decomposed as discrete sums of plane waves, and the Robin boundary condition at the polyhedral domain's boundary ∂V can be equivalently formulated as the usual reflection of planes waves on the few infinite impedance planes that contain the faces of the polyhedron. More precisely, using notations similar to those in Eq. (17), the eigenfunctions of these special polyhedra can be written in the following form: $\forall n > 0$,

$$\varphi_n(\boldsymbol{x}, k) = \int_{\boldsymbol{u}\in\mathcal{S}(0,1)} e^{\imath 2\pi \boldsymbol{\mathcal{K}}(K_n\boldsymbol{u},k)^\top \boldsymbol{x}}\, d\alpha_n(\boldsymbol{u}, k), \tag{A.1}$$

where $\boldsymbol{\mathcal{K}}(K_n\boldsymbol{u}, k)$ and $d\alpha_n(\boldsymbol{u}, k)$ are defined as follows:

- with Neumann's boundary condition: $\boldsymbol{\mathcal{K}}(K_n\boldsymbol{u}, k) = K_n\boldsymbol{u} \in \mathbb{R}^3$, and $d\alpha_n(\boldsymbol{u}, k) = da_n(\boldsymbol{u})$ is a complex Hermitian measure with discrete and finite support;
- with Robin's boundary condition: $\boldsymbol{\mathcal{K}}(K_n\boldsymbol{u}, k) \in \mathbb{C}^3$ denotes the wave vector distortion introduced in Badeau (2025c, Eq. (30)); it is an odd function of $K_n\boldsymbol{u}$, and $d\alpha_n(\boldsymbol{u}, k) = e^{\imath\phi(\boldsymbol{u},K_n,k)}\, da_n(\boldsymbol{u})$ is also a complex measure with discrete and finite support, where function ϕ represents the complex phase distortion between the Neumann and Robin boundary conditions.

Then, the boundary condition can be expressed as follows:

- Neumann's boundary condition: $\forall \boldsymbol{s} \in \partial V$,

$$\begin{array}{ll} & \left[\frac{\partial}{\partial \boldsymbol{n}(\boldsymbol{s})}\left(e^{\imath 2\pi K_n \boldsymbol{u}^\top \boldsymbol{x}}\, da_n(\boldsymbol{u}) + e^{\imath 2\pi K_n \boldsymbol{u}^\top \boldsymbol{R}(\boldsymbol{s})\boldsymbol{x}}\, da_n(\boldsymbol{R}(\boldsymbol{s})\boldsymbol{u})\right)\right](\boldsymbol{x}=\boldsymbol{s}) = 0 \\ \Leftrightarrow & e^{\imath 2\pi K_n \boldsymbol{u}^\top \boldsymbol{s}}\, da_n(\boldsymbol{u}) = e^{\imath 2\pi K_n \boldsymbol{u}^\top \boldsymbol{R}(\boldsymbol{s})\boldsymbol{s}}\, da_n(\boldsymbol{R}(\boldsymbol{s})\boldsymbol{u}), \end{array} \tag{A.2}$$

where the incident plane wave is characterized by the wave vector $K_n\boldsymbol{u}$ and the complex amplitude $da_n(\boldsymbol{u})$, the reflected plane wave is characterized by the wave vector $K_n\boldsymbol{R}(\boldsymbol{s})\boldsymbol{u}$ and the complex amplitude $da_n(\boldsymbol{R}(\boldsymbol{s})\boldsymbol{u})$, and $\boldsymbol{R}(\boldsymbol{s})$ denotes the reflection matrix w.r.t. the 2D subspace parallel to the plane tangent to ∂V at $\boldsymbol{s}$: $\boldsymbol{R}(\boldsymbol{s}) = \boldsymbol{I} - 2\,\boldsymbol{n}(\boldsymbol{s})\,\boldsymbol{n}(\boldsymbol{s})^\top$, where $\boldsymbol{I}$ denotes the identity matrix;
- Robin's boundary condition: $\forall \boldsymbol{s} \in \partial V$,

