Machine learning for multivariate and functional anomaly detection: orderings and data depth

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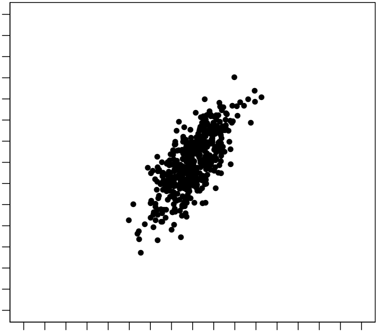
A real task

Regard two measurements during a test in a production process:



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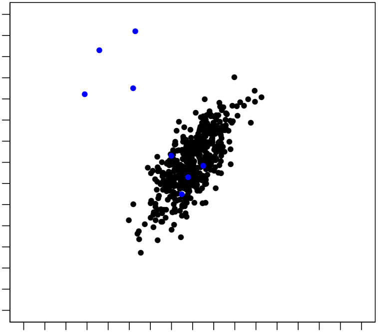

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- Whether new observations are **normal** data or **anomalies**?

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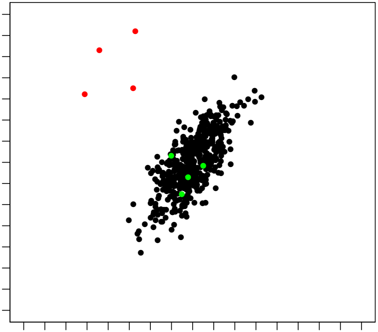

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Multivariate framework

- A training data set:

$$\boldsymbol{X} = \{\boldsymbol{x}_1, ..., \boldsymbol{x}_n\} \subset \mathbb{R}^d$$

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- Typical example: a table from a data base, with lines being observations (=individuals, items,...).
- Construct a decision function:

$$\mathbb{R}^d \to \{-1, +1\} : \boldsymbol{x} \mapsto g(\boldsymbol{x}),$$

  which attributes to any (possible) $\boldsymbol{x} \in \mathbb{R}^d$ a label whether it is an anomaly (*e.g.*, $+1$) or a normal observation (*e.g.*, -1).

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which attributes to any (possible) $\boldsymbol{x} \in \mathbb{R}^d$ a label whether it is an anomaly (*e.g.*, $+1$) or a normal observation (*e.g.*, -1).
- It is more useful to provide an ordering on $\mathbb{R}^d$:

$$\mathbb{R}^d \to \mathbb{R} : \boldsymbol{x} \mapsto g(\boldsymbol{x}),$$

such that abnormal observations obtain higher anomaly score.

Practical session

Notebooks:

- `anomdet_simulation1.Rmd`,
- `anomdet_hurricanes.Rmd`,
- `anomdet_brainimaging.Rmd`,
- `anomdet_cars.ipynb`,
- `anomdet_airbus.ipynb`.

Data sets:

- `carsanom.csv`: Data set on anomaly detection for cars.
- `airbus_data.csv`: Data set from Airbus.
- `hurdat2-1851-2019-052520.txt`: Historical hurricane data.
- `101_1_dwi_fa.nii`: Anatomical brain volume data.
- `101_1_dwi.voxelcoordsL.txt`: Left brain fiber's bundle.
- `101_1_dwi.voxelcoordsR.txt`: Right brain fiber's bundle.

Supplementary scripts:

- `depth_routines.py`: Routines for data depth calculation.
- `FIF.py`: Implementation of the functional isolation forest.
- `depth_routines.R`: Routines for curves' parametrization.
- `DTI.R`: Routines for input of brain imaging data.

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One-class support vector machines (Schölkopf, Platt, Shawe-Taylor, Smola, Williamson; 1999)

Generalized portrait:

- The method of the **generalized portrait** was introduced by Vapnik & Lerner (1963) and Vapnik & Chervonenkis (1974).

