Full-diversity Product Codes for Block Erasure and Block Fading Channels

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In collaboration with

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March 2008

Presentation outline

1. Introduction
2. Problem illustration with a simple product code
3. Channel model, notations and properties
4. Definition of rootchecks for linear codes
5. Building full diversity product codes
6. Product codes with high-order rootchecks
7. Experimental results

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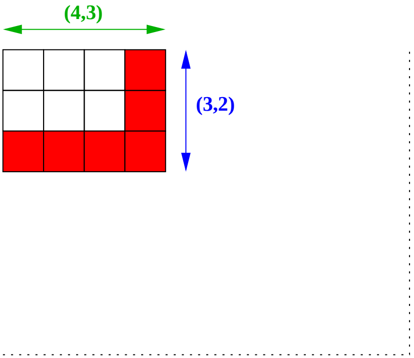
- Consider a $(3, 2) \times (4, 3)$ binary product code of rate $R = 6/12 = 1/2$.
- Color half of the boxes with white and the other half with red.
- Try erasing all red boxes. Is the code capable of finding them with the help of white boxes? Similarly, erase all white boxes. Is the code capable of finding them with the help of red boxes?



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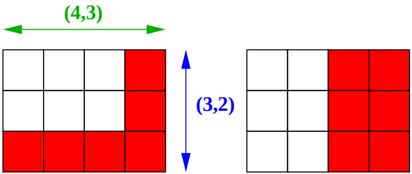

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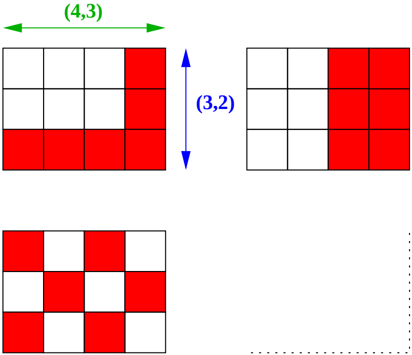

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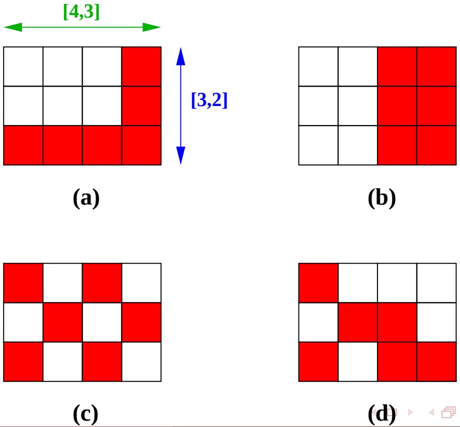

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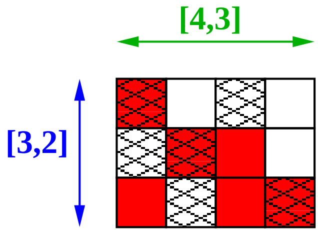

- Put a pattern on privileged bits. Those bits are said to be connected to a rootcheck.
- The standard location of information bits is the upper left 2×3 corner.
- Move the position of information bits to those connected to rootchecks.
- The new code is referred to as a root product code. It has the following properties:
  - It is full diversity on both block erasure and block fading channels.
  - Full diversity is achieved after one decoding iteration.
  - It achieves the highest rate according to the block fading Singleton bound.

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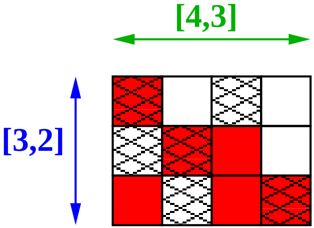

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Block erasures on a simple product code (3)

- Can we build a root product code $[16, 11]^2$?
- Exhaustive search:

  the total number of red/white configurations is $\approx 2^{250}$.
- Take into account all symmetries:

  the number of red/white configurations is $\approx 2^{160}$.
- Solution: Introduce rootchecks in the code structure.

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Channel model and notations (1)

- Linear binary coding for non-ergodic channels is considered. The channel state is assumed to be invariant for some time period, finite or infinite.

- Given the channel state α, for an input x and an output y, the channel transition probability is

$$p(y|x,\alpha) \propto \exp\left(-\frac{|y-\alpha x|^2}{2\sigma^2}\right)$$

  where $y = \alpha x + \eta$.

- The fading coefficient α belongs to $\mathbb{R}^+$: non-ergodic Rayleigh fading channel.

- The fading coefficient α belongs to $\{0, +\infty\}$: block erasure channel.

