Convex analysis

Master “Mathematics for data science and big data”

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Chapter 1

Introduction: Optimization, machine learning and convex analysis

1.1 Optimization problems in Machine Learning

Most of Machine Learning algorithms consist in solving a minimization problem. In other words, the output of the algorithm is the solution (or an approximated one) of a minimization problem. In general, non-convex problems are difficult, whereas convex ones are easier to solve. Here, we are going to focus on convex problems.
First, let's give a few examples of well-known issues you will have to deal with in supervised learning :

Example 1.1.1 (Least squares, simple linear regression or penalized linear regression).

(a) Ordinary Least Squares:

$$\min_{x \in \mathbb{R}^p} \|Zx - Y\|^2, Z \in \mathbb{R}^{n \times p}, Y \in \mathbb{R}^n$$

(b) Lasso :

$$\min_{x \in \mathbb{R}^p} \|Zx - Y\|^2 + \lambda \|x\|_1,$$

(c) Ridge :

$$\min_{x \in \mathbb{R}^p} \|Zx - Y\|^2 + \lambda \|x\|_2^2,$$

Example 1.1.2 (Linear classification).
The data consists of a training sample $\mathcal{D} = \{(w_1, y_1), \ldots, (w_n, y_n)\}$, $y_i \in \{-1, 1\}$, $w_i \in \mathbb{R}^p$, where the w_i's are the data's*features* (also called *regressors*), whereas the y_i's are the labels which represent the class of each observation i. The sample is obtained by independent realizations of a vector $(W, Y) \sim P$, of unknown distribution P. Linear classifiers are linear functions defined on the *feature space*, of the kind:

$$h : w \mapsto \text{sign}(\langle x, w \rangle + x_0) \qquad (x \in \mathbb{R}^p, x_0 \in \mathbb{R})$$

A classifier h is thus determined by a vector $\mathbf{x} = (x, x_0)$ in $\mathbb{R}^{p+1}$. The vector x is the normal vector to an hyperplane which separates the space into two regions, inside which the predicted labels are respectively "+1" and "−1".

The goal is to learn a classifier which, in average, is not wrong by much: that means that we want $\mathbb{P}(h(W) = Y)$ to be as big as possible.
To quantify the classifier's error/accuracy, the reference loss function is the '0-1 loss':

$$L_{01}(\mathbf{x}, w, y) = \begin{cases} 0 & \text{if } -y\left(\langle x, w\rangle + x_0\right) \leq 0 \quad (h(w) \text{ and } y \text{ of same sign}), \\ 1 & \text{otherwise }. \end{cases}$$

In general, the implicit goal of machine learning methods for supervised classification is to solve (at least approximately) the following problem:

$$\min_{\mathbf{x}\in\mathbb{R}^{p+1}} \frac{1}{n}\sum_{i=1}^{n} L_{0,1}(\mathbf{x}, w_i, y_i) \tag{1.1.1}$$

i.e. to minimize the *empirical risk*.
As the cost L is not convex in $\mathbf{x}$, the problem (1.1.1) is *hard*. Classical Machine learning methods consist in minimizing a function that is similar to the objective (1.1.1): the idea is to replace the cost 0-1 by a *convex substitute*, and then to add a penalty term which penalizes "complexity" of x, so that the problem becomes numerically feasible. More precisely, the problem to be solved numerically is

$$\min_{x\in\mathbb{R}^p, x_0\in\mathbb{R}} \sum_{i=1}^{n} \varphi(-y_i(x^\top w_i + x_0)) + \lambda\,\mathcal{P}(x), \tag{1.1.2}$$

where $\mathcal{P}$ is the penalty and φ is a convex substitute to the cost 0-1.
Different choices of penalties and convex subsitutes are available, yielding a range of methods for supervised classification :

- For $\varphi(z) = \max(0, 1+z)$ (Hinge loss), $\mathcal{P}(x) = \|x\|^2$, this is the SVM.
- In the separable case (*i.e.* when there is a hyperplane that separates the two classes), introduce the "infinite indicator function" (also called *characteristic function*),

$$\mathbb{I}_A(z) = \begin{cases} 0 & \text{if } z \in A, \\ +\infty & \text{if } z \in A^c, \end{cases} \qquad (A \subset \mathcal{X})$$

and set

$$\varphi(z) = \mathbb{I}_{\mathbb{R}^-}(z).$$

The solution to the problem is the maximum margin hyperplane.

To summarize, the common denominator of all these versions of example 1.1.2 is as follows:

- The risk of a classifier x is defined by $J(x) = \mathbb{E}(L(x, D))$. We are looking for x which minimizes J.
- $\mathbb{P}$ is unknown, and so is J. However, $D \sim \mathbb{P}$ is available. Therefore, the approximate problem is to find:

$$\hat{x} \in \arg\min_{x\in\mathcal{X}} J_n(x) \hat{=} \frac{1}{n}\sum_{i=1}^{n} L(x, d_i)$$

- The cost L is replaced by a convex surrogate L_φ, so that the function $J_{n,\varphi} = \frac{1}{n}\sum_{i=1}^{n} L_\varphi(x, d_i)$ convex in x.

- In the end, the problem to be solved, when a convex penalty term is incorporated, is

$$\min_{x \in \mathcal{X}} J_{n,\varphi}(x) + \lambda \mathcal{P}(x). \tag{1.1.3}$$

In the remaining of the course, the focus is on that last point: how to solve the convex minimization problem (1.1.3) ?

1.2 General formulation of the problem

In this course, we only consider optimization problems which are defined on a finite dimension space $\mathcal{X} = \mathbb{R}^n$. These problems can be written, without loss of generality, as follows:

$$\begin{gathered} \min_{x \in \mathcal{X}} f(x) \\ s.t. \text{ (such that / under constraint that)} \\ g_i(x) \le 0 \text{ for } 1 \le i \le p, \quad F_i(x) = 0 \text{ for } 1 \le i \le m\,. \end{gathered} \tag{1.2.1}$$

The function f is the *target function* (or *target*),
the vector

$$C(x) = (g_1(x), \dots, g_p(x), F_1(x), \dots, F_m(x))$$

is the (functional) constraint vector.
The region

$$K = \{x \in \mathcal{X} : \ g_i(x) \le 0, \ 1 \le i \le p, \quad F_i(x) = 0, \ 1 \le i \le m\,\}$$

is the set of *feasible* points.

- If $K = \mathbb{R}^n$, this is an *unconstrained* optimization problem.
- Problems where $p \ge 1$ and $m = 0$, are referred to as *inequality contrained* optimization problems.
- If $p = 0$ and $m \ge 1$, we speak of *equality contrained* optimization.
- When f and the contraints are regular (differentiable), the problem is called *differentiable* or *smooth*.
- If f or the contraints are not regular, the problem is called *non-differentiable* or *non-smooth*.
- If f and the contraints are convex, we have a *convex* optimization problem (more details later).

Solving the general problem (1.2.1) consists in finding

- a minimizer $x^* \in \arg\min_K f$ (if it exists, *i.e.* if $\arg\min_K f \neq \emptyset$),
- the *value* $f(x^*) = \min_{x \in K} f(x)$,

We can rewrite the constrained problem as an unconstrained problem, thanks to the infinite indicator function $\mathbb{I}$ introduced earlier. Let's name g and (resp) F the vectors of the inequality and (resp) equality contraints.
For $x, y \in \mathbb{R}^n$, we write $x \preceq y$ if $(x_1 \leq y_1, \ldots, x_n \leq y_n)$ and $x \not\preceq y$ otherwise. The problem (1.2.1) is equivalent to :

$$\min_{x \in E} f(x) + \mathbb{I}_{g \preceq 0, F=0}(x) \tag{1.2.2}$$

Let's notice that, even if the initial problem is smooth, the new problem isn't anymore !

1.3 Algorithms

Approximated solutions Most of the time, Problem (1.2.1) cannot be analytically solved. However, numerical algorithms can provide an approximate solution. Finding an ϵ-approximate solution (**ϵ-solution**) consists in finding $\hat{x} \in K$ such that, if the "true" minimum x^* exists, we have

- $\|\hat{x} - x^*\| \leq \epsilon$,

  and/or

- $|f(\hat{x}) - f(x^*)| \leq \epsilon$.

"Black box" model A standard framework for optimization is the **black box**. That is, we want to optimize a function in a situation where:

- The target f is not entirely accessible (otherwise the problem would already be solved !)
- The algorithm does not have any access to f (and to the constraints), except by successive calls to an *oracle* $\mathcal{O}(x)$.
  Typically, $\mathcal{O}(x) = f(x)$ (0-order oracle) or $\mathcal{O}(x) = (f(x), \nabla f(x))$ (1-order oracle), or $\mathcal{O}(x)$ can evaluate higher derivative of f ($\geq$ 2-order oracle).
- At iteration k, the algorithm only has the information $\mathcal{O}(x_1), \ldots, \mathcal{O}(x_k)$ as a basis to compute the next point x_{k+1}.
- The algorithm stops at time k if a criterion $T_\epsilon(x_k)$ is satisfied: the latter ensures that x_k is an ϵ-solution.

Performance of an algorithm Performance is measured in terms of computing resources needed to obtain an approximate solution.
This obviously depends on the considered problem. A **class of problems** is:

- A class of target functions (regularity conditions, convexity or other)
- A condition on the starting point x_0 (for example, $\|x - x_0\| \leq R$)
- An oracle.

Definition 1.3.1 (oracle complexity)**.** *The **oracle complexity** of an algorithm $\mathcal{A}$, for a class of problems C and a given precision ϵ, is the minimal number $N_{\mathcal{A}}(\epsilon)$ such that, for all objective functions and any initial point $(f, x_0) \in C$, we have:*

$$N_{\mathcal{A}}(f, \epsilon) \leq N_{\mathcal{A}}(\epsilon)$$

where : $N_{\mathcal{A}}(f, \epsilon)$ is the number of calls to the oracle that are needed for $\mathcal{A}$ to give an ϵ-solution. The oracle complexity, as defined here, is a *worst-case* complexity. The computation time depends on the oracle complexity, but also on the number of required arithmetical operations at each call to the oracle. The total number of arithmetic operations to achieve an ϵ-solution in the worst case, is called *arithmetic complexity*. In practice, it is the arithmetic complexity which determines the computation time, but it is easier to prove bounds on the oracle complexity .

1.4 Preview of the rest of the course

A natural idea to solve general problem (1.2.1) is to start from an arbitrary point x_0 and to propose the next point x_1 in a region where f "has a good chance" to be smaller.
If f is differentiable, one widely used method is to follow "the line of greatest slope", i.e. move in the direction given by $-\nabla f$.
What's more, if there is a local minimum x^*, we then have $\nabla f(x^*) = 0$. So a similar idea to the previous one is to set the gradient equal to zero.
Here we have made implicit assumptions of regularity, but in practice some problems can arise.

- Under which assumptions is the necessary condition '$\nabla f(x) = 0$' sufficient for x to be a local minimum?
- Under which assumptions is a local minimum a global one?
- What if f is not differentiable ?
- How should we proceed when E is a high-dimensional space?
- What if the new point x_1 leaves the admissible region K?

The appropriate framework to answer the first two questions is convex analysis. The lack of differentiability can be bypassed by introducing the concept of *subdifferential*. *Duality* methods solve a problem related to ((1.2.1)), called *dual problem*. The dual problem can often be easier to solve (*ex:* if it belongs to a space of smaller dimension). Typically, once the dual solution is known, the primal problem can be written as a unconstrained problem that is easier to solve than the initial one. For example, *proximal* methods can be used to solve constrained problems.

To go further . . .
A panorama in Boyd and Vandenberghe (2009), chapter 4, more rigor in Nesterov (2004)'s introduction chapter (easy to read !).

Chapter 2

Elements of convex analysis

Throughout this course, the functions of interest are defined on a subset of $\mathcal{X} = \mathbb{R}^n$. We will also need a Euclidean space $\mathbf{E}$, endowed with a scalar product denoted by $\langle \cdot\,, \cdot \rangle$ and an associated norm $\| \cdot \|$. In practice, the typical setting is $\mathbf{E} = \mathcal{X} \times \mathbb{R} = \mathbb{R}^{n+1}$.
Notations: For convenience, the same notation is used for the scalar product in $\mathcal{X}$ and in $\mathbf{E}$. If $a \leq b \in \mathbb{R} \cup \{-\infty, +\infty\}$, $(a, b]$ is an interval open at a, closed at b, with similar meanings for $[a, b)$, (a, b) and $[a, b]$.

N.B The proposed exercises include basic properties for you to demonstrate. You are strongly encouraged to do so ! The exercises marked with * are less essential.

2.1 Convexity

Definition 2.1.1 (Convex set). *A set $K \subset \mathbf{E}$ is **convex** if*

$$\forall (x, y) \in K^2, \forall t \in [0, 1], \quad t\, x + (1 - t)\, y \in K.$$

Exercise 2.1.1.

1. Show that a ball, a vector subspace or an affine subspace of $\mathbb{R}^n$ are convex.
2. Show that any intersection of convex sets is convex.

In constrained optimization problems, it is useful to define cost functions with value $+\infty$ outside the admissible region. For all $f : \mathcal{X} \to [-\infty, +\infty]$, the *domain* of f, denoted by $\mathrm{dom}(f)$, is the set of points x such that $f(x) < +\infty$.
A function f is called **proper** if $\mathrm{dom}(f) \neq \emptyset$ (i.e $f \not\equiv +\infty$) and if f *never* takes the value $-\infty$.

Definition 2.1.2. *Let $f : \mathcal{X} \to [-\infty, +\infty]$. The **epigraph of** f, denoted by $\mathrm{epi}\, f$, is the subset of $\mathcal{X} \times \mathbb{R}$ defined by:*

$$\mathrm{epi}\, f = \{(x, t) \in \mathcal{X} \times \mathbb{R} : \quad t \geq f(x)\ \}.$$

Beware: the "ordinates" of points in the epigraph always lie in $(-\infty, \infty)$, by definition.