$$\begin{array}{ll} & \left[\left(\frac{\partial}{\partial \boldsymbol{n}(\boldsymbol{s})} + \imath 2\pi k\widehat{\beta}(\boldsymbol{s},k)\right)\left(e^{\imath 2\pi \boldsymbol{\mathcal{K}}(K_n\boldsymbol{u},k)^\top \boldsymbol{x}}\, d\alpha_n(\boldsymbol{u},k)\right.\right. \\ & \left.\left. + e^{\imath 2\pi \boldsymbol{\mathcal{K}}(K_n\boldsymbol{u},k)^\top \boldsymbol{R}(\boldsymbol{s})\boldsymbol{x}}\, d\alpha_n(\boldsymbol{R}(\boldsymbol{s})\boldsymbol{u},k)\right)\right](\boldsymbol{x}=\boldsymbol{s}) = 0 \\ \Leftrightarrow & e^{\imath\left(2\pi \boldsymbol{\mathcal{K}}(K_n\boldsymbol{u},k)^\top \boldsymbol{s} + \phi(\boldsymbol{u},K_n,k)\right)}\left(\boldsymbol{\mathcal{K}}(K_n\boldsymbol{u},k)^\top \boldsymbol{n}(\boldsymbol{s}) + k\widehat{\beta}(\boldsymbol{s},k)\right) da_n(\boldsymbol{u}) \\ & + e^{\imath\left(2\pi \boldsymbol{\mathcal{K}}(K_n\boldsymbol{u},k)^\top \boldsymbol{R}(\boldsymbol{s})\boldsymbol{s} + \phi(\boldsymbol{R}(\boldsymbol{s})\boldsymbol{u},K_n,k)\right)}\left(-\boldsymbol{\mathcal{K}}(K_n\boldsymbol{u},k)^\top \boldsymbol{n}(\boldsymbol{s}) + k\widehat{\beta}(\boldsymbol{s},k)\right) da_n(\boldsymbol{R}(\boldsymbol{s})\boldsymbol{u}) = 0, \end{array} \tag{A.3}$$

where the incident plane wave is characterized by the distorted wave vector $\boldsymbol{\mathcal{K}}(K_n\boldsymbol{u},k)$ and the distorted complex amplitude $d\alpha_n(\boldsymbol{u},k) = e^{\imath\phi(\boldsymbol{u},K_n,k)}\, da_n(\boldsymbol{u})$, and the reflected plane wave is characterized by the distorted wave vector $\boldsymbol{R}(\boldsymbol{s})\boldsymbol{\mathcal{K}}(K_n\boldsymbol{u},k)$ and the distorted complex amplitude $d\alpha_n(\boldsymbol{R}(\boldsymbol{s})\boldsymbol{u},k) = e^{\imath\phi(\boldsymbol{R}(\boldsymbol{s})\boldsymbol{u},K_n,k)}\, da_n(\boldsymbol{R}(\boldsymbol{s})\boldsymbol{u})$.

By substituting Eq. (A.2) into (A.3), we get $\forall \boldsymbol{s} \in \partial V$,

$$\frac{e^{\imath\left(2\pi(\boldsymbol{\mathcal{K}}(K_n\boldsymbol{u},k)-K_n\boldsymbol{u})^\top \boldsymbol{s} + \phi(\boldsymbol{u},K_n,k)\right)}}{e^{\imath\left(2\pi(\boldsymbol{\mathcal{K}}(K_n\boldsymbol{u},k)-K_n\boldsymbol{u})^\top \boldsymbol{R}(\boldsymbol{s})\boldsymbol{s} + \phi(\boldsymbol{R}(\boldsymbol{s})\boldsymbol{u},K_n,k)\right)}} = \frac{\boldsymbol{\mathcal{K}}(K_n\boldsymbol{u},k)^\top \boldsymbol{n}(\boldsymbol{s}) - k\widehat{\beta}(\boldsymbol{s},k)}{\boldsymbol{\mathcal{K}}(K_n\boldsymbol{u},k)^\top \boldsymbol{n}(\boldsymbol{s}) + k\widehat{\beta}(\boldsymbol{s},k)},$$