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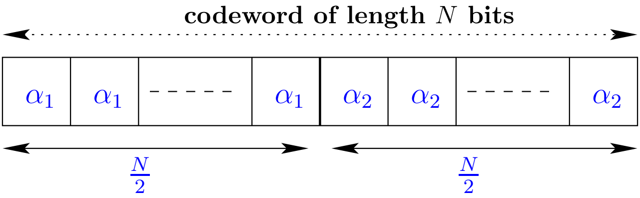

- For simplicity, we consider the case $n_c = 2$ channel states per codeword.
- Code construction and analysis is generally straightforward for $n_c \geq 3$.
- Channel coding is made via a rate-R linear binary code $C[N, K]$.
- The code C is built from a rate-r constituent $C_0[n, k]$, also referred to as a subcode of the compound code, $R = r^2$ for a product code.

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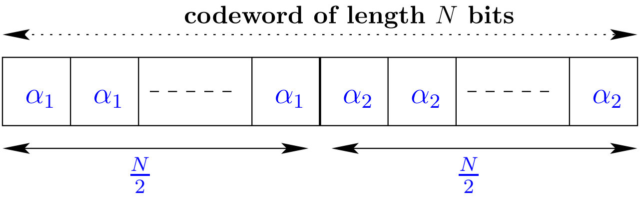

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Channel and code design properties (1)

- **Tx/Rx Diversity**: number of independent replicas of the same information.
- **Channel Diversity**: number of independent channel states.
- On a block erasure channel of diversity order $d = n_c$, for a given block erasure probability ϵ, the word error rate behavior should be

$$P_e \;=\; \epsilon^d$$

- On a block fading channel of diversity order $d = n_c$, at high SNR $\gamma = \frac{1}{\sigma^2} \gg 1$, the word error rate behavior should be

$$P_e \propto \gamma^{-d}$$

- Our objective is to design full-diversity codes capable of achieving near outage limit performance. According to the block fading Singleton bound, the highest possible rate is

$$R \leq R_{max} = \frac{1}{n_c}$$

Channel and code design properties (2)

For any code structure, under ML decoding, full diversity on block erasure channels is a sufficient condition for full diversity on non-ergodic Rayleigh fading channels.

Proposition 1:

Consider a linear binary code $C(N, K)$. If C is full diversity on a block erasure channel then it is full diversity on a block fading channel under ML decoding.

As described in the next slide, the special root structure for any compound code achieves full diversity under iterative decoding.

Proposition 2:

Consider a linear binary code $C(N, K)$ with a root structure (LDPC, Product, GLD, etc). Under iterative decoding, the code is full diversity on both block erasure and block Rayleigh fading channels.

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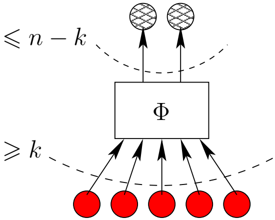
Root checknodes for linear codes (1)

Definition (rootcheck)

A rootcheck is a subcode node with all roots colored in white and all leaves colored in red. A similar definition is given after interchanging red and white.



Up to $n-k$ root vertices on state 1

Φ: BCH constituent $C_0[n,k]$

All leaves undergo state 2

- The version of C_0 defined by a parity-check matrix H_0 must satisfy: The $n-k$ root vertices are associated to $n-k$ independent columns of H_0.
- The simplest convention is to write $H_0 = [I_{n-k} \mid P_0]$ and assign the $n-k$ first columns to root bitnodes.

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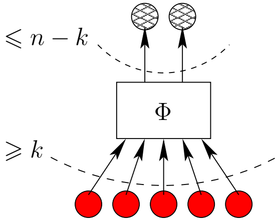

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Root checknodes for linear codes (2)

Proposition 2 revisited at the subcode level (block erasure)

A rootcheck guarantees full diversity to all its roots under block erasures.

Proof: If root bits are erased then recompute their value from leaf bits using H_0.

Root checknodes for linear codes (3)

Proposition 2 revisited at the subcode level (block fading)

A rootcheck $C_0(n,k,d)$ guarantees full diversity to all its roots under block fading.