Definition 2.1.3 (Convex function). *$f : \mathcal{X} \to [-\infty, +\infty]$ is **convex** if its epigraph is convex.*

Proposition 2.1.1. *A function $f : \mathcal{X} \to [-\infty, +\infty]$ is convex if and only if*

$$\forall (x, y) \in \mathcal{X}^2,\ \forall t \in (0, 1), \quad f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y).$$

Proof. Assume that f satisfies the inequality. Let (x, u) and (y, v) be two points of the epigraph: $u \geq f(x)$ and $v \geq f(y)$. In particular, $(x, y) \in \mathrm{dom}(f)^2$. Let $t \in]0, 1[$. The inequality implies that $f(tx + (1-t)y) \leq tu + (1-t)v$. Thus, $t(x, u) + (1-t)(y, v) \in \mathrm{epi}(f)$, which proves that $\mathrm{epi}(f)$ is convex.
Conversely, assume that $\mathrm{epi}(f)$ is convex. If $x \notin \mathrm{dom}\, f$ or $y \notin \mathrm{dom}\, f$, the inequality is trivial. So let us consider $(x, y) \in \mathrm{dom}(f)^2$. For (x, u) and (y, v) two points in $\mathrm{epi}(f)$, and $t \in [0, 1]$, the point $t(x, u) + (1-t)(y, v)$ belongs to $\mathrm{epi}(f)$. So, $f(t(x + (1-t)y) \leq tu + (1-t)v$.

- If $f(x)$ et $f(y)$ are $> -\infty$, we can choose $u = f(x)$ and $v = f(y)$, which demonstrates the inequality.
- If $f(x) = -\infty$, we can choose u arbitrary close to $-\infty$.Letting u go to $-\infty$, we obtain $f(t(x + (1-t)y) = -\infty$, which demonstrates here again the inequality we wanted to prove. □

Exercise 2.1.2. Show that:

1. If f is convex, then $\mathrm{dom}(f)$ is convex.
2. If f_1, f_2 are convex and $a, b \in \mathbb{R}_+$, then $af_1 + bf_2$ is convex.
3. If f is convex and $x, y \in \mathrm{dom}\, f$, then for all $t \geq 1$, $z_t = x + t(y - x)$ satisfies the inequality $f(z_t) \geq f(x) + t(f(y) - f(x))$.
4. If f is convex, proper, with $\mathrm{dom}\, f = \mathcal{X}$, and if f is bounded, then f is constant.

Exercise 2.1.3. *
Let f be a convex function and x, y in $\mathrm{dom}\, f$, $t \in (0, 1)$ and $z = tx + (1-t)y$. Assume that the three points $(x, f(x)), (z, f(z))$ and $(y, f(y))$ are aligned. Show that for all $u \in (0, 1)$, $f(u\, x + (1-u)\, y) = u\, f(x) + (1-u)\, f(y)$.

In the following, the **upper hull** of a family $(f_i)_{i \in I}$ of convex functions will play a key role. By definition, the upper hull of the family is the function $x \mapsto \sup_i f_i(x)$.

Proposition 2.1.2. *Let $(f_i)_{i \in I}$ be a family of convex functions $\mathcal{X} \to [-\infty, +\infty]$, with I* any *set of indices. Then* ***the upper hull of the family*** $(f_i)_{i\in I}$ ***is convex****.*

Proof. Let $f = \sup_{i \in I} f_i$ be the upper hull of the family.

(a) *epi* $f = \bigcap_{i \in I} \mathrm{epi}\, f_i$. Indeed,

$$(x, t) \in \mathrm{epi}\, f \Leftrightarrow \forall i \in I, t \geq f_i(x) \Leftrightarrow \forall i \in I, (x, t) \in \mathrm{epi}\, f_i \Leftrightarrow (x, t) \in \cap_i \mathrm{epi}\, f_i.$$

(b) Any intersection of convex sets $K = \cap_{i \in I} K_i$ is convex (exercice 2.1.1)

(a) and (b) show that $\mathrm{epi}\, f$ is convex, *i.e.* that f is convex. □

Proposition* 2.1.3. *Let $F : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ be a* **jointly** *convex function. Then the function*

$$\begin{aligned} f : \mathcal{X} &\to \mathbb{R} \\ x &\mapsto \inf_{y \in \mathcal{Y}} F(x, y) \end{aligned}$$

is convex.

Proof. Let $u, v \in \mathcal{X}$ and $\alpha \in (0,1)$. We need to show that $f(\alpha u+(1-\alpha)v) \leq \alpha f(u)+(1-\alpha)f(v)$.

$$\begin{aligned}
\alpha f(u) + (1-\alpha)f(v) &= \alpha \inf_{y_u \in \mathcal{Y}} F(u, y_u) + (1-\alpha) \inf_{y_v \in \mathcal{Y}} F(v, y_v) && \text{(definition of } f\text{)} \\
&= \inf_{y_u \in \mathcal{Y}, y_v \in \mathcal{Y}} \alpha F(u, y_u) + (1-\alpha) F(v, y_v) && \text{(separable problems)} \\
&\geq \inf_{y_u \in \mathcal{Y}, y_v \in \mathcal{Y}} F(\alpha u + (1-\alpha)v, \alpha y_u + (1-\alpha) y_v) && \text{(joint convexity of } F\text{)} \\
&= \inf_{y \in \mathcal{Y}} F(\alpha u + (1-\alpha), y) && \text{(change of variable)} \\
&= f(\alpha u + (1-\alpha)v)
\end{aligned}$$

A valid change of variable is $(y, y') = (\alpha y_u + (1-\alpha)y_v, y_v)$. It is indeed invertible since we have $\alpha \in (0,1)$. □

Definition 2.1.4 (Strong convexity)**.** *A function f is μ-strongly convex if $f - \frac{\mu}{2}\|\cdot\|^2$ is convex.*

Proposition 2.1.4. *A function $f : \mathcal{X} \to [-\infty, +\infty]$ is μ-strongly convex if and only if*

$$\forall (x,y) \in \mathcal{X}^2,\ \forall t \in (0,1), \quad f(tx + (1-t)y) \leq t f(x) + (1-t) f(y) - \frac{\mu}{2} t(1-t)\|x-y\|^2.$$

2.2 Lower semi-continuity

In this course, we will consider functions with infinite values. Such function cannot be continuous. However, some kind of continuity would be very desirable. For infinite-valued convex function, lower semi-continuity is the good generalization of continuity.

Definition 2.2.1 (Reminder: $\liminf$: **limit inferior**)**.**
*The **limit inferior** of a sequence $(u_n)_{n\in\mathbb{N}}$, where $u_n \in [-\infty, \infty]$, is*

$$\liminf(u_n) = \sup_{n \geq 0} \Big(\inf_{k \geq n} u_k \Big).$$

Since the sequence $V_n = \inf_{k \geq n} u_k$ is non decreasing, an equivalent definition is

$$\liminf(u_n) = \lim_{n \to \infty} \Big(\inf_{k \geq n} u_k \Big).$$

Definition 2.2.2 (Lower semicontinuous function)**.** *A function $f : \mathcal{X} \to [-\infty, \infty]$ is called* **lower semicontinuous (l.s.c.) at** $x \in \mathcal{X}$ *if for all sequence (x_n) which converges to x,*

$$\liminf f(x_n) \geq f(x).$$

The function f is said to be **lower semicontinuous***, if it is l.s.c. at x, for all $x \in \mathcal{X}$.*

The interest of l.s.c. functions becomes clear in the next result

Proposition 2.2.1 (epigraphical characterization)**.** *Let $f : \mathcal{X} \to [-\infty, +\infty]$, any function f is l.s.c. if and only if its epigraph is closed.*

Proof. If f is l.s.c., and if $(x_n, t_n) \in \operatorname{epi} f \to (\bar{x}, \bar{t})$, then, $\forall n, t_n \geq f(x_n)$. Consequently,

$$\bar{t} = \liminf t_n \geq \liminf f(x_n) \geq f(\bar{x}).$$

Thus, $(\bar{x}, \bar{t}) \in \operatorname{epi} f$, and $\operatorname{epi} f$ is closed.
Conversely, if f is *not* l.s.c., there exists an $x \in \mathcal{X}$, and a sequence $(x_n) \to x$, such that $f(x) > \liminf f(x_n)$, *i.e.*, there is an $\epsilon > 0$ such that $\forall n \geq 0$, $\inf_{k \geq n} f(x_k) \leq f(x) - \epsilon$. Thus, for all n, $\exists k_n \geq k_{n-1}$, $f(x_{k_n}) \leq f(x) - \epsilon$. We have built a sequence $(w_n) = (x_{k_n}, f(x) - \epsilon)$, each term of which belongs to $\operatorname{epi} f$, and which converges to a limit $\bar{w} = (f(x) - \epsilon)$ which is outside the epigraph. Consequently, $\operatorname{epi} f$ is not closed. □

There is a great variety of characterizations of l.s.c. functions, one of them is given in the following exercise.

Exercise 2.2.1. Show that a function f is l.s.c. if and only if its level sets :

$$L_{\leq \alpha} = \{x \in \mathcal{X} : f(x) \leq \alpha\}$$

are closed.
(see, *e.g.*, Rockafellar et al. (1998), Theorem 1.6.)

Lower semi-continuity is a very desirable property for a function we want to optimize thanks to the following proposition.

Proposition 2.2.2. *Let f be a l.s.c function such that $\lim_{\|x\| \to +\infty} f(x) = +\infty$. Then there exists x^* such that $f(x^*) = \inf_{x \in \mathcal{X}} f(x)$.*

Proof. Let $(x_n)_{n \geq 0}$ be a minimizing sequence, that is a sequence of $\mathcal{X}$ such that we have $\lim_{n\to\infty} f(x_n) = \inf_{x \in \mathcal{X}} f(x)$.
Suppose that (x_n) were unbounded. Then there would exist a subsequence $(x_{\phi(n)})$ such that $\lim_{n\to\infty} \|x_{\phi(n)}\| \to +\infty$. By the assumptions on f, this implies that $\lim_{n\to\infty} f(x_n) = +\infty$ which contradicts the fact that $(x_n)_{n\geq 0}$ is a minimizing sequence.
Thus (x_n) is bounded and we can extract from is a subsequence $(x_{\phi(n)})$ converging to, say, x_*. As f is l.s.c., we get $\inf_{x \in \mathcal{X}} f(x) = \lim_{n\to\infty} f(x_{\phi(n)}) = \liminf f(x_{\phi(n)}) \geq f(x^*) \geq \inf_{x \in \mathcal{X}} f(x)$. □

A nice property of the family of l.s.c. functions is its stability with respect to point-wise suprema.

Lemma 2.2.1. *Let $(f_i)_{i \in I}$ a family of l.s.c. functions. Then, the upper hull $f = \sup_{i \in I} f_i$ is l.s.c.*

Proof. Let C_i denote the epigraph of f_i and $C = \operatorname{epi} f$. As already shown (proof of proposition 2.1.2), $C = \cap_{i \in I} C_i$. Each C_i is closed, and any intersection of closed sets is closed, so C is closed and f is l.s.c. □

2.3 Separating hyperplanes

Separation theorems stated in this section are easily proved in finite dimension, using the existence of the "orthogonal projection" of a point x onto a closed convex set, which is stated below.

Proposition 2.3.1 (Projection)**.** *Let $C \subset \mathbf{E}$ be a convex, closed set, and let $x \in \mathbf{E}$.*

1. *There is a unique point in C, denoted by $P_C(x)$, such that*

$$\text{for all } y \in C,\ \|y - x\| \geq \|P_C(x) - x\|.$$

   The point $P_C(x)$ satisfies:

2. $\forall y \in C$, $\langle y - P_C(x), x - P_C(x) \rangle \leq 0$.
3. $\forall (x, y) \in \mathbf{E}^2$, $\|P_C(y) - P_C(x)\| \leq \|y - x\|$.

The point $P_C(x)$ is called projection of x on C.

Proof.

1. Let $d_C(x) = \inf_{y\in C} \|y - x\|$. As $(x \mapsto x^2)$ is increasing on $[0, +\infty)$, we can also write $d_C(x)^2 = \inf_{y\in C} \|y - x\|^2 = \inf_{y\in \mathbf{E}} g(y)$, where $g(y) = \|y - x\|^2 + \mathbb{I}_C(y)$.

   g is l.s.c. and strongly convex, so there exists a unique minimizer y^* s.t. $d_C(x)^2 = g(y^*)$. Applying the square root, we get that $d_C(x) = \|y^* - x\| \leq \|y - x\|, \forall y \in C$.

2. Let $p = P_C(x)$ and let $y \in C$. For $\epsilon \in [0, 1]$, let $z_\epsilon = p + \epsilon(y - p)$. By convexity, $z_\epsilon \in C$. Consider the function 'squared distance from x':

$$\varphi(\epsilon) = \|z_\epsilon - x\|^2 \;=\; \|\epsilon(y - p) + p - x\|^2.$$

   For $0 < \epsilon \leq 1$, $\varphi(\epsilon) \geq d_C(x)^2 = \varphi(0)$. Furthermore, for ϵ sufficiently close to zero,

$$\varphi(\epsilon) = d_C(x)^2 - 2\epsilon \langle y - p, x - p\rangle + o(\epsilon),$$

   whence $\varphi'(0) = -2 \langle y - p, x - p\rangle$. In the case $\varphi'(0) < 0$, we would have, for ϵ close to 0, $\varphi(\epsilon) < \varphi(0) = d_C(x)$, which is impossible. So $\varphi'(0) \geq 0$ and the result follows.

3. Adding the inequalities

$$\begin{aligned}&\langle P_C(y) - P_C(x), x - P_C(x)\rangle \leq 0 \;, \text{ and}\\ &\langle P_C(x) - P_C(y), y - P_C(y)\rangle \leq 0 \;,\end{aligned}$$

   yields $\langle P_C(y) - P_C(x), y - x\rangle \geq \|P_C(x) - P_C(y)\|^2$. The conclusion follows using Cauchy-Schwarz inequality. □

The existence of a projection allows to explicitly obtain the "separating hyperplanes". First, let's give two definitions, illustrated in Figure 2.1.

Definition 2.3.1 (strong separation, proper separation)**.** *Let $A, B \subset \mathbf{E}$, and H an affine hyperplane, $H = \{x \in \mathbf{E} : \langle x, w\rangle = \alpha\}$, where $w \neq 0$.*

- *H **properly separates** A **and** B if,*

$$\begin{aligned}&\forall x \in A, \langle w, x\rangle \leq \alpha \;, \quad \textit{and}\\ &\forall x \in B, \langle w, x\rangle \geq \alpha.\end{aligned}$$

- *H **strongly separates** A **and** B if, for some $\delta > 0$,*

$$\begin{aligned}&\forall x \in A, \langle w, x\rangle \leq \alpha - \delta \;, \quad \textit{and}\\ &\forall x \in B, \langle w, x\rangle \geq \alpha + \delta \;,\end{aligned}$$

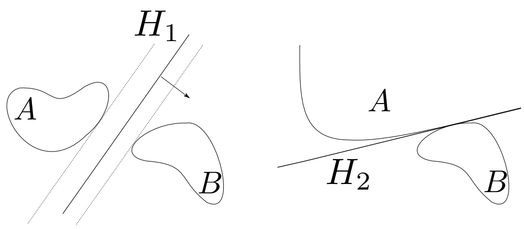

Figure 2.1: strong separation of A and B by H_1, proper separation by H_2.

