I just realized that the requirement that the degree function is bounded from above, in Definition 1 (page 5), is redundant. So I propose to simplify Definition 1 as follows.

**Definition 1.** A Harder-Narasimhan lattice is a modular lattice $L$ of finite length equipped with a lower semimodular function $\deg: L \to \mathbb{R}$.

I think this is nice since it provides a very minimalistic combinatorial framework for Harder-Narasimhan theory. (Besides this boundedness condition, previously I also got rid of the Northcott-type condition, as explained at the bottom of page 6.)

In order to prove that boundedness is implied by this new Definition 1, I then propose to insert the following lemma between the first and second paragraphs in page 6, just before the canonical polygon is constructed, which is precisely where it is needed.

**Lemma.** Let $(L, \deg)$ be a Harder-Narasimhan lattice. Then the degree function $\deg$ is bounded from above.

**Proof.** We proceed by induction on the length $n = \text{rk}(L)$ of $L$. The result is obvious if $n = 0$ or $1$, so we assume $L$ has length $n \geq 2$ and make the following *Induction hypothesis*: on every Harder-Narasimhan lattice of length at most $n - 1$, the degree function is bounded from above.

By contradiction suppose $\deg$ is not bounded on $L$, and choose $r$ maximal such that the set

$$E_r = \{ \deg(x) : x \in L, \text{rk}(x) = r \}$$

is unbounded (observe $r \leq n - 1$ since $E_n = \{ \deg(1_L) \}$ is finite).

Now fix an $a \in L$ of rank $\text{rk}(a) = 1$, and let $x \in L$ vary with $\text{rk}(x) = r$ and $\deg(x)$ arbitrarily large. The sublattice $L/a$ has length $n - 1$, so by our *Induction hypothesis* we will have $x \not\in L/a$ as soon as $\deg(x)$ is large enough. This means $a \not\subseteq x$, and forces $a \land x = 0_L$ and

$$\text{rk}(a \lor x) = r + 1.$$

But then, by semimodularity,

$$\deg(a \lor x) \geq \deg(x) + \deg(a) - \deg(0_L)$$

can be arbitrarily large when $\deg(x)$ is. This means precisely that $E_{r+1}$ is unbounded, contradicting the maximality of $r$. \qed