which can be rewritten $\forall \boldsymbol{s} \in \partial V$,

$$\begin{array}{ll} & \phi(\boldsymbol{u},K_n,k) - \phi(\boldsymbol{R}(\boldsymbol{s})\boldsymbol{u},K_n,k) \\ = & -2\pi\left(\boldsymbol{\mathcal{K}}(K_n\boldsymbol{u},k) - K_n\boldsymbol{u}\right)^\top 2\left(\boldsymbol{n}(\boldsymbol{s})\,\boldsymbol{n}(\boldsymbol{s})^\top \boldsymbol{s}\right) \\ & + \imath \ln\left(\frac{\boldsymbol{\mathcal{K}}(K_n\boldsymbol{u},k)^\top \boldsymbol{n}(\boldsymbol{s}) + k\widehat{\beta}(\boldsymbol{s},k)}{\boldsymbol{\mathcal{K}}(K_n\boldsymbol{u},k)^\top \boldsymbol{n}(\boldsymbol{s}) - k\widehat{\beta}(\boldsymbol{s},k)}\right). \end{array} \tag{A.4}$$

By multiplying by λ and integrating over $\boldsymbol{s} \in \partial V$, Eq. (A.4) implies

$$\begin{array}{ll} & \lambda \int_{\boldsymbol{s}\in\partial V} \phi(\boldsymbol{u},K_n,k) - \phi(\boldsymbol{R}(\boldsymbol{s})\boldsymbol{u},K_n,k)dS(\boldsymbol{s}) \\ = & -2\pi\left(\boldsymbol{\mathcal{K}}(K_n\boldsymbol{u},k) - K_n\boldsymbol{u}\right)^\top \times 2\lambda \int_{\boldsymbol{s}\in\partial V} \boldsymbol{n}(\boldsymbol{s})\ \boldsymbol{n}(\boldsymbol{s})^\top \boldsymbol{s}\ dS(\boldsymbol{s}) \\ & + \imath\lambda \int_{\boldsymbol{s}\in\partial V} \ln\left(\frac{\boldsymbol{\mathcal{K}}(K_n\boldsymbol{u},k)^\top \boldsymbol{n}(\boldsymbol{s}) + k\widehat{\beta}(\boldsymbol{s},k)}{\boldsymbol{\mathcal{K}}(K_n\boldsymbol{u},k)^\top \boldsymbol{n}(\boldsymbol{s}) - k\widehat{\beta}(\boldsymbol{s},k)}\right) dS(\boldsymbol{s}). \end{array} \tag{A.5}$$

However, we will show below that the solution ϕ to Eq. (A.5) is uniquely defined, so Eqs. (A.4) and (A.5) are actually equivalent. Therefore, Eq. (A.5) completely characterizes the Robin boundary condition in the particular case of the special polyhedra.

In other respects, we have shown in Badeau (2025c, 2026a,b) that the general equations of the statistical wave field theory include the exact closed form expressions satisfied by the special polyhedra as particular cases. The main difference is that these equations hold at all frequencies in the particular case of the special polyhedra, whereas they hold only asymptotically at high frequencies in the general case of semi-ergodic billiards, because the high frequency assumption is necessary for the boundary surface to be locally approximated by its tangent plane. Knowing that, we will admit that Eq. (A.5) completely characterizes the Robin boundary condition in all semi-ergodic billiards, asymptotically at high frequency.