The following theorem is one of the two major results of this section (with the existence of a supporting hyperplane). It is the direct consequence of the proposition 2.3.1.

Theorem 2.3.1 (Strong separation of closed convex set and a point)**.** *Let $C \subset \mathbf{E}$ convex, closed, and let $x \notin C$. Then, there is an affine hyperplane which strongly separates x and C.*

Proof. Let $p = p_C(x)$, $w = x - p$. For $y \in C$, according to the proposition 2.3.1, 2., we have $\langle w, y - p\rangle \leq 0$, i.e

$$\forall y \in C, \langle w, y\rangle \leq \langle p, w\rangle .$$

Further, $\langle w, x - p\rangle = \|w\|^2 > 0$, so that

$$\langle w, x\rangle = \langle w, p\rangle + \|w\|^2.$$

Now, define $\delta = \|w\|^2/2 > 0$ and $\alpha = \langle p, w\rangle + \delta$, so that the inequality defining strong separation is satisfied. $\square$

An immediate consequence, which will be repeatedly used thereafter :

Corollary 2.3.1 (Consequence of the strong separation)**.** *Let $C \subset \mathbf{E}$ be convex, closed. And let $x_0 \notin C$. Then there is $w \in \mathbf{E}$, such that*

$$\forall y \in C,\ \langle w, y\rangle < \langle w, x_0\rangle$$

In the following, we denote by $\mathrm{cl}(A)$ the closure of a set A and by $\mathrm{int}(A)$ its interior.The following lemma is easily proved:

Lemma 2.3.1. *If A is convex, then $\mathrm{cl}(A)$ and $\mathrm{int}(A)$ are convex.*

Exercise 2.3.1. Show Lemma 2.3.1.
Hint: construct two sequences in A converging towards two points of the closure of A; Envelop two points of its interior inside two balls.

Lemma 2.3.2. *If A is convex, $x \in \mathrm{int}(A)$ and $y \in \mathrm{cl}(A)$, then $[x, y) \subseteq A$.*

Proof. There is a sequence (y_n) such that $y_n \in A$ and $y_n \to y$. On the other hand, there exists $r > 0$ such that if $\|u\| \leq r$, then $x + u \in A$.
Let us consider $z = (1 - t)x + ty$ for $t \in [0, 1)$.

$$z = ty_n + (1 - t)\Big(x + \frac{t}{1 - t}(y - y_n)\Big)$$

where $y_n \in A$ and $x + \frac{t}{1-t}(y - y_n) \in A$ as soon as $\|\frac{t}{1-t}(y - y_n)\| \leq r$. As $y_n \to y$, there exists a n such that this inequality holds true and so $z \in A$. $\square$

Lemma 2.3.3. *If A is convex, then $\mathrm{int}(\mathrm{cl}(A)) = \mathrm{int}(A)$.*

Proof. $\mathrm{int}(A) \subseteq \mathrm{int}(\mathrm{cl}(A))$ since $A \subseteq \mathrm{cl}(A)$.
Let us consider $x \in \mathrm{int}(\mathrm{cl}(A))$. Let us also consider $z \in \mathrm{int}(A)$. As $x \in \mathrm{int}(\mathrm{cl}(A))$, there exists $r > 0$ such that if $\|y - x\| \leq r$ then $y \in \mathrm{cl}\, A$, and so $x - r/\|z - x\|(z - x) \in \mathrm{cl}\, A$.
By the previous lemma, for all $t \in [0, 1)$, $(1 - t)z + t(x - r/\|z - x\|(z - x)) \in A$. We just need to choose $t = \frac{1}{1+r/\|z-x\|}$ to deduce that $x \in A$.
Hence $\mathrm{int}(\mathrm{cl}(A)) \subseteq A$ which implies that $\mathrm{int}(\mathrm{cl}(A)) \subseteq \mathrm{int}(A)$. $\square$

The second major result is the following:

Theorem 2.3.2 (supporting hyperplane)**.** *Let $C \subset \mathbf{E}$ be a convex set and let x_0 be a point of its boundary, $x_0 \in \partial(C) = \mathrm{cl}(C) \setminus \mathrm{int}(C)$. There is an affine hyperplane that properly separates x_0 and C,* i.e.,

$$\exists w \in \mathbf{E}\ :\ \forall y \in C,\ \langle w, y\rangle \leq \langle w, x_0\rangle .$$

Proof. Let C and x_0 be as in the statement.
As $x_0 \notin \operatorname{int} C$, $x_0 \in (\operatorname{int} C)^c = \operatorname{cl}(C^c) = \operatorname{cl}\Big(\big(\operatorname{cl}(C)\big)^c\Big)$. (The last equality uses Lemma 2.3.3.) Thus, there is a sequence (x_n) with $x_n \in \big(\operatorname{cl}(C)\big)^c$ and $x_n \to x_0$. Each x_n can be strongly separated from $\operatorname{cl}(C)$, according to Theorem 2.3.1. Furthermore, corollary 2.3.1 implies :

$$\forall n,\ \exists w_n \in \mathbf{E}\ :\ \forall y \in C, \langle w_n, y\rangle < \langle w_n, x_n\rangle$$

In particular, each w_n is non-zero, so that the corresponding unit vector $u_n = w_n/\|w_n\|$ is well defined. We get:

$$\forall n, \forall y \in C,\ \langle u_n, y\rangle < \langle u_n, x_n\rangle\,. \tag{2.3.1}$$

Since the sequence (u_n) is bounded, we can extract a subsequence $(u_{k_n})_n$ that converges to some $u \in \mathbf{E}$. Since each u_n belongs to the unit sphere, which is closed, so does the limit u, so $u \neq 0$. Letting n tend to infinity in (2.3.1) (for y fixed), and using the linearity of scalar product, we get:

$$\forall y \in C,\ \langle u, y\rangle \le \langle u, x_0\rangle$$

□

Remark 2.3.1. *In infinite dimension, Theorems 2.3.1 and 2.3.2 remain valid if $\mathbf{E}$ is a Hilbert space (or even a Banach space). This is the "Hahn-Banach theorem", the proof of which may be found, for example, within the first few pages of Brezis (1987)*

2.4 Subdifferential

As a consequence (Proposition 2.4.1) of Theorem 2.3.2, the following definition is 'non-empty':

Definition 2.4.1 (Subdiffential)**.** *Let $f : \mathcal{X} \to [-\infty, +\infty]$ and $x \in \operatorname{dom}(f)$. A vector $\phi \in \mathcal{X}$ is called a **subgradient** of f at x if:*

$$\forall y \in \mathcal{X},\quad f(y) - f(x) \ge \langle \phi, y - x\rangle\ .$$

*The **subdifferential** of f in x, denoted by $\partial f(x)$, is the whole set of the subgradients of f at x. By convention, $\partial f(x) = \emptyset$ if $x \notin \operatorname{dom}(f)$.*

Interest: Gradient methods in optimization can still be used in the non-differentiable case, choosing a subgradient in the subdifferential.
In order to clarify in what cases the subdifferential is non-empty, we need two more definitions:

Definition 2.4.2. *A set $A \subset \mathcal{X}$ is called an **affine space** if, for all $(x, y) \in A^2$ and for all $t \in \mathbb{R}$, $x + t(y - x) \in A$. The **affine hull** $\mathcal{A}(C)$ of a set $C \subset \mathcal{X}$ is **the smallest affine space** that contains C.*

Definition 2.4.3. *Let $C \subset \mathbf{E}$. The **topology relative to** C is a topology on $\mathcal{A}(C)$. The open sets in this topology are the sets of the kind $\{V \cap \mathcal{A}(C)\}$, where V is open in $\mathbf{E}$.*

Definition 2.4.4. *Let $C \subset \mathcal{X}$. The **relative interior** of C, denoted by $\operatorname{relint}(C)$, is the interior of C for the topology relative to C. In other words, it consists of the points x that admit a neighborhood V, open in $\mathbf{E}$, such that $V \cap \mathcal{A}(C) \subset C$.*

Clearly, $\operatorname{int}(C) \subset \operatorname{relint}(C)$. What's more, if C is convex, $\operatorname{relint}(C) \neq \emptyset$. Indeed :

- if C is reduced to a singleton $\{x_0\}$, then $\operatorname{relint}\{x_0\} = \{x_0\}$. ($\mathcal{A}(C) = \{x_0\}$ and for an open set $U \subset \mathcal{X}$, such that $x_0 \subset U$, we indeed have $x_0 \in U \cap \{x_0\}$) ;

- if C contains at least two points x, y, then any other point within the open segment $\{x + t(y - x), t \in (0, 1)\}$ is in $\mathcal{A}(C)$.

Proposition 2.4.1. *Let $f : \mathcal{X} \to [-\infty, +\infty]$ be a convex function and $x \in \operatorname{relint}(\operatorname{dom} f)$. Then $\partial f(x)$ is non-empty.*

Proof.* Let $x_0 \in \operatorname{relint}(\operatorname{dom} f)$. We assume that $f(x_0) > -\infty$ (otherwise the proof is trivial). We may restrict ourselves to the case $x_0 = 0$ and $f(x_0) = 0$ (up to replacing f by the function $x \mapsto f(x + x_0) - f(x_0)$).
In this case, for all vector $\phi \in \mathcal{X}$,

$$\phi \in \partial f(0) \quad \Leftrightarrow \quad \forall x \in \operatorname{dom} f,\ \langle \phi, x\rangle \leq f(x)\,.$$

Let $\mathcal{A} = \mathcal{A}(\operatorname{dom} f)$. $\mathcal{A}$ contains the origin, so it is an Euclidean vector space.
Let C be the closure of $\operatorname{epi} f \cap (\mathcal{A} \times \mathbb{R})$. The set C is a convex closed set in $\mathcal{A} \times \mathbb{R}$, which is endowed with the scalar product $\langle (x, u), (x', u')\rangle = \langle x, x'\rangle + uu'$.
The pair $(0, 0) = (x_0, f(x_0))$ belongs to the boundary of C, so that Theorem 2.3.2 applies in $\mathcal{A} \times \mathbb{R}$: There is a vector $w \in \mathcal{A} \times \mathbb{R}$, $w \neq 0$, such that

$$\forall z \in C,\ \langle w, z\rangle \leq 0$$

Write $w = (\phi, u) \in \mathcal{A} \times \mathbb{R}$. For $z = (x, t) \in C$, we have

$$\langle \phi, x\rangle + u\,t \leq 0.$$

Let $x \in \operatorname{dom}(f)$. In particular $f(x) < \infty$ and for all $t \geq f(x)$, $(x, t) \in C$. Thus,

$$\forall x \in \operatorname{dom}(f),\ \forall t \geq f(x), \quad \langle \phi, x\rangle + u\,t \leq 0. \tag{2.4.1}$$

Letting t tend to $+\infty$, we obtain $u \leq 0$.
Let us prove by contradiction that $u < 0$. Suppose not (*i.e.* $u = 0$). Then $\langle \phi, x\rangle \leq 0$ for all $x \in \operatorname{dom}(f)$. As $0 \in \operatorname{relint} \operatorname{dom}(f)$, there is a set $\tilde{V}$, open in $\mathcal{A}$, such that $0 \in \tilde{V} \subset \operatorname{dom} f$. Thus for $x \in \mathcal{A}$, there is an $\epsilon > 0$ such that $\epsilon x \in \tilde{V} \subset \operatorname{dom}(f)$. According to (2.4.1), $\langle \phi, \epsilon x\rangle \leq 0$, so $\langle \phi, x\rangle \leq 0$. Similarly, $\langle \phi, -x\rangle \leq 0$. Therefore, $\langle \phi, x\rangle \equiv 0$ on $\mathcal{A}$. Since $\phi \in \mathcal{A}$, $\phi = 0$ as well. Finally $w = 0$, which is a contradiction.
As a result, $u < 0$. Dividing inequality (2.4.1) by $-u$, and taking $t = f(x)$, we get

$$\forall x \in \operatorname{dom}(f),\ \forall t \geq f(x), \quad \langle \frac{-1}{u}\phi, x\rangle \leq f(x)\,.$$

So $\frac{-1}{u}\phi \in \partial f(0)$. □

Remark 2.4.1 (the question of $-\infty$ values)**.**
If $f : \mathcal{X} \to [-\infty, +\infty]$ is convex and if relint dom f *contains a point x such that $f(x) > -\infty$, then f* never *takes the value $-\infty$. So f is proper.*

Exercise 2.4.1. Show this point, using proposition 2.4.1.