Proof: Local optimal probabilistic decoding is given by

$$APP(b) \propto \sum_{c \in C_0|b} p(y|c)$$

where b is a root. The sub-optimal log-map decoder considers the dominant likelihoods for $b=0$ and $b=1$. Its log-ratio message is

$$\Lambda \;=\; \|y-\alpha\bar{x}\|^2 - \|y-\alpha x\|^2 \;=\; 2Y+\nu$$

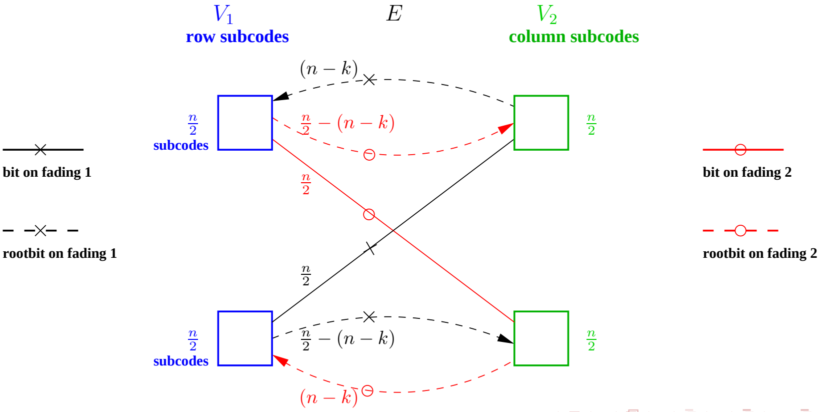
$$Y = \alpha_1^2 \sum_{i=1}^{n-k} (x_i - \bar{x}_i) \;+\; \alpha_2^2 \sum_{i=n-k+1}^{n} (x_i - \bar{x}_i) \;=\; \omega_1\alpha_1^2 + \omega_2\alpha_2^2$$

At high SNR, we have $\omega_1 \geq 1$ and $\omega_2 \geq 1$. The exact values for ω_i depend on the weight distribution of C_0. The 4th order χ^2 distribution of Y guarantees the double diversity.

Similar results hold in presence of a priori messages at the input of the rootcheck.

Full-Diversity Product Codes (1)

- Use a complete graph representation for a product code $C(N, K) = C_0(n, k)^{\otimes 2}$.
- Put row subcodes on the left and column subcodes on the right.
- Bits are associated to edges linking row subcodes to column subcodes.
- Arrows are pointing to rootchecks of the corresponding bit.



Full-Diversity Product Codes (2)

- Let us assume that all row subcodes are rootchecks, then the code guarantees full diversity for all its information bits if $n(n-k) \geq k^2$, i.e. $r^2 + r - 1 \leq 0$.

Proposition 3

A full-diversity product code $C_0(n,k)^{\otimes 2}$ with all its rootchecks being row subcodes satisfies $r = \frac{k}{n} \leq \frac{\sqrt{5}-1}{2}$.

- Consider a non-ergodic channel with 2 states, α_1 and α_2. A root-product code is graph-encodable if the subcode dimension does not exceed the number of incoming and outgoing roots, $k \leq 2(n-k)$.

Proposition 4

A graph-encodable full-diversity root-product exists if $r = \frac{k}{n} \leq \frac{2}{3}$.

- From the above propositions, we conclude that a good range for the rate of the product code constituent is

$$\frac{\sqrt{5}-1}{2} \leq r \leq \frac{2}{3}$$

Full-Diversity Product Codes (3)

Corollary 1

A graph-encodable full-diversity root-product code $[16, 11, 4]^{\otimes 2}$ does not exist.

The compact graph of a graph-encodable root-product code is built as follows:

- The number of edges linking two groups of subcodes should be equal to the number of subcodes since the graph is complete.
- To render a graph-encodable code, the number of supernodes cannot be equal to 2 as in the previous graph representation. The number of non-root bits must be less than $n - k$. In order to let rootchecks cover all information bits, the number of supernodes should be taken equal to

$$\lceil \frac{n}{n-k} \rceil$$

on each side of the compact graph representation.

- Colors should be selected in order to maximize the number of root bits. After color selection, information bits are placed on root edges.