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Generalized portrait:

- The method of the **generalized portrait** was introduced by Vapnik & Lerner (1963) and Vapnik & Chervonenkis (1974).
- Generalized portrait is the vector:

$$\psi = \frac{\boldsymbol{\varphi}}{\min_{\boldsymbol{x} \in \boldsymbol{X}} \langle \boldsymbol{x}, \boldsymbol{\varphi} \rangle} \quad \text{with } \boldsymbol{\varphi} \text{ from} \quad \max_{\|\boldsymbol{\varphi}\|=1} \min_{\boldsymbol{x} \in \boldsymbol{X}} \langle \boldsymbol{x}, \boldsymbol{\varphi} \rangle .$$

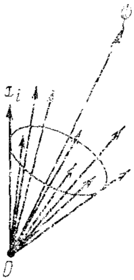

Рис. 24.

One-class support vector machines (Schölkopf, Platt, Shawe-Taylor, Smola, Williamson; 1999)

Kernel trick (Boser, Guyon, Vapnik; 1992):

- Let Φ be a feature map: $\mathbb{R}^d \mapsto \mathcal{H}$.

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- Let Φ be a feature map: $\mathbb{R}^d \mapsto \mathcal{H}$.
- Due to the kernel trick, the dot product in the image of φ can be computed by evaluation of a kernel K:

$$K(\boldsymbol{x}_i, \boldsymbol{x}_j) = \langle \Phi(\boldsymbol{x}_i), \Phi(\boldsymbol{x}_j) \rangle .$$

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- Example: Gaussian kernel

$$K(\boldsymbol{x}_i, \boldsymbol{x}_j) = e^{\gamma \|\boldsymbol{x}_i, \boldsymbol{x}_j\|}$$

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Soft margin (Cortes, Vapnik; 1995):

- Allow for a portion of points from $\boldsymbol{X}$ to be beyond the margin, label points far from the origin by "1", those close by "-1".

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- Controlled by a parameter $\nu \in (0, 1)$ (Schölkopf, Platt, Shawe-Taylor, Smola, Williamson; 1999).

One-class support vector machines (Schölkopf, Platt, Shawe-Taylor, Smola, Williamson; 1999)

Idea 1: Separate points from the origin.

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This can be formulated as a quadratic programming problem

$$\min_{\boldsymbol{\psi}\in\mathcal{H},\boldsymbol{\xi}\in\mathbb{R}^n,\rho\in\mathbb{R}} \quad \frac{1}{2}\|\boldsymbol{\psi}\|^2 + \frac{1}{\nu n}\sum_{i=1}^{n}\xi_i - \rho$$

$$\text{subject to} \quad \langle \boldsymbol{\xi}, \Phi(\boldsymbol{x}_i)\rangle \geq \rho - \xi_i\,,\ \xi_i \geq 0 \text{ for } i = 1, \dots, n\,,$$

with $\boldsymbol{\xi} = (\xi_1, \dots, \xi_n)^\top$.

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The solution $(\boldsymbol{\psi}^*, \boldsymbol{\xi}^*, \rho^*)$ yields the following decision function:

$$g_{OCSVM}(\boldsymbol{x}) = \operatorname{sgn}(\langle \boldsymbol{\xi}^*, \Phi(\boldsymbol{x})\rangle - \rho^*)\,.$$

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One can reformulate the optimization problem to employ the kernel trick.

One-class support vector machines (Schölkopf *et al.*, 1999)

In dual formulation, using the Lagrangian, one can restate the optimization problem as follows:

$$\min_{\boldsymbol{\alpha}} \quad \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\alpha_j K(\boldsymbol{x}_i, \boldsymbol{x}_j)$$

$$\text{subject to} \quad 0 \leq \alpha_i \leq \frac{1}{\nu n} \text{ for } i = 1, ..., n, \ \sum_{i=1}^{n}\alpha_i = 1\,,$$

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with $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)^\top$.

The decision function is then:

$$g_{OCSVM}(\boldsymbol{x}) = \operatorname{sgn}\left(\sum_{i=1}^{n}\alpha_i K(\boldsymbol{x}_i, \boldsymbol{x}) - \rho\right),$$

where ρ can be recovered from any $\boldsymbol{x}_j$ such that $0 < \alpha_j < \frac{1}{\nu n}$:

$$\rho = \langle \boldsymbol{\psi}, \Phi(\boldsymbol{x}_j)\rangle = \sum_{i=1}^{n}\alpha_i K(\boldsymbol{x}_i, \boldsymbol{x}_j)\,.$$

One-class support vector machines (Schölkopf *et al.*, 1999)