When f is differentiable at $x \in \operatorname{dom} f$, we denote by $\nabla f(x)$ its gradient at x. The link between differentiation and subdifferential is given by the following proposition:

Proposition 2.4.2. *Let $f : \mathcal{X} \to (-\infty, \infty]$ be a convex function, differentiable in x. Then $\partial f(x) = \{\nabla f(x)\}$.*

Proof. If f is differentiable at x, the point x necessarily belongs to $\operatorname{int}\big(\operatorname{dom}(f)\big)$. Let $\phi \in \partial f(x)$ and $t \neq 0$. Then for all $y \in \operatorname{dom}(f)$, $f(y) - f(x) \geq \langle \phi, y - x\rangle$. Applying this inequality to $y = x + t(\phi - \nabla f(x))$ (which belongs to $\operatorname{dom}(f)$ for t small enough) leads to :

$$\frac{f\left(x + t(\phi - \nabla f(x))\right) - f(x)}{t} \geq \langle \phi, \phi - \nabla f(x)\rangle \ .$$

The left term converges to $\langle \nabla f(x), \phi - \nabla f(x)\rangle$. Finally,

$$\langle \nabla f(x) - \phi, \phi - \nabla f(x)\rangle \geq 0,$$

i.e. $\phi = \nabla f(x)$. □

Example 2.4.1. The absolute-value function $x \mapsto |x|$ defined on $\mathbb{R} \to \mathbb{R}$ admits as a subdifferential the sign application, defined by :

$$\operatorname{sign}(x) = \begin{cases} \{1\} & \text{if } x > 0 \\ [-1, 1] & \text{if } x = 0 \\ \{-1\} & \text{ifi } x < 0\,. \end{cases}$$

Exercise 2.4.2. Determine the subdifferentials of the following functions, at the considered points :

1. In $\mathcal{X} = \mathbb{R}$, $f(x) = \mathbb{I}_{[0,1]}$, at $x = 0, x = 1$ and $0 < x < 1$.
2. In $\mathcal{X} = \mathbb{R}^2$, $f(x) = \mathbb{I}_{B(0,1)}$ (closed Euclidian ball), at $\|x\| < 1$, $\|x\| = 1$.
3. In $\mathcal{X} = \mathbb{R}^2$, $f(x_1, x_2) = \mathbb{I}_{x_1 < 0}$, at x such that $x_1 = 0$, $x_1 < 0$.
4. $\mathcal{X} = \mathbb{R}$,

   $$f(x) = \begin{cases} +\infty & \text{if } x < 0 \\ -\sqrt{x} & \text{if } x \geq 0 \end{cases}$$

   at $x = 0$, and $x > 0$.
5. $\mathcal{X} = \mathbb{R}^n$, $f(x) = \|x\|$, determine $\partial f(x)$, for any $x \in \mathbb{R}^n$.
6. $\mathcal{X} = \mathbb{R}$, $f(x) = x^3$. Show that $\partial f(x) = \emptyset$, $\forall x \in \mathbb{R}$. Explain this result.
7. $\mathcal{X} = \mathbb{R}^n$, $C = \{y : \|y\| \leq 1\}$, $f(x) = \mathbb{I}_C(x)$. Give the subdifferential of f at x such that $\|x\| < 1$ and at x such that $\|x\| = 1$.

   Hint: For $\|x\| = 1$:

   - Show that $\partial f(x) = \{\phi : \ \forall y \in C, \langle \phi, y - x\rangle \leq 0\}$.
   - Show that $x \in \partial f(x)$ using Cauchy-Schwarz inequality. Deduce that the cone $\mathbb{R}_+ x = \{t\, x \ : \ t \geq 0\} \subset \partial f(x)$.
   - To show the converse inclusion : Fix $\phi \in \partial f$ and pick $u \in \{x\}_\perp$ (*i.e.*, u s.t. $\langle u, x\rangle = 0$). Consider the sequence $y_n = \|x + t_n u\|^{-1}(x + t_n u)$, for some sequence $(t_n)_n, t_n > 0, t_n \to 0$. What is the limit of y_n ?

     Consider now $u_n = t_n^{-1}(y_n - x)$. What is the limit of u_n ? Conclude about the sign of $\langle \phi, u\rangle$.

     Do the same with $-u$, conclude about $\langle \phi, u\rangle$. Conclude.
8. Let $f : \mathbb{R}^n \to \mathbb{R}$, differentiable. Show that: f is convex, if and only if

   $$\forall (x, y) \in \mathbb{R}^n \times \mathbb{R}^n, \langle \nabla f(y) - \nabla f(x), y - x\rangle \geq 0.$$

2.5 Operations on subdifferentials

Until now, we have seen examples of subdifferential computations on basic functions, but we haven't mentioned how to derive the subdifferentials of more complex functions, such as sums or linear transforms of basic ones. A basic fact from differential calculus is that, when all the terms are differentiable, $\nabla(f+g) = \nabla f + \nabla g$. Also, if M is a linear operator, $\nabla(g \circ M)(x) = M^* \nabla g(Mx)$. Under qualification assumptions, these properties are still valid in the convex case, up to replacing the gradient by the subdifferential and point-wise operations by set operations. But first, we need to define operations on sets.

Definition 2.5.1 (addition and transformations of sets)**.** *Let $A, B \subset \mathcal{X}$. The Minkowski sum and difference of A and B are the sets*

$$A + B = \{x \in \mathcal{X} : \ \exists a \in A, \exists b \in B, \ x = a + b\}$$
$$A - B = \{x \in \mathcal{X} : \ \exists a \in A, \exists b \in B, \ x = a + b\}$$

Let $\mathcal{Y}$ another space and M any mapping from $\mathcal{X}$ to $\mathcal{Y}$. Then MA is the image of A by M,

$$MA = \{y \in \mathcal{Y} : \ \exists a \in A, \ y = Ma\}.$$

Proposition 2.5.1. *Let $f : \mathcal{X} \to (-\infty, +\infty]$, $g : \mathcal{Y} \to (-\infty, \infty]$ two convex functions and let $M : \mathcal{X} \to \mathcal{Y}$ a linear operator.*

$$\forall x \in \mathcal{X}, \partial f(x) + M^* \partial g(Mx) \subseteq \partial(f + g \circ M)(x)$$

Moreover, if $0 \in \operatorname{relint}(\operatorname{dom} g - M \operatorname{dom} f)$*, then*

$$\forall x \in \mathcal{X}, \partial(f + g \circ M)(x) = \partial f(x) + M^* \partial g(Mx)$$

Proof. Let us show first that $\partial f(\,\cdot\,) + M^* \partial g(M\,\cdot\,) \subseteq \partial(f + g \circ M)(\,\cdot\,)$. Let $x \in \mathcal{X}$ and $\phi \in \partial f(x) + M^* \partial g \circ M(x)$, which means that $\phi = u + M^* v$ where $u \in \partial f(x)$ and $v \in \partial g \circ M(x)$. In particular, none of the latter subdifferentials is empty, which implies that $x \in \operatorname{dom} f$ and $x \in \operatorname{dom}(g \circ M)$. By definition of u and v, for $y \in \mathcal{X}$,

$$\begin{cases} f(y) - f(x) \geq \langle u, y - x \rangle \\ g(My) - g(Mx) \geq \langle v, M(y - x) \rangle = \langle M^* v, y - x \rangle \,. \end{cases}$$

Adding the two inequalities,

$$(f + g \circ M)(y) - (f + g \circ M)(x) \geq \langle \phi, y - x \rangle \,.$$

Thus, $\phi \in \partial(f + g \circ M)(x)$ and $\partial f(x) + M^* \partial g(Mx) \subset \partial(f + g \circ M)(x)$.

The proof of the converse inclusion requires to use Fenchel-Rockafellar theorem 4.4.1, and **may be skipped at first reading.**
Notice first that $\operatorname{dom}(f + g \circ M) = \{x \in \operatorname{dom} f \ : \ Mx \in \operatorname{dom} g\}$. The latter set is non empty: to see this, use the assumption $0 \in \operatorname{relint}(\operatorname{dom} g - M \operatorname{dom} f)$. This means in particular that $0 \in \operatorname{dom} g - M \operatorname{dom} f$, so that $\exists \ (y, x) \in \operatorname{dom} g \times \operatorname{dom} f \ : \ 0 = y - Mx$.
Thus, let $x \in \operatorname{dom}(f + g \circ M)$. Then $x \in \operatorname{dom} f$ and $Mx \in \operatorname{dom} g$.
Assume $\phi \in \partial(f + g \circ M)(x)$. For $y \in \mathcal{X}$,

$$f(y) + g(My) - (f(x) + g(Mx)) \geq \langle \phi, y - x \rangle \,,$$

thus, x is a minimizer of the function $\varphi : y \mapsto f(y) - \langle \phi, y\rangle + g(My)$, which is convex. Using Fenchel-Rockafellar theorem 4.4.1, where $f - \langle \phi, \cdot\rangle$ replaces f, the dual value is attained: there exists $\psi \in \mathcal{Y}$, such that

$$f(x) - \langle \phi, x\rangle + g(Mx) = -(f - \langle \phi, .\rangle)^*(-M^*\psi) - g^*(\psi)\,.$$

It is easily verified that $(f - \langle \phi, \cdot\rangle)^* = f^*(\cdot + \phi)$. Thus,

$$f(x) - \langle \phi, x\rangle + g(Mx) = -f^*(-M^*\psi + \phi) - g^*(\psi)\,.$$

In other words,

$$f(x) + f(-M^*\psi + \phi) - \langle \phi, x\rangle + g(Mx) + g^*(\psi) = 0\,,$$

so that

$$[f(x) + f(-M^*\psi + \phi) - \langle -M^*\psi + \phi, x\rangle] + [g(Mx) + g^*(\psi) - \langle \psi, Mx\rangle] = 0\,.$$

Each of the terms within brackets is non negative (from Fenchel-Young inequality, Proposition4.2.1). Thus, both are null. Equality in Fenchel-Young implies that $\psi \in \partial g(Mx)$ and $-M^*\psi + \phi \in \partial f(x)$. This means that $\phi \in \partial f(x) + M^*\partial g(Mx)$, which concludes the proof. □

2.6 Fermat's rule, optimality conditions.

A point x is called a **minimizer** of f if $f(x) \leq f(y)$ for all $y \in \mathcal{X}$. The set of minimizers of f is denoted $\arg\min(f)$.

Theorem 2.6.1 (Fermat's rule)**.** $x \in \arg\min f \;\Leftrightarrow\; 0 \in \partial f(x)$.

Proof.

$$x \in \arg\min f \;\Leftrightarrow\; \forall y, f(y) \geq f(x) + \langle 0, y - x\rangle \Leftrightarrow 0 \in \partial f(x).$$

□

Recall that, in the differentiable, non convex case, a *necessary* condition (not a sufficient one) for $\bar{x}$ to be a local minimizer of f, is that $\nabla f(\bar{x}) = 0$. Convexity allows handling non differentiable functions, and turns the necessary condition into a sufficient one.
Besides, local minima for any function f are not necessarily global ones. In the convex case, everything works fine:

Proposition 2.6.1. *Let x be a local minimum of a convex function f. Then, x is a global minimizer.*

Proof. The local minimality assumption means that there exists an open ball $V \subset \mathcal{X}$, such that $x \in V$ and that, for all $u \in V$, $f(x) \leq f(u)$.
Let $y \in \mathcal{X}$ and t such that $u = x + t(y - x) \in V$. Then using convexity of f, $f(u) \leq tf(y) + (1 - t)f(x)$. Re-organizing, we get

$$f(y) \geq t^{-1}\big(f(u) - (1 - t)f(x)\big) \geq f(x).$$

□

Chapter 3

Lagrangian duality

3.1 Lagrangian function, Lagrangian duality

In this chapter, we consider the convex optimization problem

$$\text{minimize over } \mathbb{R}^n : \quad f(x) + \mathbb{I}_{g \preceq 0}(x) \ . \tag{3.1.1}$$

($i.e.$ minimize $f(x)$ over $\mathbb{R}^n$, under the constraint $g(x) \preceq 0$),
where $\mathbb{I}_{g \preceq 0} = \mathbb{I}_{g^{-1}(\mathbb{R}^p_-)}$, $f : \mathbb{R}^n \to (-\infty, \infty]$ is a convex, proper function; $g(x) = (g_1(x), \dots, g_p(x))$, and each $g_i : \mathbb{R}^n \to (-\infty, +\infty)$ is a convex function ($1 \le i \le p$). $g(x) \notin \{+\infty, -\infty\}$, for the sake of simplicity. This condition may be however replaced by a weaker one:

$$0 \in \operatorname{relint}(\operatorname{dom} f - \cap_{i=1}^p \operatorname{dom} g_i) \ . \tag{3.1.2}$$

See Bauschke and Combettes (2011) for more details on how to deal with functions with infinite values.

It is easily verified that under these conditions, the function $x \mapsto f(x) + \mathbb{I}_{g \preceq 0}(x)$ is convex.

Definition 3.1.1 (primal value, primal optimal point)**.** *The **primal value** associated to (3.1.1) is the infimum*

$$p = \inf_{x \in \mathbb{R}^n} f(x) + \mathbb{I}_{g \preceq 0}(x).$$

A point $x^* \in \mathbb{R}^n$ is called **primal optimal** *if*

$$p = f(x^*) + \mathbb{I}_{g \preceq 0}(x^*).$$

Notice that, under our assumption, $p \in [-\infty, \infty]$. Also, there is no guarantee about the existence of a primal optimal point, $i.e.$ that the primal value be attained.

Since (3.1.3) may be difficult to solve, it is useful to see this as an 'inf sup' problem, and solve a 'sup inf' problem instead (see definition 3.1.3 below). To make this precise, we introduce the Lagrangian function.

Definition 3.1.2. *The* **Lagrangian function** *associated to problem (3.1.1) is the function*

$$\begin{aligned} L : \mathbb{R}^n \times \mathbb{R}^p_+ &\longrightarrow [-\infty, +\infty] \\ (x, \phi) &\mapsto f(x) + \langle \phi, g(x) \rangle \end{aligned}$$

(where $\mathbb{R}^p_+ = \{\phi \in \mathbb{R}^p, \phi \succeq 0\}$).

The link with the initial problem comes next:

Lemma 3.1.1 (constrained objective as a supremum)**.** *The constrained objective is the supremum (over ϕ) of the Lagrangian function,*

$$\forall x \in \mathbb{R}^n,\ f(x) + \mathbb{I}_{g\preceq 0}(x) = \sup_{\phi\in\mathbb{R}^p_+} L(x,\phi)$$

Proof. Distinguish the case $g(x) \preceq 0$ and $g(x) \npreceq 0$.
(a) If $g(x) \npreceq 0, \exists i \in \{1,\dots,p\} : g_i(x) > 0$. Choosing $\phi_t = te_i$ (where $\mathbf{e} = (e_1,\dots,e_p)$ is the canonical basis of $\mathbb{R}^p$)), $t \geq 0$, then $\lim_{t\to\infty} L(x,\phi_t) = +\infty$, whence $\sup_{\phi\in\mathbb{R}^p_+} L(x,\phi) = +\infty$. On the other hand, in such a case, $\mathbb{I}_{g(x)\preceq 0} = +\infty$, whence the result.

(b) If $g(x) \preceq 0$, then $\forall \phi \in \mathbb{R}^p_+, \langle \phi, g(x)\rangle \leq 0$, and the supremum is attained at $\phi = 0$. Whence, $\sup_{\phi\succeq 0} L(x,\phi) = f(x)$.
On the other hand, $\mathbb{I}_{g(x)\leq 0} = 0$, so $f(x) + \mathbb{I}_{g(x)\preceq 0} = f(x)$. The result follows. □

Equipped with lemma 3.1.1, the primal value associated to problem (3.1.1) writes

$$p = \inf_{x\in\mathbb{R}^n} \sup_{\phi\in\mathbb{R}^p_+} L(x,\phi). \tag{3.1.3}$$

One natural idea is to exchange the order of inf and sup in the above problem. Before proceeding, the following simple lemma allows to understand the consequence of such an exchange.