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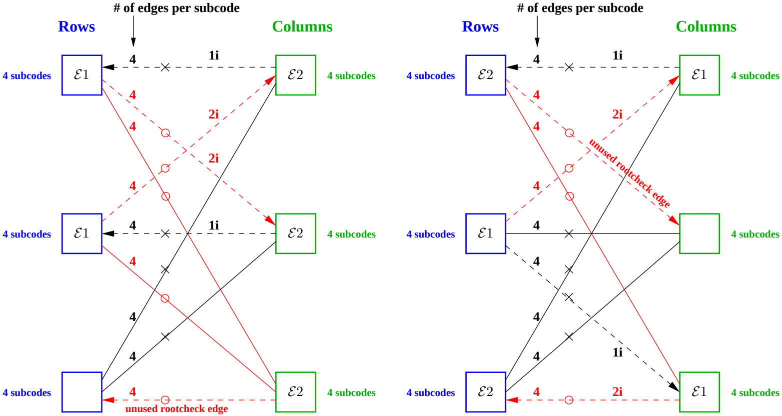
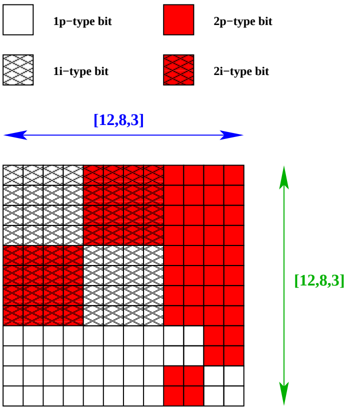
Full-Diversity Product Codes (4)



Two compact graph representations of a full-diversity product code $[12, 8, 3]^{\otimes 2}$, $r = \frac{2}{3}$ and $R = \frac{4}{9} = 0.4444$.

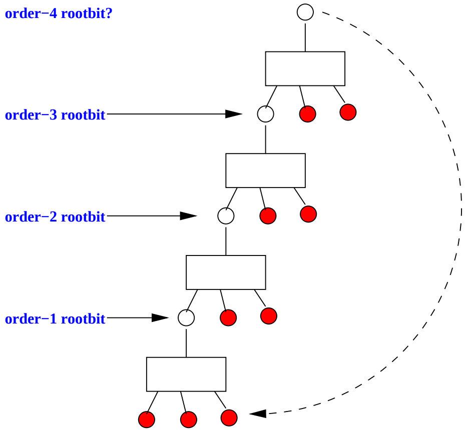
Full-Diversity Product Codes (5)



Two matrix representations of full-diversity product codes $[12, 8]^{\otimes 2}$ defined by a compact graph with 3 supernodes on each side, each supernode containing 4 subcodes. All matrix patterns with $r = \frac{2}{3}$ are identical.

High-Order Root-Product Codes (1)

Can we exceed $r = \frac{2}{3}$, i.e. $R = \frac{4}{9}$?



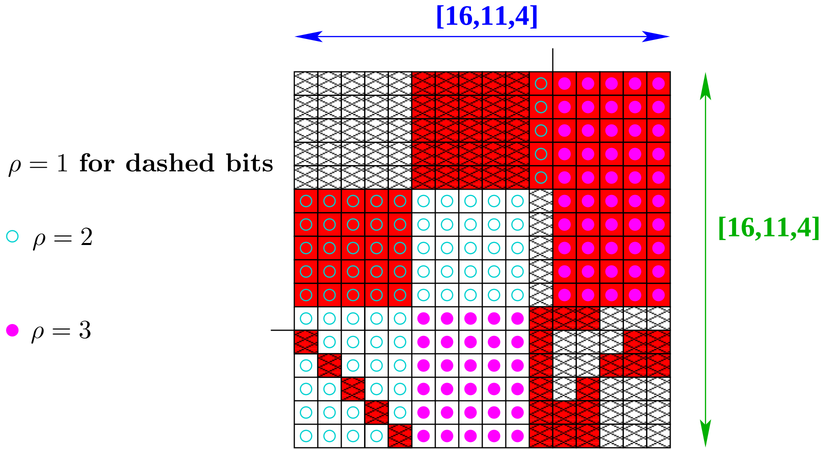
Tree structure for defining rootbits of order $\rho = 1, 2, 3$. By convention, a bit that never achieves full diversity under iterative decoding will be assigned $\rho = +\infty$.

High-Order Root-Product Codes (2)

Proposition 5

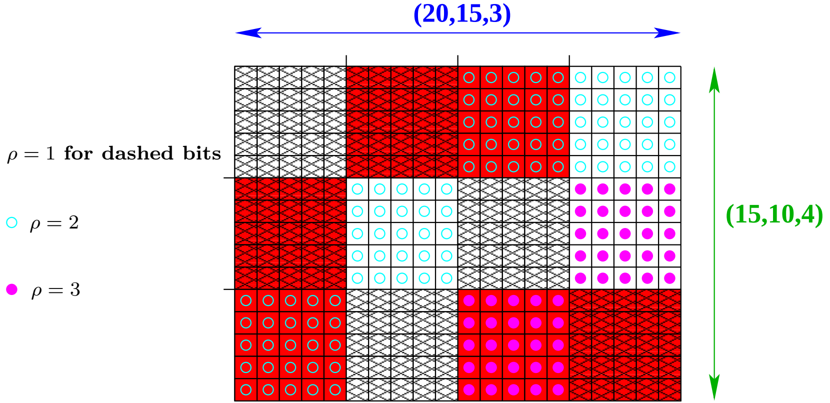
A root-product code attains full diversity order (or equivalently recovers all information bits) after at most 3 decoding iterations.

