Galois descent for vector spaces in half a page

Hugues Randriam

September 1, 2019

Let $L/K$ be a finite Galois extension, of degree $n$ and Galois group $G = \{\sigma_1, \ldots, \sigma_n\}$. Let $V$ be a finite dimensional $L$-vector space equipped with a skew-linear action of $G$, which means $\sigma(lv) = \sigma(l)\sigma(v)$ for all $\sigma \in G$, $l \in L$, $v \in V$. Let $V^G \subseteq V$ be the $K$-subspace of vectors fixed by $G$.

**Theorem 1.** Under these hypotheses, the natural $L$-linear map

$$V^G \otimes_K L \to V$$

compatible with the action of $G$, is an isomorphism.

**Proof.** By linear independence of characters and equality of dimension, the map

$$L \otimes_K L \to L^n$$

$$x \otimes y \mapsto (x\sigma_1(y), \ldots, x\sigma_n(y))$$

is an isomorphism. Thus we can find $x_1, \ldots, x_n, y_1, \ldots, y_n \in L$ such that

$$\sum_i x_i y_i = 1, \quad \sum_i x_i \sigma(y_i) = 0 \text{ for } \sigma \neq \text{id}.$$ But then we get

$$U \otimes_K L \sim V.$$ But then we get $U = (U \otimes_K L)^G = V^G$, which finishes the proof. $\square$