Lemma 3.1.2. *Let $F : A\times B \to [-\infty,\infty]$ any function. Then,*

$$\sup_{y\in B}\inf_{x\in A} F(x,y) \leq \inf_{x\in A}\sup_{y\in B} F(x,y).$$

Proof. $\forall(\bar{x},\bar{y}) \in A\times B$,

$$\inf_{x\in A} F(x,\bar{y}) \leq F(\bar{x},\bar{y}) \leq \sup_{y\in B} F(\bar{x},y).$$

Taking the supremum over $\bar{y}$ in the left-hand side we still have

$$\sup_{\bar{y}\in B}\inf_{x\in A} F(x,\bar{y}) \leq \sup_{y\in B} F(\bar{x},y).$$

Now, taking the infimum over $\bar{x}$ in the right-hand side yields

$$\sup_{\bar{y}\in B}\inf_{x\in A} F(x,\bar{y}) \leq \inf_{\bar{x}\in A}\sup_{y\in B} F(\bar{x},y).$$

up to to a simple change of notation, this is

$$\sup_{y\in B}\inf_{x\in A} F(x,y) \leq \inf_{x\in A}\sup_{y\in B} F(x,y).$$

□

Definition 3.1.3 (Dual problem, dual function, dual value)**.**
The **dual value** *associated to (3.1.3) is*

$$d = \sup_{\phi\in\mathbb{R}^p_+}\inf_{x\in\mathbb{R}^n} L(x,\phi).$$

The function

$$\mathcal{D}(\phi) = \inf_{x\in\mathbb{R}^n} L(x,\phi)$$

is called the **Lagrangian dual function**. *Thus, the* **dual problem** *associated to the primal problem (3.1.1) is*

$$\text{maximize over } \mathbb{R}^p_+ : \quad \mathcal{D}(\phi).$$

A vector $\lambda \in \mathbb{R}_+p$ is called **dual optimal** *if*

$$d = \mathcal{D}(\lambda).$$

Without any further assumption, there is no reason for the two values (primal and dual) to coincide. However, as a direct consequence of lemma 3.1.2, we have :

Proposition 3.1.1 (Weak duality). *Let p and d denote respectively the primal and dual value for problem (3.1.1). Then,*

$$d \leq p\ .$$

Proof. Apply lemma 3.1.2. □

One interesting property of the dual function, for optimization purposes, is:

Lemma 3.1.3. *The dual function $\mathcal{D}$ is concave and upper semi-continuous.*

Proof. For each fixed $x \in \mathbb{R}^n$, the function

$$h_x : \phi \mapsto L(x,\phi) = f(x) + \langle \phi, g(x)\rangle$$

is affine, whence concave on $\mathbb{R}^p_+$. In other words, the negated function $-h_x$ is convex. Thus, its upper hull $h = \sup_x(-h_x)$ is convex. There remains to notice that

$$\mathcal{D} = \inf_x h_x = -\sup_x(-h_x) = -h,$$

so that $\mathcal{D}$ is concave, as required.
Using the same decomposition, $-\mathcal{D}(\phi) = \sup_{x\in\mathbb{R}^n}(-h_x)$, so $-\mathcal{D}$ is the supremum over a family indexed by x of l.s.c. functions (in ϕ). Hence, by Lemma 2.2.1, $-\mathcal{D}$ is itself l.s.c. □

3.2 Saddle points and KKT theorem

Introductory remark (Reminder: KKT theorem in smooth convex optimization). You may have already encountered the KKT theorem, in the smooth convex case: If f and the g_i's ($1 \leq i \leq p$) are convex, differentiable, and if the constraints are qualified in some sense (*e.g.*, Slater) it is a well known fact that, x^* is primal optimal if and only if, there exists a Lagrange multiplier vector $\lambda \in \mathbb{R}^p$, such that

$$\lambda \succeq 0, \quad \langle \lambda, g(x^*)\rangle = 0, \quad \nabla f(x^*) = -\sum_{i\in I} \lambda_i \nabla g_i(x^*).$$

(where I is the set of active constraints, *i.e.* the i's such that $g_i(x) = 0$.)
The last condition of the statement means that, if only one g_i is involved, and if there is no minimizer of f within the region $g_i < 0$, the gradient of the objective and that of the constraint are colinear, in opposite directions.
The objective of this section is to obtain a parallel statement in the convex, non-smooth case, with subdifferentials instead of gradients.

Definition 3.2.1 (Slater conditions)**.** *Consider the convex optimization problem (3.1.1). Assume that*

$$\exists \bar{x} \in \operatorname{dom} f: \ \forall i \in \{1, \dots, p\}, g_i(\bar{x}) < 0.$$

Then, we say that the constraints are qualified, in the sense of Slater.

Proposition 3.2.1. *If Slater's constraint qualification condition holds (Def 3.2.1), then there exist $\lambda \in \mathbb{R}^p$ such that*

$$\mathcal{D}(\lambda) = \sup_{\phi \in \mathbb{R}^m} \mathcal{D}(\phi) \tag{3.2.1}$$

Proof. By Slater's condition, there exists $\bar{x} \in \operatorname{dom} f$ such that $g_i(\bar{x}) < 0$ for all i. Hence, either $\phi \not\geq 0$ and $-\mathcal{D}(\phi) = +\infty$ or

$$\begin{aligned}-\mathcal{D}(\phi) &= \sup_{x \in \mathbb{R}^n} -f(x) - \langle \phi, x \rangle + \mathbb{I}_{\mathbb{R}^m_+}(\phi) = \sup_{x \in \mathbb{R}^n} -f(x) - \langle \phi, g(x) \rangle \geq -f(\bar{x}) - \langle \phi, g(\bar{x}) \rangle \\ &\geq -f(\bar{x}) + \max_j \phi_j \times (-g_j(\bar{x})) \geq -f(\bar{x}) + \|\phi\|_\infty \min_j(-g_j(\bar{x}))\end{aligned}$$

As $\min_j(-g_j(\bar{x})) > 0$ and $f(\bar{x}) < +\infty$, we conclude that $\lim_{\|\phi\|_\infty \to +\infty} -\mathcal{D}(\phi) = +\infty$.
Using Proposition 2.2.2, we known that there exist $\lambda \in \mathbb{R}^p$ such that

$$\mathcal{D}(\lambda) = \sup_{\phi \in \mathbb{R}^m} \mathcal{D}(\phi)$$ □

First, we shall prove that, under the constraint qualification condition (Def. 3.2.1), the solutions for problem (3.1.1) correspond to saddle points of the Lagrangian function.

Definition 3.2.2 (Saddle point)**.** *Let $F : A \times B \to [-\infty, \infty]$ any function, and A, B two sets. The point $(x^*, y^*) \in A \times B$ is called a* **saddle point** *of F if, for all $(x, y) \in A \times B$,*

$$F(x^*, y) \leq F(x^*, y^*) \leq F(x, y^*).$$

Proposition 3.2.2. *Assume that Slater's constraint qualification condition holds (Def 3.2.1). Then*

$$p = \inf_{x \in \mathbb{R}^n} f(x) + \mathbb{I}_{g \leq 0}(x) = \sup_{\phi \in \mathbb{R}^p} \mathcal{D}(\phi) = d$$

Proof. Let us introduce the function

$$\mathcal{V}(b) = \inf_{x \in \mathbb{R}^n} f(x) + \mathbb{I}_{\{(x', b') : g(x') \leq b'\}}(x, b).$$

called the value function of the problem. Note that $\mathcal{V}(0)$ is the value of the primal problem:

$$\mathcal{V}(0) = \inf_{x \in \mathbb{R}^n} f(x) + \mathbb{I}_{\{x' : g(x') \leq 0\}}(x).$$

As $\{(x', b') : g(x') \leq b'\}$ is convex and f is convex, $(x, b) \mapsto f(x) + \mathbb{I}_{\{(x', b') : g(x') \leq b'\}}(x, b)$ is convex. $\mathcal{V}$ is convex since it is the infimum of a jointly convex function (Proposition 2.1.3).
$\operatorname{dom} \mathcal{V} = \{b \in \mathbb{R}^p \ : \ \exists x \in \operatorname{dom} f, g(x) \leq b\}$. Hence, Slater's condition implies that for all $b \geq g(x_0)$, $b \in \operatorname{dom} \mathcal{V}$. As $g(x_0) < 0$, this implies that $0 \in \operatorname{int}(\operatorname{dom} \mathcal{V})$ and so $0 \in \operatorname{relint}(\operatorname{dom} \mathcal{V})$.
By Proposition 2.4.1, $\exists \lambda \in \partial \mathcal{V}(0)$. Using the definition of the subgradient,

$$\begin{gathered}\forall b \in \mathbb{R}^p, \mathcal{V}(b) \geq \mathcal{V}(0) + \langle \lambda, b - 0 \rangle \\ \mathcal{V}(b) - \langle \lambda, b \rangle \geq \mathcal{V}(0)\end{gathered}$$

Taking the infimum over b, we get a formula involving what we will call in Chapter 4 the Fenchel-Legendre conjugate of $\mathcal{V}$:

$$\inf_{b\in\mathbb{R}^p} \mathcal{V}(b) - \langle\lambda, b\rangle \geq \mathcal{V}(0). \tag{3.2.2}$$

We are going to show that $\inf_{b\in\mathbb{R}^p} \mathcal{V}(b) - \langle\lambda, b\rangle = \mathcal{D}(-\lambda)$.

$$\begin{aligned}\inf_{b\in\mathbb{R}^p} \mathcal{V}(b) - \langle\lambda, b\rangle &= \inf_{b\in\mathbb{R}^p} \inf_{x\in\mathbb{R}^n} f(x) + \mathbb{I}_{\{(x',b'):g(x')\leq b'\}}(x,b) - \langle\lambda, b\rangle \\ &= \inf_{x\in\mathbb{R}^n} \Big(f(x) + \inf_{b\in\mathbb{R}^p} \mathbb{I}_{\{(x',b'):g(x')\leq b'\}}(x,b) - \langle\lambda, b\rangle \Big) \end{aligned} \tag{3.2.3}$$

Let us fix x for the moment. If $\lambda \leq 0$, minimizing $\mathbb{I}_{\{(x',b'):g(x')\leq b'\}}(x,b) - \langle\lambda, b\rangle$ amounts to finding b such that $b \geq g(x)$ and which has the smallest possible coordinates. Hence, the optimum is given at $b = g(x)$ and $\inf_{b\in\mathbb{R}^p} \mathbb{I}_{\{(x',b'):g(x')\leq b'\}}(x,b) - \langle\lambda, b\rangle = \langle\lambda, g(x)\rangle$.
If $\lambda \not\leq 0$, $\exists j$ such that $\lambda_j > 0$. Choosing $b(t) = g(x) + te_j$ with $t > 0$, we get $b(t) \geq g(x)$ and

$$\mathbb{I}_{\{(x',b'):g(x')\leq b'\}}(x,b(t)) - \langle\lambda, b\rangle - \langle\lambda, b(t)\rangle = -\langle\lambda, g(x)\rangle - t\lambda_j \underset{t\to\infty}{\longrightarrow} -\infty.$$

To conclude,

$$\inf_{b\in\mathbb{R}^p} \mathbb{I}_{\{(x',b'):g(x')\leq b'\}}(x,b) - \langle\lambda, b\rangle = \langle-\lambda, g(x)\rangle - \mathbb{I}_{\mathbb{R}^p_+}(-\lambda)$$

and so by (3.2.3),

$$\inf_{b\in\mathbb{R}^p} \mathcal{V}(b) - \langle\lambda, b\rangle = \inf_{x\in\mathbb{R}^n} f(x) + \langle-\lambda, g(x)\rangle - \mathbb{I}_{\mathbb{R}^p_+}(-\lambda) = \mathcal{D}(-\lambda).$$

Hence, by (3.2.2), $\mathcal{D}(-\lambda) \geq \inf_{x\in\mathbb{R}^n} f(x) + \mathbb{I}_{g\leq 0}(x)$. Moreover, by Proposition 3.1.1, $\mathcal{D}(-\lambda) \leq \sup_{\phi\in\mathbb{R}^p} \mathcal{D}(\phi) \leq \inf_{x\in\mathbb{R}^n} f(x) + \mathbb{I}_{g\leq 0}(x)$, so $\mathcal{D}(-\lambda) = \sup_{\phi\in\mathbb{R}^p} \mathcal{D}(\phi) = \inf_{x\in\mathbb{R}^n} f(x) + \mathbb{I}_{g\leq 0}(x)$. ☐

Proposition 3.2.3 (primal attainment and saddle point)**.**
Consider the convex optimization problem (3.1.1) and assume that Slater's constraint qualification condition holds (Def 3.2.1). The following statements are equivalent:

(i) The point x^* is primal-optimal,

(ii) $\exists\lambda \in \mathbb{R}^p_+$, such that the pair (x^*, λ) is a saddle point of the Lagrangian function L.