However, Eq. (A.5) can be rewritten

$$\int_{\boldsymbol{v}\in\mathcal{S}(0,1)} A(\boldsymbol{u},\boldsymbol{v})\,\phi(\boldsymbol{v},K_n,k)\,dS(\boldsymbol{v}) = b(K_n\boldsymbol{u},k), \tag{A.6}$$

where the linear operator A was defined in Badeau (2026a, Eq. (A3)):

$$A(\boldsymbol{u},\boldsymbol{v}) = \lambda \int_{\boldsymbol{s}\in\partial V} \left(\delta\left(\boldsymbol{u}-\boldsymbol{v}\right) - \delta(\boldsymbol{u}-\boldsymbol{v}+2(\boldsymbol{n}(\boldsymbol{s})^\top\boldsymbol{v})\boldsymbol{n}(\boldsymbol{s}))\right) dS(\boldsymbol{s}), \tag{A.7}$$

and

$$\begin{aligned} b(\boldsymbol{k},k) = \ & -2\pi\left(\boldsymbol{\mathcal{K}}(\boldsymbol{k},k)-\boldsymbol{k}\right)^\top \\ & \times 2\lambda \int_{\boldsymbol{s}\in\partial V} \boldsymbol{n}(\boldsymbol{s})\ \boldsymbol{n}(\boldsymbol{s})^\top\boldsymbol{s}\ dS(\boldsymbol{s}) \\ & + \imath\lambda \int_{\boldsymbol{s}\in\partial V} \ln\left(\frac{\boldsymbol{\mathcal{K}}(\boldsymbol{k},k)^\top\boldsymbol{n}(\boldsymbol{s})+k\widehat{\beta}(\boldsymbol{s},k)}{\boldsymbol{\mathcal{K}}(\boldsymbol{k},k)^\top\boldsymbol{n}(\boldsymbol{s})-k\widehat{\beta}(\boldsymbol{s},k)}\right) dS(\boldsymbol{s}). \end{aligned} \tag{A.8}$$

We have shown in Badeau (2026a, Subsection 1 of Appendix A) that in the general case of semi-ergodic billiards, the operator A is positive semidefinite, and its kernel $\mathrm{Ker}(A)$ is non-zero. Therefore the Hilbert space $L^2(\mathcal{S}(0,1))$ can be decomposed as the direct sum of the range space $\mathrm{Span}(A)$, and the kernel $\mathrm{Ker}(A)$, which are orthogonal complements.

Let us now prove that function $b(K_n\boldsymbol{u},k)$ in the right member of Eq. (A.6) belongs to $\mathrm{Span}(A)$, which will imply that Eq. (A.6) does admit a unique solution $\phi=\theta$ in $\mathrm{Span}(A)$. Indeed, the orthogonal projection of $b(K_n\boldsymbol{u},k)$ onto $\mathrm{Ker}(A)$ is zero: $\forall\boldsymbol{v}\in\mathcal{S}(0,1)$, $\forall\boldsymbol{s}\in\partial V$,

$$\left(\mathrm{Proj}_{\mathrm{Ker}(A)}(b(K_n.,k))\right)(\boldsymbol{v}) = \int_{\boldsymbol{u}\in\mathcal{S}(0,1)} b(K_n\boldsymbol{u},k)\,d\sigma(\boldsymbol{u}|\boldsymbol{v}) = 0,$$

where σ denotes the *directional measure* introduced in Badeau (2026a, Sec. III). The last equality is implied by the fact that function $\boldsymbol{u}\mapsto b(K_n\boldsymbol{u},k)$ in Eq. (A.8) is odd, whereas $\boldsymbol{u}\mapsto d\sigma(\boldsymbol{u}|\boldsymbol{v})$ is even. We can thus conclude that $b(K_n\boldsymbol{u},k)$ does belong to $\mathrm{Ker}(A)^\perp=\mathrm{Span}(A)$.

Consequently, any solution to Eq. (A.6) can be decomposed as

$$\phi(\boldsymbol{u},K_n,k) = \theta(\boldsymbol{u},K_n,k) - \imath\ln M(\boldsymbol{u},K_n,k), \tag{A.9}$$

where $\theta\in\mathrm{Span}(A)$ is uniquely defined, and $\ln M\in\mathrm{Ker}(A)$.