Idea 2: Put points into a small ball.

$$\min_{R \in \mathbb{R}, \boldsymbol{\xi} \in \mathbb{R}^n, \boldsymbol{c} \in \mathcal{H},} \quad R^2 + \frac{1}{\nu n} \sum_{i=1}^{n} \xi_i$$

$$\text{subject to} \quad \|\Phi(\boldsymbol{x}_i) - \boldsymbol{c}\| \leq R^2 + \xi_i, \ \xi_i \geq 0 \text{ for } i = 1, ..., n\,.$$

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This leads to the dual:

$$\min_{\boldsymbol{\alpha}} \quad \sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\alpha_j K(\boldsymbol{x}_i, \boldsymbol{x}_j) - \sum_{i=1}^{n}\alpha_i K(\boldsymbol{x}_i, \boldsymbol{x}_i)$$

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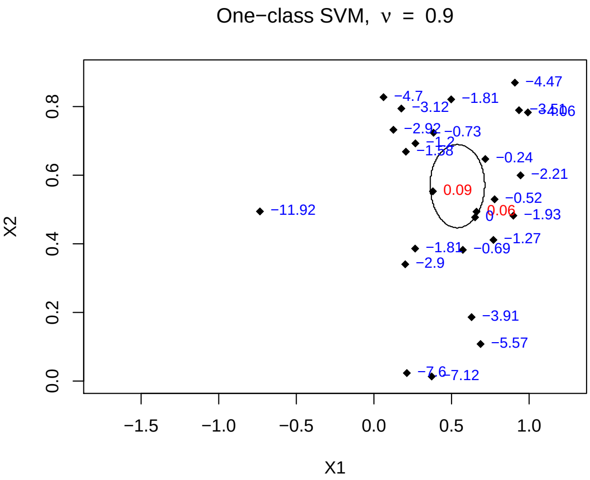
which leads to the decision function:

$$g_{OCSVM}(\boldsymbol{x}) = \left(R^2 - \sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\alpha_j K(\boldsymbol{x}_i, \boldsymbol{x}_j) + 2\sum_{i=1}^{n}\alpha_i K(\boldsymbol{x}_i, \boldsymbol{x}) - K(\boldsymbol{x}, \boldsymbol{x})\right),$$

with $R^2 = \sum_{i,j}\alpha_i\alpha_j K(\boldsymbol{x}_i, \boldsymbol{x}_j) - 2\sum_i \alpha_i K(\boldsymbol{x}_i, \boldsymbol{x}_k) + K(\boldsymbol{x}_k, \boldsymbol{x}_k)$ for any $\boldsymbol{x}_k$ such that $0 < \alpha_k < 1/(\nu n)$.

One-class support vector machines (Schölkopf, Platt, Shawe-Taylor, Smola, Williamson; 1999)

Illustration: Case 1



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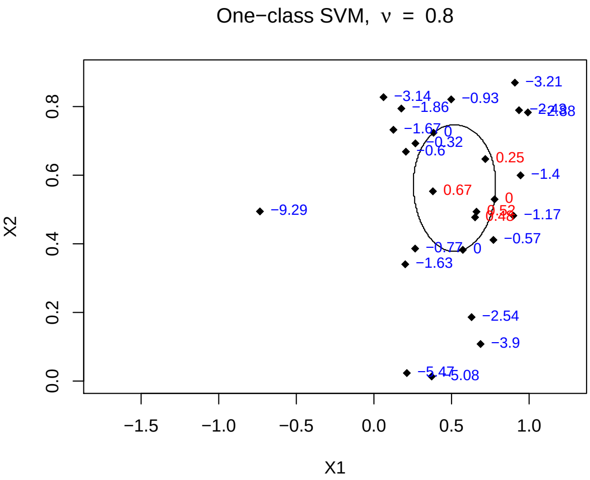