Furthermore, if (i) or (ii) holds, then

$$p = d = L(x^*, \lambda).$$

Proof. **i) ⇒ ii)** First of all, by Proposition 3.2.1, $\exists\lambda$ such that $\mathcal{D}(\lambda) = d$. Let us assume that the point x^* is primal-optimal. This implies that for all $x \in \mathbb{R}^n$,

$$L(x^*, \lambda) \overset{\text{def of sup}}{\leq} \sup_{\phi\in\mathbb{R}^p} L(x^*, \phi) = p$$

Similarly, for all $\phi \in \mathbb{R}^p$,

$$L(x^*, \lambda) \overset{\text{def of inf}}{\geq} \inf_{x\in\mathbb{R}^n} L(x, \lambda) = \mathcal{D}(\lambda) = d$$

By Proposition 3.2.2, $p = d$, so both of the above inequalities are equalities and for all $x \in \mathbb{R}^n$ and all $\phi \in \mathbb{R}^p$,

$$L(x^*, \phi) \leq \sup_{\phi'\in\mathbb{R}^p} L(x^*, \phi') = L(x^*, \lambda) = \inf_{x'\in\mathbb{R}^n} L(x', \lambda) \leq L(x', \lambda)$$

ii) ⇒ i) Let us assume that $\exists\lambda \in \mathbb{R}^p$, such that the pair (x^*, λ) is a saddle point of the Lagrangian function L.

$$f(x^*) + \mathbb{I}_{g\le 0}(x^*) = \sup_{\phi\in\mathbb{R}^p} L(x^*, \phi) \overset{(x^*,\lambda)\text{ saddle p.}}{\le} \sup_{\phi\in\mathbb{R}^p} L(x^*, \lambda) = L(x^*, \lambda) \overset{(x^*,\lambda)\text{ saddle p.}}{\le} L(x, \lambda)$$

This is true for all $x \in \mathbb{R}^n$, so

$$f(x^*) + \mathbb{I}_{g\le 0}(x^*) \le \inf_{x\in\mathbb{R}^n} L(x, \lambda) \overset{\text{def of sup}}{\le} \sup_{\phi\in\mathbb{R}^p} \inf_{x\in\mathbb{R}^n} L(x, \phi) \overset{\text{def of inf}}{\le} \sup_{\phi\in\mathbb{R}^p} L(x', \phi) = f(x') + \mathbb{I}_{g\le 0}(x')$$

where x' is any element of $\mathbb{R}$. This shows that x^* is primal-optimal. □

The last ingredient of KKT theorem is the complementary slackness properties of λ. If (x^*, λ) is a saddle point of the Lagrangian, and if the constraints are qualified, then $g(x^*) \preceq 0$. Call $I = \{i_1, \dots, i_k\}$, $k \le p$, the set of **active constraints** at x^*, *i.e.*,

$$I = \Big\{ i \in \{1, \dots, p\} \ : \ g_i(x^*) = 0 \Big\}.$$

the indices i such that $g_i(x^*) = 0$.

Proposition 3.2.4. *Consider the convex problem (3.1.1) and assume that the constraints satisfiability condition "$\exists\bar{x} \in \operatorname{dom} f$ such that $g(\bar{x}) < 0$" is satisfied.*
The pair (x^*, λ) is a saddle point of the Lagrangian, if an only if

$$\begin{cases} g(x^*) \preceq 0 & & \textit{(admissibility)} \\ \lambda \succeq 0, \quad \langle \lambda, g(x^*) \rangle = 0, & (i) & \textit{(complementary slackness)} \\ 0 \in \partial f + \sum_{i\in I} \lambda_i \partial g_i(x^*). & (ii) & \end{cases} \tag{3.2.4}$$

Remark 3.2.1. *The condition (3.2.4) (ii) may seem complicated at first view. However, notice that, in the differentiable case, this the usual 'colinearity of gradients' condition in the KKT theorem :*

$$\nabla f(x^*) = -\sum_{i\in I} \lambda_i \nabla g_i(x^*).$$

Proof. Assume that (x^*, λ) is a saddle point of L. By definition of the Lagrangian function, $\lambda \succeq 0$. The first inequality in the saddle point property implies $\forall\phi \in \mathbb{R}_+p, L(x^*, \phi) \le L(x^*, \lambda)$, which means

$$f(x^*) + \langle \phi, g(x^*) \rangle \le f(x^*) + \langle \lambda, g(x^*) \rangle\,,$$

i.e.

$$\forall\phi \in \mathbb{R}^p_+, \quad \langle \phi - \lambda, g(x^*) \rangle \le 0.$$

Since x^* is primal optimal, and the constraints are qualified, $g(x^*) \preceq 0$. For $i \in \{1, \dots, p\}$,

- If $g_i(x) < 0$, then choosing $\phi = \begin{cases} \lambda_j & (j \ne i) \\ 0 & (j = i) \end{cases}$ yields $-\lambda_i g_i(x^*) \le 0$, whence $\lambda_i \le 0$, and finally $\lambda_i = 0$. Thus, $\lambda_i g_i(x^*) = 0$.
- If $g_i(x) = 0$, then $\lambda_i g_i(x^*) = 0$ as well.

As a consequence, $\lambda_j g_j(x^*) = 0$ for all j, and (3.2.4 (i)) follows.
Furthermore, the saddle point condition implies that x^* is a minimizer of the function $L_\lambda : x \mapsto L(x, \lambda) = f(x) + \sum_{i\in I} \lambda_i g_i(x)$. (the sum is restricted to the active set of constraint, due to (i)). From Fermat's rule,

$$0 \in \partial \left[f + \sum \lambda_i g_i \right]$$

Since $\operatorname{dom} g_i = \mathbb{R}^n$, the condition for the subdifferential calculus rule 2.5.1 is met and an easy recursion yield $0 \in \partial f(x^*) + \sum_{i\in I} \partial(\lambda_i g_i(x^*))$. As easily verified, $\partial \lambda_i g_i = \lambda_i \partial g_i$, and (3.2.4 (ii)) follows.

Conversely, assume that λ satisfies (3.2.4). By Fermat's rule, and the subdifferential calculus rule 2.5.1, condition (3.2.4) (ii) means that x^* is a minimizer of the function $h_\lambda : x \mapsto f(x) + \sum_{i\in I} \lambda_i g_i(x)$. Using the complementary slackness condition ($\lambda_i = 0$ for $i \notin I$), $h_\lambda(x) = L(x, \lambda)$, so that the second inequality in the definition of a saddle point holds:

$$\forall x, L(x^*, \lambda) \leq L(x, \lambda).$$

Furthermore, for any $\phi \succeq 0 \in \mathbb{R}^p$,

$$L(x^*, \phi) - L(x^*, \lambda) = \langle \phi, g(x^*) \rangle - \langle \lambda, g(x^*) \rangle = \langle \phi, g(x^*) \rangle \leq 0,$$

since $g(x^*) \preceq 0$. This is the second inequality in the saddle point condition, and the proof is complete. □

Definition 3.2.3. *Any vector $\lambda \in \mathbb{R}^p$ which satisfies (3.2.4) is called a* **Lagrange multiplier** *at x^* for problem (3.1.1).*

The following theorem summarizes the arguments developed in this section

Theorem 3.2.1 (KKT (Karush, Kuhn,Tucker) conditions for optimality)**.** *Assume that Slater's constraint qualification condition (Def. 3.2.1) is satisfied for the convex problem (3.1.1). Let $x^* \in \mathbb{R}^n$. The following assertions are equivalent:*

(i) x^* is primal optimal.

(ii) There exists $\lambda \in \mathbb{R}_+p$, such that (x^*, λ) is a saddle point of the Lagrangian function.

(iii) There exists a Lagrange multiplier vector λ at x^*, i.e. *a vector $\lambda \in \mathbb{R}^p$, such that the* **KKT conditions***:*

$$\begin{cases} g(x^*) \preceq 0 & \textit{(admissibility)} \\ \lambda \succeq 0, \quad \langle \lambda, g(x^*) \rangle = 0, & \textit{(complementary slackness)} \\ 0 \in \partial f + \sum_{i\in I} \lambda_i \partial g_i(x^*). & \textit{('colinearity of subgradients')} \end{cases}$$

are satisfied.

Proof. The equivalence between (ii) and (iii) is proposition 3.2.4; the one between (i) and (ii) is proposition 3.2.3. □

Exercise 3.2.1 (Examples of duals, Borwein and Lewis (2006), chap.4)**.** Compute the dual of the following problems. In other words, calculate the dual function $\mathcal{D}$ and write the problem of maximizing the latter as a convex minimization problem.

1. Linear program
$$\inf_{x \in \mathbb{R}^n} \langle c, x \rangle$$
$$\text{under constraint } Gx \preceq b$$
where $c \in \mathbb{R}^n$, $b \in \mathbb{R}^p$ and $G \in \mathbb{R}^{p \times n}$.

Hint: you should find that the dual problem is again a linear program, with equality constraints.

2. Quadratic program
$$\inf_{x \in \mathbb{R}^n} \frac{1}{2} \langle x, Cx \rangle$$
$$\text{under constraint } Gx \preceq b$$
where C is symmetric, positive, definite.

Hint : you should obtain an unconstrained quadratic problem.

- Assume in addition that the constraints are linearly independent, *i.e.* $\mathrm{rank}(G) = p$, *i.e.* $G = \begin{pmatrix} w_1^\top \\ \vdots \\ w_p^\top \end{pmatrix}$, where $(w_1, \dots, w_p)$ are linearly independent. Compute then the dual value.

Exercise 3.2.2 (dual gap)**.** Consider the two examples in exercise 3.2.1, and assume, as in example 2., that the constraints are linearly independent. Show that the duality gap is zero under this conditions.
Hint: Slater.

Chapter 4

Fenchel-Legendre transformation, Fenchel Duality

We introduce now an important tool of convex analysis, especially useful for duality approaches: the Fenchel-Legendre transform.

4.1 Fenchel-Legendre Conjugate

Definition 4.1.1. *Let $f : \mathcal{X} \to [-\infty, +\infty]$. The* **Fenchel-Legendre conjugate** *of f is the function $f^* : \mathcal{X} \to [-\infty, \infty]$, defined by*

$$f^*(\phi) = \sup_{x \in \mathcal{X}} \langle \phi, x \rangle - f(x) \,, \qquad \forall \phi \in \mathcal{X}.$$

Notice that

$$f^*(0) = - \inf_{x \in \mathcal{X}} f(x).$$

Figure 4.1 provides a graphical representation of f^*. You should get the intuition that, in the differentiable case, if the maximum is attained in the definition of f^* at point x_0, then $\phi = \nabla f(x_0)$, and $f^*(\phi) = \langle \nabla f(x_0), x_0 \rangle - f(x_0)$. This intuition will be proved correct in proposition 4.2.1.

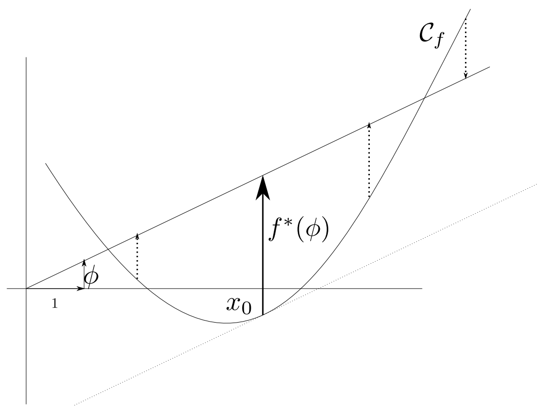

Figure 4.1: Fenchel Legendre transform of a smooth function f. The maximum positive difference between the line with slope $\tan(\phi)$ and the graph $\mathcal{C}_f$ of f is reached at x_0.

Exercise 4.1.1.
Prove the following statements.
General hint: If $h_\phi : x \mapsto \langle \phi, x\rangle - f(x)$ reaches a maximum at x^*, then $f^*(\phi) = h_\phi(x^*)$. Furthermore, h_ϕ is concave (if f is convex). If h_ϕ is differentiable, it is enough to find a zero of its gradient to obtain a maximum.
Indeed, $x \in \arg\min(-h_\phi) \Leftrightarrow 0 \in \partial(-h_\phi)$, and, if $-h_\phi$ is differentiable, $\partial(-h_\phi) = \{-\nabla h_\phi\}$.

1. If $\mathcal{X} = \mathbb{R}$ and f is a quadratic function (of the kind $f(x) = (x-a)^2 + b$), then f^* is also quadratic.

2. In $\mathbb{R}^n$, let A by a symmetric, definite positive matrix and $f(x) = \langle x, Ax\rangle$ (a quadratic function). Show that f^* is also quadratic.

3. $f : \mathcal{X} \to [-\infty, +\infty]$. Show that $f = f^* \Leftrightarrow f(x) = \frac{1}{2}\|x\|^2$.

   Hint: For the 'if' part: show first that $f(\phi) \geq \langle \phi, \phi\rangle - f(\phi)$.

   Then, show that $f(\phi) \leq \sup_x \langle \phi, x\rangle - \frac{1}{2}\|x\|^2$. Conclude.

4. $\mathcal{X} = \mathbb{R}$, for
$$f(x) = \begin{cases} 1/x & \text{if } x > 0; \\ +\infty & \text{otherwise} . \end{cases}$$
   we have,
$$f^*(\phi) = \begin{cases} -2\sqrt{-\phi} & \text{if } \phi \leq 0; \\ +\infty & \text{otherwise} . \end{cases}$$

5. $\mathcal{X} = \mathbb{R}$, if $f(x) = \exp(x)$, then
$$f^*(\phi) = \begin{cases} \phi \ln(\phi) - \phi & \text{if } \phi > 0; \\ 0 & \text{if } \phi = 0; \\ +\infty & \text{if } \phi < 0. \end{cases}$$

Notice that, if $f(x) = -\infty$ for some x, then $f^* \equiv +\infty$.
Nonetheless, under 'reasonable' conditions on f, the Legendre transform enjoys nice properties, and even f can be recovered from f^* (through the equality $f = f^{**}$, see Proposition 4.2.3).

Proposition 4.1.1 (Properties of f^*)**.**
Let $f : \mathcal{X} \to [-\infty, +\infty]$ be any function.

1. *f^* is always convex, and l.s.c.*

2. *If* $\operatorname{dom} f \neq \emptyset$, *then* $-\infty \notin f^*(\mathcal{X})$

3. *If f is convex and proper, then f^* is convex, l.s.c., proper.*

Proof.

1. Fix $x \in \mathcal{X}$ and consider the function $h_x : \phi \mapsto \langle \phi, x\rangle - f(x)$. From the definition, $f^* = \sup_{x\in\mathcal{X}} h_x$. Each h_x is affine, whence convex. Using proposition 2.1.2, f^* is also convex. Furthermore, each h_x is continuous, whence l.s.c, so that its epigraph is closed. Lemma 2.2.1 thus shows that f^* is l.s.c.

2. From the hypothesis, there is an x_0 in $\operatorname{dom} f$. Let $\phi \in \mathcal{X}$. The result is immediate:

$$f^*(\phi) \geq h_{x_0}(\phi) = f(x_0) - \langle \phi, x_0 \rangle > -\infty.$$

3. In view of points 1. and 2., it only remains to show that $f^* \not\equiv +\infty$. Let $x_0 \in \operatorname{relint}(\operatorname{dom} f)$. According to proposition 2.4.1, there exists a subgradient ϕ_0 of f at x_0. Moreover, since f is proper, $f(x_0) < \infty$. From the definition of a subgradient,

$$\forall x \in \operatorname{dom} f, \langle \phi_0, x - x_0 \rangle \leq f(x) - f(x_0).$$

Whence, for all $x \in \mathcal{X}$,

$$\langle \phi_0, x \rangle - f(x) \leq \langle \phi_0, x_0 \rangle - f(x_0),$$

thus, $\sup_x \langle \phi_0, x \rangle - f(x) \leq \langle \phi_0, x_0 \rangle - f(x_0) < +\infty$.