Then, in the particular case of the special polyhedra, function M is uniquely identified by normalizing the eigenfunctions, i.e., the integral $\int_V \varphi_n(\boldsymbol{x},k)^2 d\boldsymbol{x}$ has to be equal to 1. By calculating this integral in closed form, the resulting normalizing factor always leads to the following asymptotic expansion[2]:

$$M(\boldsymbol{u}, K_n, k) = \begin{array}{l} 1 - \frac{\imath\lambda}{8\pi} \int_{\boldsymbol{s}\in\partial V} \int_{\boldsymbol{v}\in\partial\mathcal{S}(0,1)} \Big(\frac{1}{\boldsymbol{\mathcal{K}}(K_n\boldsymbol{u},k)^\top \boldsymbol{v} - k\widehat{\beta}(\boldsymbol{s},k)} \\ \quad - \frac{1}{\boldsymbol{\mathcal{K}}(K_n\boldsymbol{u},k)^\top \boldsymbol{v} + k\widehat{\beta}(\boldsymbol{s},k)} \Big) d\sigma(\boldsymbol{v}|\boldsymbol{n}(\boldsymbol{s})) dS(\boldsymbol{s}). \end{array} \tag{A.10}$$

In Eq. (A.10), the directional measure σ was characterized in Badeau (2025c, Remark 3):

$$\sigma(\boldsymbol{v}|\boldsymbol{n}) = \frac{1}{|\mathrm{D}_{n\mathrm{h}}|} \sum_{\boldsymbol{Q}\in\mathrm{D}_{n\mathrm{h}}} \delta_{\mathcal{S}(0,1)}(\boldsymbol{v}, \boldsymbol{Q}\,\boldsymbol{n}),$$

where $\delta_{\mathcal{S}(0,1)}$ denotes the Dirac distribution on the unit sphere, and $\mathrm{D}_{n\mathrm{h}}$ is the finite dihedral point group of isometries $\boldsymbol{Q}$ generated by all the reflections w.r.t. the polyhedron's faces. Consequently, $\ln M$ with M defined as in Eq. (A.10) does belong to $\mathrm{Ker}(A)$, because it is invariant to all the reflections w.r.t. all the polyhedron's faces. Finally, Eq. (A.10) actually holds for all semi-ergodic billiards, by considering the general expression of the directional measure σ introduced in Badeau (2025c, Sec. III).

In other respects, note that function $\boldsymbol{u} \mapsto M(\boldsymbol{u}, K_n, k)$ in Eq. (A.10) is even, because function $\boldsymbol{u} \mapsto \boldsymbol{\mathcal{K}}(K_n\boldsymbol{u}, k)$ is odd. In the same way, the unique solution θ to Eq. (A.6) in $\mathrm{Span}(A)$ is odd, because function $\boldsymbol{u} \mapsto b(K_n\boldsymbol{u}, k)$ defined in Eq. (A.8) is odd. Therefore, Eq. (A.9) decomposes the general solution ϕ to Eq. (A.6) as the sum of an odd function θ and an even function $-\imath\ln(M)$.

Appendix A.1. Invariance to a change of the origin of the coordinate system

Note that Eqs. (A.1) and (A.5) are invariant to a change of the origin of the coordinate system. Indeed, if a vector $\boldsymbol{o}$ (which stands for "origin") is subtracted to the position vectors $\boldsymbol{x}$ and $\boldsymbol{s}$ in Eqs. (A.1) and (A.5), respectively, then these equations yield:

$$\varphi_n^o(\boldsymbol{x}, k) = \int_{\boldsymbol{u}\in\mathcal{S}(0,1)} e^{\imath\left(2\pi\boldsymbol{\mathcal{K}}(K_n\boldsymbol{u},k)^\top(\boldsymbol{x}-\boldsymbol{o}) + \phi^o(\boldsymbol{u},K_n,k)\right)} da_n(\boldsymbol{u}),$$

where

$$\phi^o(\boldsymbol{u}, K_n, k) = \phi(\boldsymbol{u}, K_n, k) + 2\pi\left(\boldsymbol{\mathcal{K}}(K_n\boldsymbol{u}, k) - K_n\boldsymbol{u}\right)^\top \boldsymbol{o}.$$