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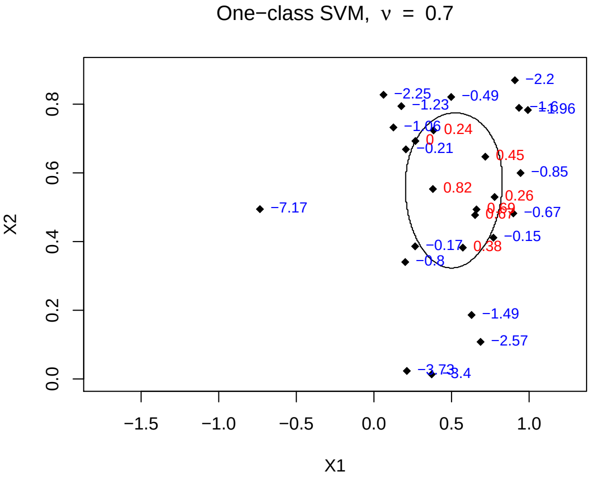

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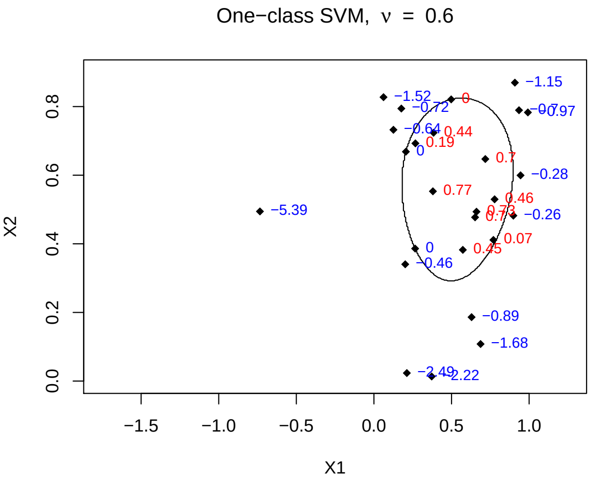

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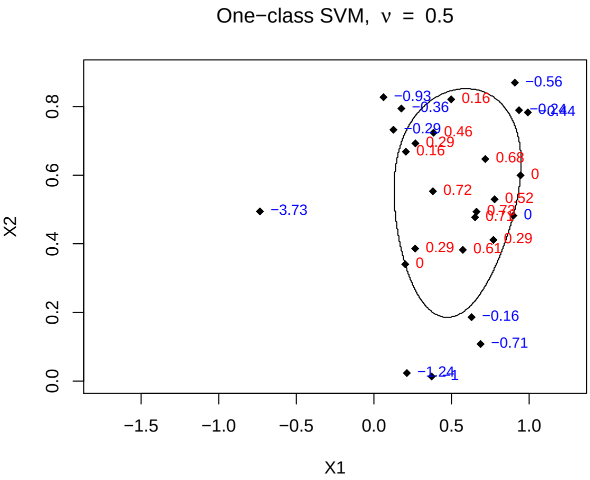

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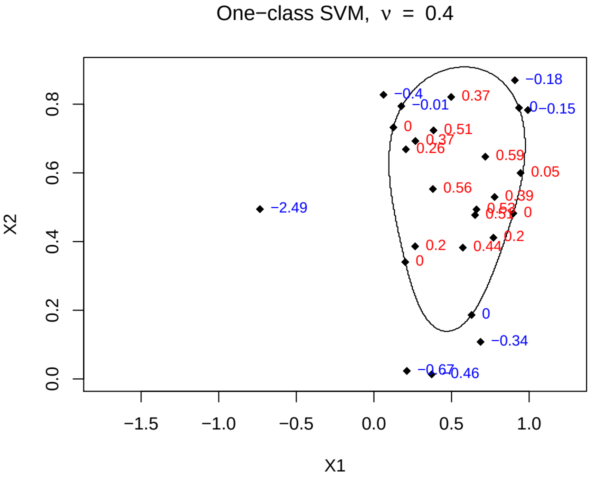

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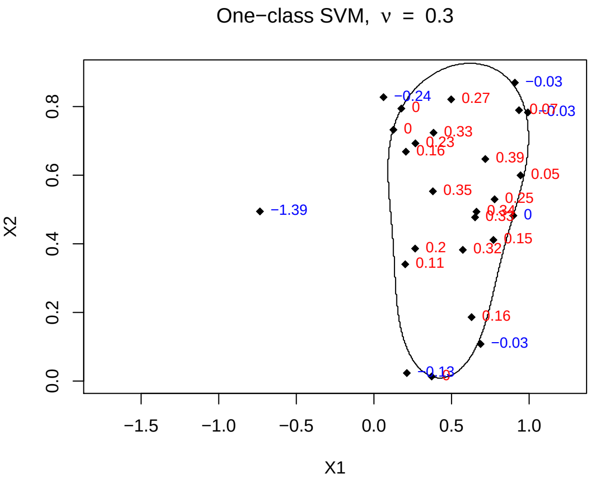