Therefore, $f^*(\phi_0) < +\infty$. □

4.2 Affine minorants**

Proposition 4.2.1 (Fenchel-Young)**.** *Let $f : \mathcal{X} \to [-\infty, \infty]$. For all $(x, \phi) \in \mathcal{X}^2$, the following inequality holds:*

$$f(x) + f^*(\phi) \geq \langle \phi, x \rangle ,$$

With equality if and only if $\phi \in \partial f(x)$.

Proof. The inequality is an immediate consequence of the definition of f^*. The condition for equality to hold (*i.e.*, for the converse inequality to be valid), is obtained with the equivalence

$$f(x) + f^*(\phi) \leq \langle \phi, x \rangle \;\Leftrightarrow\; \forall y,\; f(x) + \langle \phi, y \rangle - f(y) \leq \langle \phi, x \rangle \;\Leftrightarrow\; \phi \in \partial f(x) .$$

□

An **affine minorant** of a function f is any affine function $h : \mathcal{X} \to \mathbb{R}$, such that $h \leq f$ on $\mathcal{X}$. Denote $\mathcal{AM}(f)$ the set of affine minorants of function f. One key result of dual approaches is encapsulated in the next result: under regularity conditions, if the affine minorants of f are given, then f is entirely determined !

Proposition 4.2.2 (duality, episode 0)**.** *Let $f : \mathcal{X} \to (-\infty, \infty]$ a convex, l.s.c., proper function. Then f is the upper hull of its affine minorants.*

Proof. For any function f, denote E_f the upper hull of its affine minorants, $E_f = \sup_{h \in \mathcal{AM}(f)} h$. For $\phi \in \mathcal{X}$ and $b \in \mathbb{R}$, denote $h_{\phi,b}$ the affine function $x \mapsto \langle \phi, x \rangle + b$. With these notations,

$$E_f(x) = \sup\{\langle \phi, x \rangle - b \;:\; h_{\phi,b} \in \mathcal{AM}(f)\}.$$

Clearly, $E_f \leq f$.
To show the converse inequality, we proceed in two steps. First, we assume that f is non negative. The second step consists in finding a 'change of basis' under which f is replaced with non negative function.

1. Case where f is non-negative, i.e. $f(\mathcal{X}) \subset [0, \infty]$) :
Assume the existence of some $x_0 \in \mathcal{X}$, such that $t_0 = E_f(x_0) < f(x_0)$ to come up with a

contradiction. The point (x_0, t_0) does not belong to the convex closed set $\operatorname{epi} f$. The strong separation theorem 2.3.1 provides a vector $\mathbf{w} = (\phi, b) \in \mathcal{X} \times \mathbb{R}$, and scalars α, b, such that

$$\forall (x,t) \in \operatorname{epi} f, \quad \langle \phi, x \rangle + bt < \alpha < \langle \phi, x_0 \rangle + bt_0. \tag{4.2.1}$$

In particular, the inequality holds for all $x \in \operatorname{dom} f$, and for all $t \geq f(x)$. Consequently, $b \leq 0$ (as in the proof of proposition 2.4.1). Here, we cannot conclude that $b < 0$: if $f(x_0) = +\infty$, the hyperplane may be 'vertical'. However, using the non-negativity of f, if $(x,t) \in \operatorname{epi} f$, then $t \geq 0$, so that, for all $\epsilon > 0$, $(b-\epsilon)\, t \leq b\, t$. Thus, (4.2.1) implies

$$\forall (x,t) \in \operatorname{epi} f, \quad \langle \phi, x \rangle + (b-\epsilon)t < \alpha.$$

Now, $b - \epsilon < 0$ and, in particular, for $x \in \operatorname{dom} f$, and $t = f(x)$,

$$f(x) > \frac{1}{b-\epsilon}(\langle -\phi, x \rangle + \alpha) := h^\epsilon(x).$$

Thus, the function h^ϵ is an affine minorant of f. Since $t_0 \geq h^\epsilon(x_0)$ (by definition of t_0),

$$t_0 > \frac{1}{b-\epsilon}(\langle -\phi, x_0 \rangle + \alpha),$$

i.e.

$$(b-\epsilon)t_0 \leq \langle -\phi, x_0 \rangle + \alpha$$

Letting ϵ go to zero yields

$$b\, t_0 \leq -\langle \phi, x_0 \rangle + \alpha$$

which contradicts (4.2.1)

2. General case. Since f is proper, its domain is non empty. Let $x_0 \in \operatorname{relint}(\operatorname{dom} f)$. According to proposition 2.4.1, $\partial f(x_0) \neq \emptyset$. Let $\phi_0 \in \partial f(x_0)$. Using Fenchel-Young inequality (Proposition4.2.1), for all $x \in \mathcal{X}$, $\varphi(x) := f(x) + f^*(\phi_0) - \langle \phi_0, x \rangle \geq 0$. The function φ is non negative, convex, l.s.c., proper (because equality in Fenchel-Young ensures that $f^*(\phi_0) \in \mathbb{R}$). Part *1.* applies :

$$\forall x \in \mathcal{X}, \varphi(x) = \sup_{(\phi,b):h_{\phi,b} \in \mathcal{AM}(\varphi)} \langle \phi, x \rangle + b\,. \tag{4.2.2}$$

Now, for $(\phi, b) \in \mathcal{X} \times \mathbb{R}$,

$$\begin{aligned} h_{\phi,b} \in \mathcal{AM}(\varphi) &\Leftrightarrow \forall x \in \mathcal{X}, \langle \phi, x \rangle + b \leq f(x) + f^*(x_0) - \langle \phi_0, x \rangle \\ &\Leftrightarrow \forall x \in \mathcal{X}, \langle \phi + \phi_0, x \rangle + b - f^*(x_0) \leq f(x) \\ &\Leftrightarrow h_{\phi+\phi_0, b - f^*(x_0)} \in \mathcal{AM}(f). \end{aligned}$$

Thus, (4.2.2) writes as

$$\forall x \in \mathcal{X}, f(x) + f^*(\phi_0) - \langle \phi_0, x \rangle = \sup_{(\phi,b) \in \Theta(f)} \langle \phi - \phi_0, x \rangle + b + f^*(x_0)\,.$$

In other words, $x \in \mathcal{X}, f(x) = E_f(x)$. □

The announced result comes next:

Definition 4.2.1 (Fenchel Legendre biconjugate). *Let $f : \mathcal{X} \to [-\infty, \infty]$, any function. The biconjugate of f (under Fenchel-Legendre conjugation), is*

$$\begin{aligned} f^{**} :\ & \mathcal{X} \to [-\infty, \infty] \\ & x \mapsto (f^*)^*(x) = \sup_{\phi \in \mathcal{X}} \langle \phi, x \rangle - f^*(\phi). \end{aligned}$$

Proposition 4.2.3 (Involution property, Fenchel-Moreau). *If f is convex, l.s.c., proper, then $f = f^{**}$.*

Proof. Using proposition 4.2.2, it is enough to show that $f^{**}(x) = E_f(x)$

1. From Fenchel-Young inequality (Proposition4.2.1), for all $\phi \in \mathcal{X}$, the function $x \mapsto h_\phi(x) = \langle \phi, x\rangle - f^*(\phi)$ belongs to $\mathcal{AM}(f)$. Thus,
$$\mathcal{AM}^* = \{h_\phi, \phi \in \mathcal{X}\} \subset \mathcal{AM}(f),$$
so that
$$f^{**}(x) = \sup_{h \in \mathcal{AM}^*} h(x) \leq \sup_{h \in \mathcal{AM}(f)} h(x) = E_f(x).$$

2. Conversely, let $h_{\phi,b} \in \mathcal{AM}(f)$. Then, $\forall x$, $\langle \phi, x\rangle - f(x) \leq -b$, so
$$f^*(\phi) = \sup_x \langle \phi, x\rangle - f(x) \leq -b.$$
Thus,
$$\forall x, \quad \langle \phi, x\rangle - f^*(\phi) \geq \langle \phi, x\rangle + b = h(x).$$
In particular, $f^{**}(x) \geq h(x)$. Since this holds for all $h \in \mathcal{AM}(f)$, we obtain
$$f^{**}(x) \geq \sup_{h \in \mathcal{AM}(f)} h(x) = E_f(x).$$

□

One local condition to have $f(x) = f^{**}(x)$ at some point x is the following.

Proposition 4.2.4. *Let $f : \mathcal{X} \to [-\infty, \infty]$ a convex function, and let $x \in \operatorname{dom} f$.*
$$\textit{If } \partial f(x) \neq \emptyset, \textit{ then } f(x) = f^{**}(x).$$

Proof. Let $\lambda \in \partial f(x)$. This is the condition for equality in Fenchel-Young inequality (proposition 4.2.1), *i.e.*
$$f(x) + f^*(\lambda) - \langle \lambda, x\rangle = 0 \tag{4.2.3}$$
Consider the function $h_x(\phi) = f^*(\phi) - \langle \phi, x\rangle$. Equation (4.2.3) writes as
$$h_x(\lambda) = -f(x).$$
The general case in Fenchel Young writes
$$\forall \phi \in \mathcal{X}, \;\; h_x(\phi) \geq -f(x) = h_x(\lambda).$$
Thus, λ is a minimizer of h_x,
$$\lambda \in \arg\min_{\phi \in \mathcal{X}} h_x(\phi) = \arg\max_{\phi \in \mathcal{X}} (-h_x(\phi))$$
In other words,
$$f(x) = -h_x(\lambda) = \sup_\phi -h_x(\phi) = \sup_\phi \langle \phi, x\rangle - f^*(\phi) = f^{**}(x).$$

□

Exercise 4.2.1. Let $f : \mathcal{X} \to (-\infty, +\infty]$ a proper, convex, l.s.c. function. Show that
$$\partial(f^*) = (\partial f)^{-1}$$
where, for $\phi \in \mathcal{X}$, $(\partial f)^{-1}(\phi) = \{x \in \mathcal{X} : \; \phi \in \partial f(x)\}$.
Hint: Use Fenchel-Young inequality to show one inclusion, and the property $f = f^{**}$ for the other one.

4.3 Equality constraints

Theorem 4.3.1. *Let A be an affine function and f be a convex function. Let us consider the problem with equality constraints*

$$\min_{x\in\mathbb{R}^n} f(x) + \mathbb{I}_{A=0}(x),$$

the associated Lagrangian

$$L(x,\phi) = f(x) + \langle \phi, A(x)\rangle$$

and the dual problem

$$\sup_{\phi\in\mathbb{R}^p} \mathcal{D}(\phi)$$

where $\mathcal{D}(\phi) = \inf_{x\in\mathbb{R}^n} f(x) + \langle \phi, A(x)\rangle$.
If $0 \in \operatorname{relint}(A(\operatorname{dom} f))$ (constraint qualification condition), then

1. $\inf_{x\in\mathbb{R}^n} f(x) + \mathbb{I}_{A=0}(x) < +\infty$
2. $\inf_{x\in\mathbb{R}^n} f(x) + \mathbb{I}_{A=0}(x) = \sup_{\phi\in\mathbb{R}^p} \mathcal{D}(\phi)$ *(*i.e.*, the duality gap is zero).*
3. *(Dual attainment at some $\lambda \succeq 0$):*

$$\exists \lambda \in \mathbb{R}^p, \lambda \succeq 0, \textit{such that} \;\; d = \mathcal{D}(\lambda).$$

Note that if $\operatorname{dom} f = \mathbb{R}^n$, then the constraints are automatically qualified.

Proof.* The idea of the proof is to apply Propositions 2.4.1 and 4.2.4 on the value function

$$\mathcal{V}(b) = \inf_{x\in\mathbb{R}^n} f(x) + \mathbb{I}_{\{b\}}(A(x)).$$

Note that $\mathcal{V}(0)$ is the value of the primal problem. $\mathcal{V}$ is convex since it is the infimum of a jointly convex function (Proposition 2.1.3). To apply Proposition 4.2.4, we need to calculate $\mathcal{V}^*$ and $\mathcal{V}^{**}$.
For $\phi \in \mathbb{R}^p$, by definition of the Fenchel conjugate,

$$\begin{aligned}
\mathcal{V}^*(-\phi) &= \sup_{y\in\mathbb{R}^p} \langle -\phi, y\rangle - \mathcal{V}(y) \\
&= \sup_{y\in\mathbb{R}^p} \langle -\phi, y\rangle - \inf_{x\in\mathbb{R}^n} \left[f(x) + \mathbb{I}_{\{y\}}(A(x))\right] \\
&= \sup_{y\in\mathbb{R}^p} \langle -\phi, y\rangle + \sup_{x\in\mathbb{R}^n} \left[-f(x) - \mathbb{I}_{\{y\}}(A(x))\right] \\
&= \sup_{y\in\mathbb{R}^p} \sup_{x\in\mathbb{R}^n} \langle -\phi, y\rangle - f(x) - \mathbb{I}_{\{y\}}(A(x)) \\
&= \sup_{x\in\mathbb{R}^n} \left[\sup_{y\in\mathbb{R}^p} \underbrace{\langle -\phi, y\rangle - \mathbb{I}_{\{y\}}(A(x))}_{\varphi_x(y)} \right] - f(x).
\end{aligned} \tag{4.3.1}$$

For a fixed $x \in \operatorname{dom} f$, consider the function $\varphi_x : y \mapsto \langle -\phi, y\rangle - \mathbb{I}_{\{y\}}(A(x))$. As

$$\varphi(y) = \begin{cases} -\infty & \text{if } y \neq A(x) \\ \langle -\phi, A(x)\rangle & \text{otherwise} \end{cases}$$

Thus, (4.3.1) becomes

$$\begin{aligned}\mathcal{V}^*(-\phi) &= \sup_{x\in\mathbb{R}^n} \langle -\phi, A(x)\rangle - f(x),\\ &= -\inf_{x\in\mathbb{R}^n} \underbrace{f(x) + \langle \phi, A(x)\rangle}_{L(x,\phi)}\\ &= -\mathcal{D}(\phi)\end{aligned}$$