We then retrieve the same expression of the eigenfunctions as in Eq. (A.1):

$$\varphi_n^o(\boldsymbol{x}, k) = \int_{\boldsymbol{u}\in\mathcal{S}(0,1)} e^{\imath\left(2\pi\boldsymbol{\mathcal{K}}(K_n\boldsymbol{u},k)^\top\boldsymbol{x} + \phi(\boldsymbol{u},K_n,k)\right)} da_n^o(\boldsymbol{u}),$$

where the change of the origin has been incorporated into the complex amplitudes: $da_n^o(\boldsymbol{u}) = e^{-\imath 2\pi K_n\boldsymbol{u}^\top\boldsymbol{o}}\, da_n(\boldsymbol{u})$.

[2]Calculations are not included here due to space constraints. They are straightforward in the particular case of the rectangular cuboid, by pursuing the mathematical developments initiated in Badeau (2025c, Sec. III D 2). The case of the equilateral triangular right prism is far more tricky, but Eq. (A.10) can be derived from the equations presented in McCartin (2004). Calculations for the two other special polyhedra, i.e., the isosceles right triangular right prism and the hemiequilateral triangular right prism, are then straightforward, given the formulas of the rectangular cuboid and the equilateral triangular right prism.

Appendix A.2. Numerical computation of the phase distortion function

We proved in Badeau (2026a, Subsection 2 of Appendix A) that the linear operator A is block-diagonalizable in a basis of real spherical harmonics (Courant and Hilbert, 1989). Therefore, Eq. (A.6) can be solved in practice by decomposing function θ as a series of spherical harmonics, which results in Eqs. (18) to (21). More precisely, substituting Eq. (18) into Eq. (A.6) with $\phi = \theta$ yields

$$\sum_{l'\in\mathbb{N}} \left(\int_{\boldsymbol{v}\in\mathcal{S}(0,1)} A(\boldsymbol{u},\boldsymbol{v})\,\boldsymbol{Y}_{l'}(\boldsymbol{v})^\top\,dS(\boldsymbol{v})\right)\boldsymbol{\theta}_{l'}(K,k) = b(K\boldsymbol{u},k).$$

Then, left-multiplying the two members of this equation by $\boldsymbol{Y}_l(\boldsymbol{u})$, and integrating w.r.t. $\boldsymbol{u}\in\mathcal{S}(0,1)$ yields Eq. (19), where we have defined

$$\boldsymbol{A}_l = \int_{\boldsymbol{u}\in\mathcal{S}(0,1)}\int_{\boldsymbol{v}\in\mathcal{S}(0,1)} \boldsymbol{Y}_l(\boldsymbol{u})\,A(\boldsymbol{u},\boldsymbol{v})\boldsymbol{Y}_l(\boldsymbol{v})^\top\,dS(\boldsymbol{v})dS(\boldsymbol{u}), \tag{A.11}$$

and

$$\boldsymbol{b}_l(K,k) = \int_{\boldsymbol{u}\in\mathcal{S}(0,1)} \boldsymbol{Y}_l(\boldsymbol{u})\,b(K\boldsymbol{u},k)\,dS(\boldsymbol{u}), \tag{A.12}$$

and where we have used Badeau (2026a, Eq. A.12):

$$\begin{aligned}\forall l\neq l',\quad & \int_{\boldsymbol{u}\in\mathcal{S}(0,1)}\int_{\boldsymbol{v}\in\mathcal{S}(0,1)}\\ & \boldsymbol{Y}_l(\boldsymbol{u})A(\boldsymbol{u},\boldsymbol{v})\,\boldsymbol{Y}_{l'}(\boldsymbol{v})^\top\,dS(\boldsymbol{v})dS(\boldsymbol{u}) = 0.\end{aligned}$$