High-Order Root-Product Codes (3)



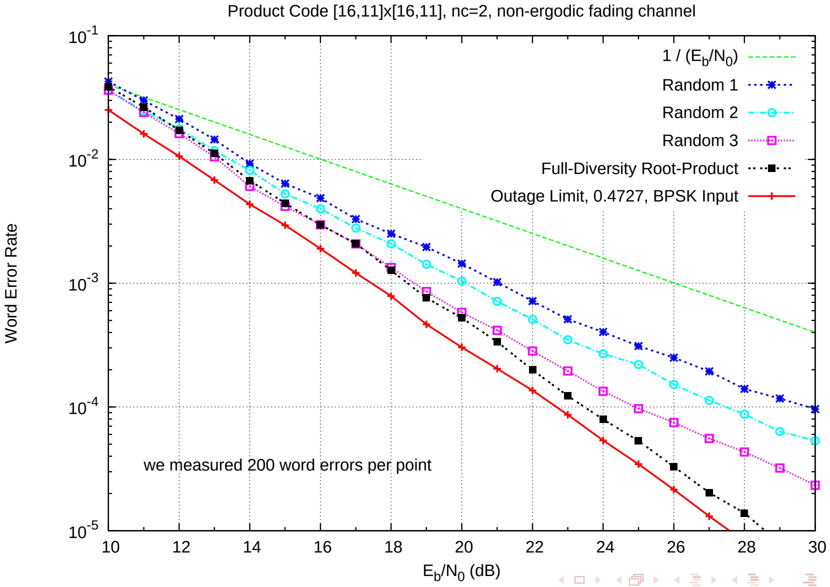
Full-diversity root-product code $[16, 11, 4]^{\otimes 2}$ with high order rootchecks.
Overall coding rate is $R = 0.4726$.
3 iterations are needed to fill all erasures or to reach full diversity.

High-Order Root-Product Codes (4)

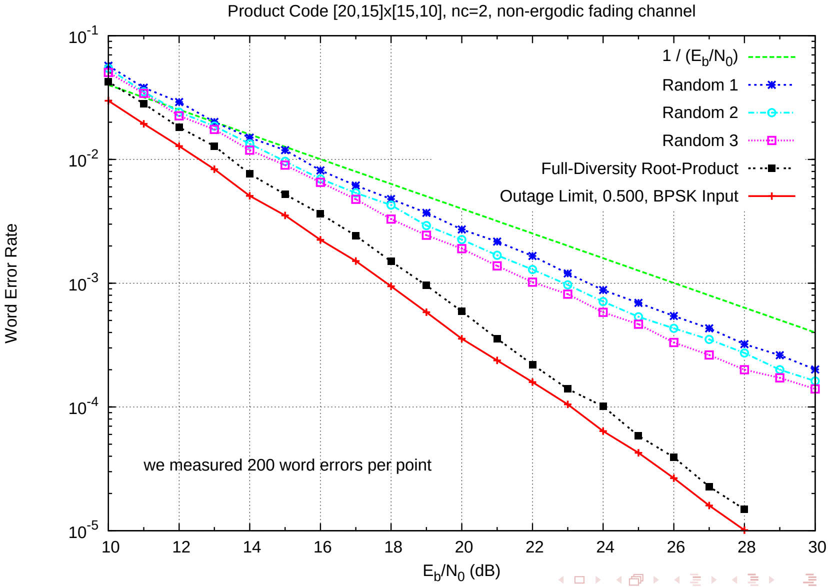


Matrix representation of a full-diversity product code $[20, 15, 3] \otimes [15, 10, 4]$ with high order rootchecks, where the order ρ of each bit is indicated.
Overall coding rate is $R = 1/2$.

Performance of product codes (1)



Performance of product codes (2)



Summary

- A finite-length design of bi-dimensional binary product codes suitable for block fading channels has been proposed.
- The study is based on graphical tools and some simple algebraic properties of product codes.
- Codes at several coding rates capable of achieving the highest diversity order have been found.
- This work should be enhanced via the analysis of the coding gain and the general asymptotic performance behavior of product codes on non-ergodic channels.