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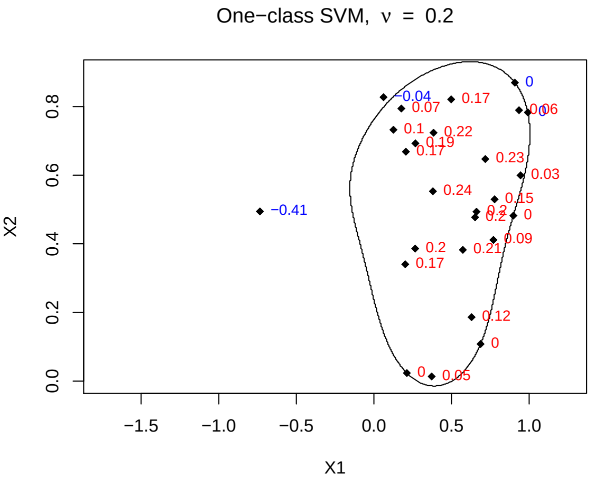

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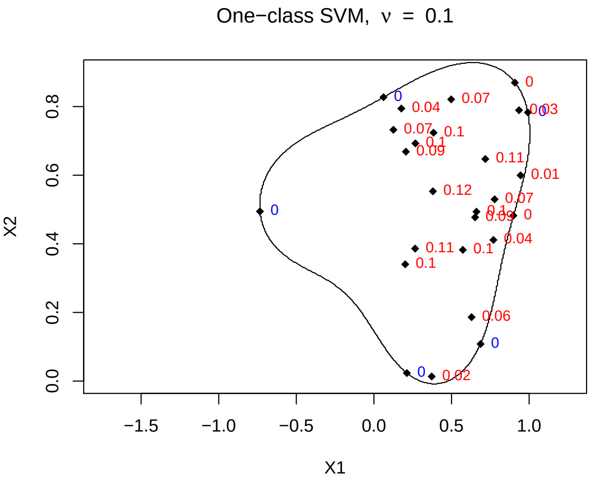

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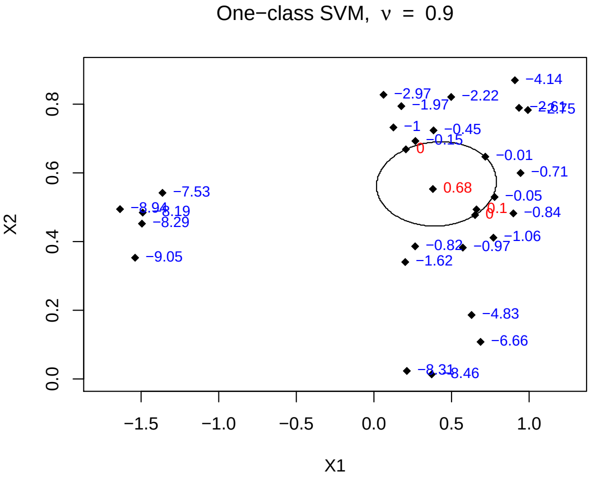

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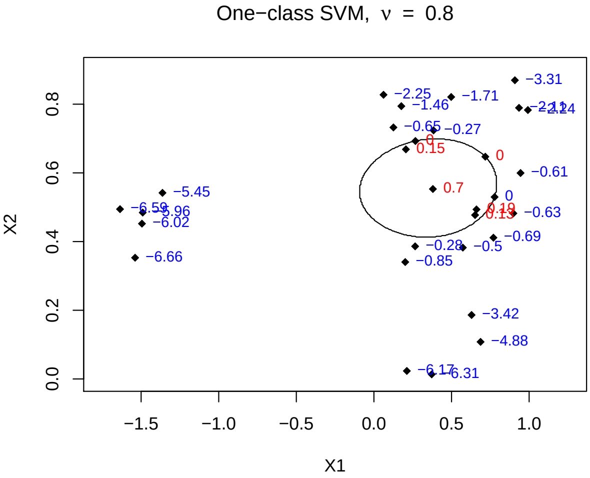