Finally, Eq. (20) is obtained by substituting Eq. (A.7) into Eq. (A.11), and the expression of $\boldsymbol{b}_l(K,k)$ is obtained by substituting Eq. (A.8) into Eq. (A.12):

$$\begin{aligned}\boldsymbol{b}_l(K,k) = \;& -\left(2\pi\int_{\boldsymbol{u}\in\mathcal{S}(0,1)}\boldsymbol{Y}_l(\boldsymbol{u})\,(\boldsymbol{\mathcal{K}}(K\boldsymbol{u},k)-K\boldsymbol{u})^\top\,dS(\boldsymbol{u})\right)\\ & \times\left(2\lambda\int_{\boldsymbol{s}\in\partial V}\boldsymbol{n}(\boldsymbol{s})\left(\boldsymbol{n}(\boldsymbol{s})^\top\boldsymbol{s}\right)\,dS(\boldsymbol{s})\right)\\ & +\int_{\boldsymbol{u}\in\mathcal{S}(0,1)}\boldsymbol{Y}_l(\boldsymbol{u})\Bigg(\imath\lambda\\ & \times\int_{\boldsymbol{s}\in\partial V}\ln\left(\frac{\boldsymbol{\mathcal{K}}(K\boldsymbol{u},k)^\top\boldsymbol{n}(\boldsymbol{s})+k\widehat{\beta}(\boldsymbol{s},k)}{\boldsymbol{\mathcal{K}}(K\boldsymbol{u},k)^\top\boldsymbol{n}(\boldsymbol{s})-k\widehat{\beta}(\boldsymbol{s},k)}\right)dS(\boldsymbol{s})\Bigg)dS(\boldsymbol{u}).\end{aligned} \tag{A.13}$$

Appendix A.3. Ergodic billiards

In the particular case of ergodic billiards, we get two simplifications:

- The wave vector distortion is isotropic (Badeau, 2025c, Sec. VII):

$$\boldsymbol{\mathcal{K}}(K\boldsymbol{u},k) = \mathcal{K}(K,k)\boldsymbol{u}, \tag{A.14}$$

  where the wave number distortion $\mathcal{K}(K,k)$ was expressed in Eq. (15), therefore Eq. (A.1) yields Eq. (17), Eq. (A.5) yields Eq. (16), and Eq. (A.13) yields Eq. (21);
- any function in $\mathrm{Ker}(A)$ is constant (Badeau, 2026a, Remark 2), thus $\boldsymbol{u}\mapsto M(\boldsymbol{u},K,k)$ is equal to a constant $M(\mathcal{K}(K,k),k)$. Applying Eq. (A.10) to the isotropic directional measure $d\sigma(\boldsymbol{v}|\boldsymbol{n}) = \frac{1}{4\pi}dS(\boldsymbol{v})\;\forall\boldsymbol{n}\in\mathcal{S}(0,1)$ yields

$$M(\kappa,k) = 1+\frac{\imath\lambda}{8\pi\kappa}\int_{\boldsymbol{s}\in\partial V}\ln\left|\frac{k\widehat{\beta}(\boldsymbol{s},k)+\kappa}{k\widehat{\beta}(\boldsymbol{s},k)-\kappa}\right|dS(\boldsymbol{s})$$

if $k\widehat{\beta}(\boldsymbol{s},k) \in \mathbb{R}$, or

$$M(\kappa,k) = 1 + \tfrac{\imath\lambda}{8\pi\kappa} \int_{\boldsymbol{s}\in\partial V} \ln\left(\tfrac{k\widehat{\beta}(\boldsymbol{s},k)+\kappa}{k\widehat{\beta}(\boldsymbol{s},k)-\kappa}\right) dS(\boldsymbol{s})$$

if $k\widehat{\beta}(\boldsymbol{s},k) \in \mathbb{C}\backslash\mathbb{R}$. In all cases, we remark that $\lim_{\kappa\to+\infty} M(\kappa,k) = 1$: at high frequency, the magnitude distortion is negligible.

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