From that, we deduce

$$\begin{aligned}\mathcal{V}^{**}(0) &= \sup_{\phi\in\mathbb{R}^p} -\mathcal{V}^*(\phi) = \sup_{\phi\in\mathbb{R}^p} -\mathcal{V}^*(-\phi) \quad \text{(by symmetry of } \mathbb{R}^p)\\ &= \sup_{\phi\in\mathbb{R}^p} \mathcal{D}(\phi)\end{aligned}$$

Hence, $\mathcal{V}^{**}(0)$ is the value of the dual problem.
Now remark that $\operatorname{dom}\mathcal{V} = \{b \in \mathbb{R}^p \ : \ \exists x \in \operatorname{dom} f, A(x) = b\} = A(\operatorname{dom} f)$. So the constraint qualification condition $0 \in \operatorname{relint}(A(\operatorname{dom} f))$ is equivalent to $0 \in \operatorname{relint}(\operatorname{dom}\mathcal{V})$ and we can apply Proposition 2.4.1: $\partial\mathcal{V}(0) \neq \emptyset$.
By Proposition 4.2.4, this implies that $\mathcal{V}(0) = \mathcal{V}^{**}(0)$. Said otherwise, $\inf_{x\in\mathbb{R}^n} f(x) + \mathbb{I}_{A=0}(x) = \sup_{\phi\in\mathbb{R}^p} \mathcal{D}(\phi)$.
To show the dual attainment, we take $\lambda \in \partial\mathcal{V}(0) \neq \emptyset$. Equality in Fenchel-Young (Proposition 4.2.1) writes: $\mathcal{V}(0) + \mathcal{V}^*(\lambda) = \langle \lambda, 0\rangle = 0$. Thus, we have

$$\begin{aligned}\mathcal{V}(0) &= -\mathcal{V}^*(\lambda)\\ &= \mathcal{D}(-\lambda)\\ &\leq \sup_{\phi} \mathcal{D}(\phi) = \mathcal{V}^{**}(0) = \mathcal{V}(0) = \mathcal{D}(-\lambda)\end{aligned}$$

□

Exercise 4.3.1. The goal of this exercise is to study the following semi-definite program where the variables are semi-definite matrices (the set of semi-definite matrices is denoted S_+^n)

$$\begin{aligned}&\inf_{x\in\mathbb{R}^{3\times 3}} X_{3,3} + \mathbb{I}_{S_+^3}(X)\\ \text{under the constraints: } &X_{1,2} + X_{2,1} + X_{3,3} = 1\\ &X_{2,2} = 0\end{aligned}$$

We will denote $E_{3,3} = \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$, $f(x) = \langle E_{3,3}, X\rangle + \mathbb{I}_{S_+^3}(X)$, $G(X) = [X_{1,2} + X_{2,1} + X_{3,3}; X_{2,2}]$, $e_1 = [1; 0]$ and $A(X) = G(X) - e_1$.

1. Give an optimal solution to this problem.
   Hint: determine the set of feasible points.
2. Compute the dual of this problem and solve it. What do you observe?
3. What does $0 \in A(\operatorname{dom} f)$ mean for the feasibility of the optimization problem?
4. Show that $[0; \epsilon] \in A(\operatorname{dom} f)$ if and only if $\epsilon \geq 0$. Deduce that the constraints are not qualified.

4.4 Fenchel duality

Many Machine Learning problems take the form

$$\inf_{x\in\mathcal{X}} f(x) + g(M\,x). \tag{4.4.1}$$

where $f : \mathcal{X} \to [-\infty, \infty]$ and $g : \mathcal{Y} :\to [-\infty, \infty]$ are two convex functions and $M : \mathcal{X} \to \mathcal{Y}$ is a linear operator.
Two examples

- The Lasso problem:
$$\min_{x\in\mathbb{R}^n} \frac{1}{2}\|Zx - Y\|_2^2 + \lambda\|x\|_1$$
$M = Z$ is the data matrix, $g(z) = \frac{1}{2}\|z - Y\|_2^2$ and $f(x) = \lambda\|x\|_1$.

- The support vector machine problem:
$$\min_{x\in\mathbb{R}^n, x_0\in\mathbb{R}} C\sum_{i=1}^{p} \max(0, 1 - Y_i((Zx)_i + x_0)) + \frac{1}{2}\|x\|_2^2$$
$M = [Z, e]$, $g(z) = C\sum_{i=1}^{p} \max(0, 1 - Y_i z_i)$ and $f(x, x_0) = \frac{1}{2}\|x\|^2$.

Deriving the dual of such problems is of great algorithmic interest. The following theorem deals exactly with that.

Theorem 4.4.1 (Fenchel-Rockafellar)**.** *If* $0 \in \operatorname{relint}(\operatorname{dom} g - M \ \operatorname{dom} f)$*, then*

$$\inf_{x\in\mathcal{X}} \Big(f(x) + g(M\,x)\Big) = -\inf_{\phi\in\mathcal{Y}} \Big(f^*(M^*\phi) + g^*(-\phi)\Big)\,.$$

Besides, the dual value is attained as soon as it is finite.

Proof. Problem (4.4.1) is equivalent to

$$\min_{x\in\mathbb{R}^n, y\in\mathbb{R}^p} f(x) + g(y) + \mathbb{I}_{\{0\}}(y - Mx).$$

We shall apply Theorem 4.3.1 to this problem with equality constraints. The Lagrangian is

$$L(x, y, \phi) = f(x) + g(y) + \langle\phi, y - Mx\rangle.$$

Hence, the dual function is

$$\mathcal{D}(\phi) = \inf_{x,y} L(x, y, \phi) = \inf_{x} f(x) + \langle\phi, Mx\rangle + \inf_{y} g(y) - \langle\phi, y\rangle = -f^*(M^*\phi) - g^*(-\phi)$$

The constraint qualification condition "$0 \in \operatorname{relint}(A(\operatorname{dom} f))$" exactly transcripts into $0 \in \operatorname{relint}(\operatorname{dom} g - M \ \operatorname{dom} f)$ since we are using the linear operator $[-M, I]$. □

Exercise 4.4.1. Find the dual of the Lasso problem.

Exercise 4.4.2. Find the dual of the Support Vector Machine problem.

4.5 Examples, Exercises and Problems

In addition to the following exercises, a large number of feasible and instructive exercises can be found in Boyd and Vandenberghe (2009), chapter 5, pp 273-287.

Exercise 4.5.1 (Gaussian Channel, Water filling.)**.** In signal processing, a *Gaussian channel* refers to a transmitter-receiver framework with Gaussian noise: the transmitter sends an information X (real valued), the receiver observes $Y = X + \epsilon$, where ϵ is a noise.

A Channel is defined by the joint distribution of (X, Y). If it is Gaussian, the channel is called *Gaussian.* In other words, if X and ϵ are Gaussian, we have a Gaussian channel.

Say the transmitter wants to send a word of size p to the receiver. He does so by encoding each possible word w of size p by a a certain vector of size n, $\mathbf{x}_n^w = (x_1^w, \dots, x_n^w)$. To stick with the Gaussian channel setting, we assume that the x_i^w's are chosen as $i.i.d.$ replicates of a Gaussian, centered random variable, with variance x.
The receiver knows the code (the dictionary of all 2^p possible $\mathbf{x}_n^w$'s) and he observes $\mathbf{y}_n = \mathbf{x}_n^w + \boldsymbol{\varepsilon}$, where $\boldsymbol{\varepsilon} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I}_n))$. We wants to recover w.
The *capacity* of the channel, in information theory, is (roughly speaking) the maximum ratio $C = n/p$, such that it is possible (when n and p tend to ∞ while $n/p \equiv C$), to recover a word w of size p using a code $\mathbf{x}_n^w$ of length n.

For a Gaussian Channel , $C = \log(1 + x/\sigma^2)$. (x/σ^2 is the ratio signal/noise). For n Gaussian channels in parallel, with $\alpha_i = 1/\sigma_i^2$, then

$$C = \sum_{i=1}^{n} \log(1 + \alpha_i x_i).$$

The variance x_i represents a *power* affected to channel i. The aim of the transmitter is to maximize C under a *total power constraint*: $\sum_{i=1}^n x_i \le P$. In other words, the problem is

$$\max_{x \in \mathbb{R}^n} \sum_{i=1}^{n} \log(1 + \alpha_i x_i) \quad \text{under constraints: } \forall i, x_i \ge 0, \quad \sum_{i=1}^{n} x_i \le P. \tag{4.5.1}$$

1. Write problem (4.5.1) as a minimization problem under constraint $g(x) \preceq 0$. Show that this is a convex problem (objective and constraints both convex).

2. Show that the constraints are qualified. (hint: Slater).

3. Write the Lagrangian function

4. Using the KKT theorem, show that a primal optimal x^* exists and satisfies:

   - $\exists K > 0$ such that $x_i = \max(0, K - 1/\alpha_i)$.
   - K is given by
   $$\sum_{i=1}^{n} \max(K - 1/\alpha_i, 0) = P$$

5. Justify the expression *water filling*

Exercise 4.5.2 (Max-entropy)**.** Let $p = (p_1, \dots, p_n)$, $p_i > 0$, $\sum_i p_i = 1$ a probability distribution over a finite set. If $x = (x_1, \dots, x_n)$ is another probability distribution ($x_i \geq 0$), an if we use the convention $0 \log 0 = 0$, the entropy of x with respect to p is

$$H_p(x) = -\sum_{i=1}^{n} x_i \log \frac{x_i}{p_i}.$$

To deal with the case $x_i < 0$, introduce the function $\psi : \mathbb{R} \to (-\infty, \infty]$:

$$\psi(u) = \begin{cases} u \log(u) & \text{if } u > 0 \\ 0 & \text{if } u = 0 \\ +\infty & \text{otherwise .} \end{cases}$$

If $g : \mathbb{R}^n \to \mathbb{R}^p$, the general formulation of the max-entropy problem under constraint $g(x) \preceq 0$ is

$$\begin{aligned} \texttt{maximize}_{\text{ over } \mathbb{R}^n} \quad & \sum_i \big(-\psi(x_i) + x_i \log(p_i) \big) \\ \text{under constraints } & \sum x_i = 1; g(x) \preceq 0. \end{aligned}$$

In terms of minimization, the problem writes

$$\inf_{x \in \mathbb{R}^n} \sum_{i=1}^{n} \psi(x_i) - \langle x, c \rangle + \mathbb{I}_{\langle \mathbf{1}_n, \cdot \rangle = 1}(x) + \mathbb{I}_{g \preceq 0}(x). \tag{4.5.2}$$

with $c = \log(p) = (\log(p_1), \dots, \log(p_n))$ and $\mathbf{1}_n = (1, \dots, 1)$ (the vector of size n which coordinates are equal to 1).

A: preliminary questions

1. Show that
$$\partial \mathbb{I}_{\langle \mathbf{1}_n, \cdot \rangle}(x) = \begin{cases} \{\lambda_0 \mathbf{1}_n \ : \ \lambda_0 \in \mathbb{R}\} := \mathbb{R}\mathbf{1}_n & \text{if } \sum_i x_i = 1 \\ \emptyset & \text{otherwise.} \end{cases}$$

2. Show that ψ is convex
hint: compute first the Fenchel conjugate of the function exp, then use proposition 4.1.1.

   Compute $\partial\psi(u)$ for $u \in \mathbb{R}$.

3. Show that
$$\partial(\sum_i \psi(x_i)) = \begin{cases} \sum_i (\log(x_i) + 1)\, \mathbf{e}_i & \text{if } x \succ 0 \\ \emptyset & \text{otherwise,} \end{cases}$$
where $(\mathbf{e}_i, \dots, \mathbf{e}_n)$ is the canonical basis of $\mathbb{R}^n$.

4. Check that, for any set A, $A + \emptyset = \emptyset$.

5. Consider the unconstrained optimization problem, (4.5.2) where the term $\mathbb{I}_{g(x) \preceq 0}$ has been removed. Show that there exists a unique primal optimal solution, which is $x^* = p$.

   Hint: Do not use Lagrange duality, apply Fermat's rule (section 2.6) instead. Then, check that the conditions for subdifferential calculus rules (proposition 2.5.1) apply.

B: Linear inequality constraints In the sequel, we assume that the constraints are linear, independent, and independent from $\mathbf{1}_n$, *i.e.*: $g(x) = Gx - b$, where $b \in \mathbb{R}^p$, and G is a $p \times n$ matrix,

$$G = \begin{pmatrix} (\mathbf{w}^1)^\top \\ \vdots \\ (\mathbf{w}^p)^\top \end{pmatrix},$$

where $\mathbf{w}^j \in \mathbb{R}^n$, and the vectors $(\mathbf{w}^1, \dots, \mathbf{w}^p, \mathbf{1}_n)$ are linearly independent. We also assume the existence of some point $\hat{x} \in \mathbb{R}^n$, such that

$$\forall i, \hat{x}_i > 0, \ \sum_i \hat{x}_i = 1, \ G\hat{x} = b. \tag{4.5.3}$$

1. Show that the constraints are qualified.

   Hint: Show that $0 \in \mathrm{int}(G(\Sigma_n) - \{b\} + \mathbb{R}^p_+)$, where $\Sigma_n = \{x \in \mathbb{R}^n \ : \ x \succeq 0, \sum_i x_i = 1\}$ is the n-dimensional simplex. In other words, show that for all $y \in \mathbb{R}^p$ close enough to 0, there is some small $\bar{u} \in \mathbb{R}^n$, such that $x = \hat{x} + \bar{u}$ is admissible for problem (4.5.2), and $G\,x \leq b + y$.
   To do so, exhibit some $u \in \mathbb{R}^n$ such that $Gu = -\mathbf{1}_p$ and $\sum u_i = 0$ (why does it exist ?) Pick t such that $\hat{x} + tu > 0$. Finally, consider the 'threshold' $Y = -t\mathbf{1}_p \prec 0$ and show that, if $y \succ Y$, then $G(\hat{x} + tu) \leq b + y$. Conclude.

2. Using the KKT conditions, show that any primal optimal point x^* must satisfy:
   $\exists Z > 0, \exists \lambda \in \mathbb{R}_+p$:

   $$x^*_i = \frac{1}{Z}\, p_i \ \exp\big[-\sum_{j=1}^p \lambda_j \mathbf{w}^j_i\big] \quad (i \in \{1, \dots, n\})$$

   (this is a Gibbs-type distribution).

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