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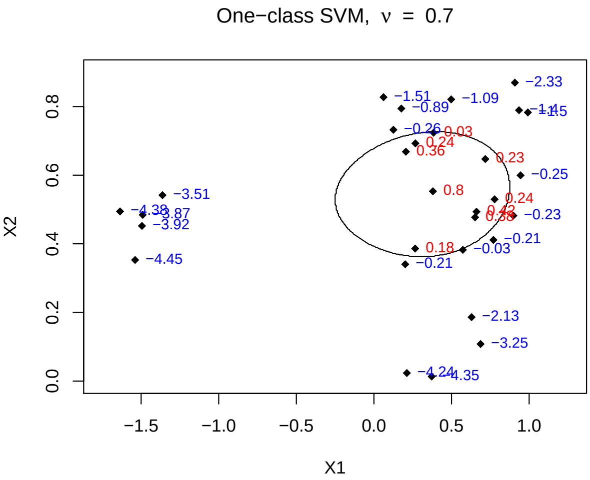

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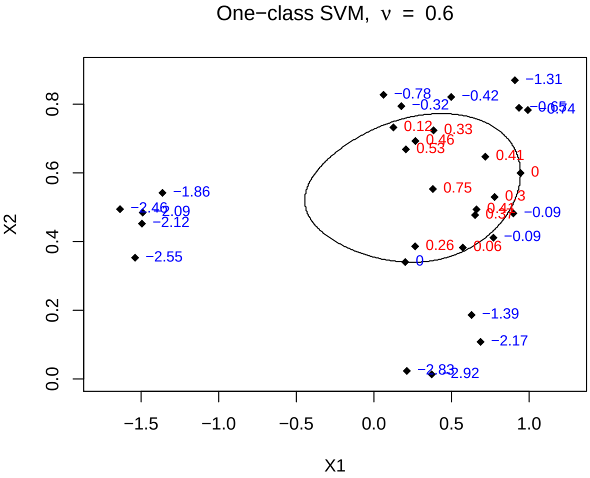

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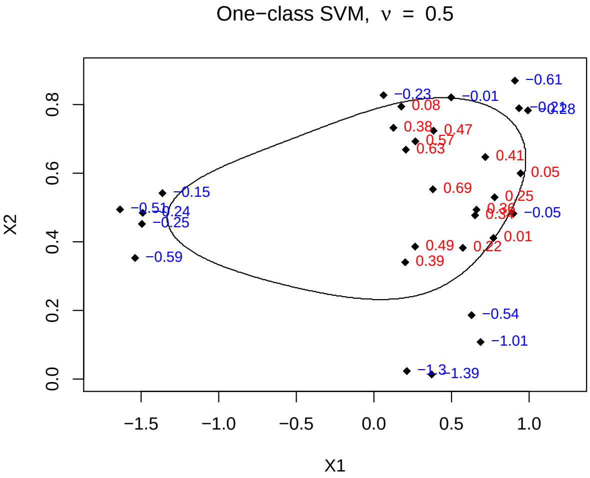

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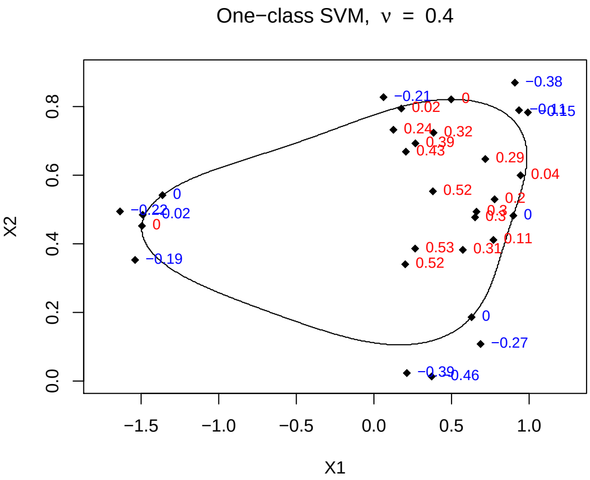

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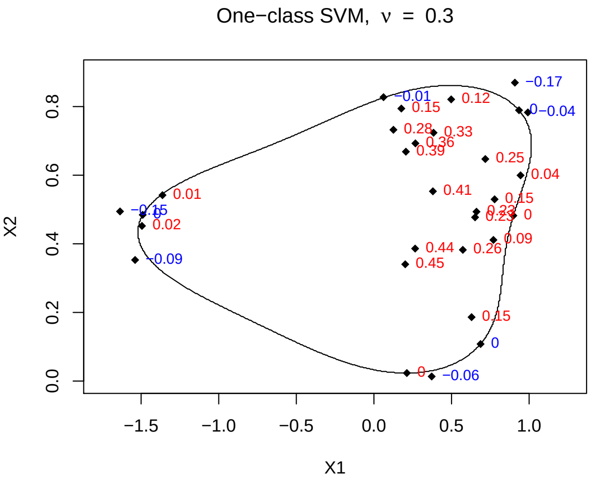

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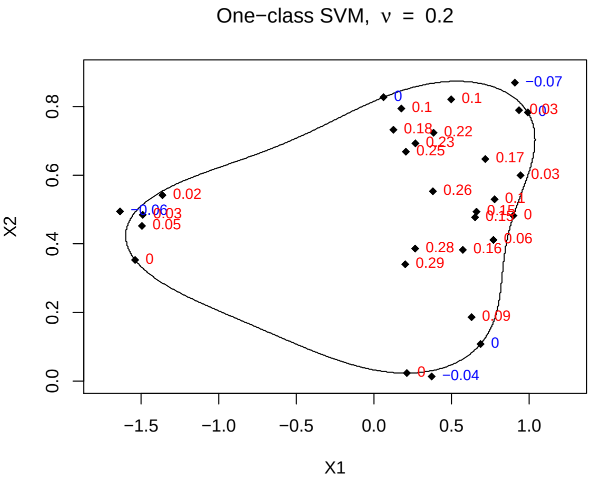

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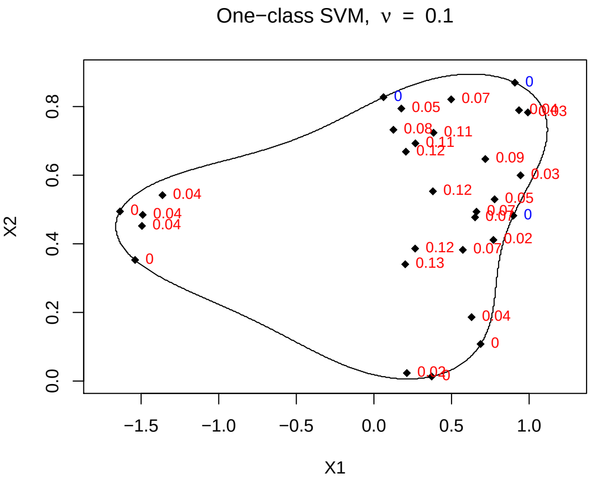

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Local outlier factor (Breunig, Kriegel, Ng, Sander; 2000)

k-**distance** of a point $\boldsymbol{x}$:

For any integer $k > 0$, the k-distance of point $\boldsymbol{x}$, denoted as $k\text{-}dist(\boldsymbol{x})$, is defined as the distance $d(\boldsymbol{x}, \boldsymbol{o})$ between $\boldsymbol{x}$ and a point $\boldsymbol{o} \in \boldsymbol{X}$ such that:

- for at least k points $\boldsymbol{o}' \in \boldsymbol{X} \setminus \{\boldsymbol{x}\}$ it holds that $d(\boldsymbol{x}, \boldsymbol{o}') \leq d(\boldsymbol{x}, \boldsymbol{o})$, and
- for at most $k - 1$ points $\boldsymbol{o}' \in \boldsymbol{X} \setminus \{\boldsymbol{x}\}$ it holds that $d(\boldsymbol{x}, \boldsymbol{o}') < d(\boldsymbol{x}, \boldsymbol{o})$.

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(=Distance from $\boldsymbol{x}$ to its kth neighbor.)

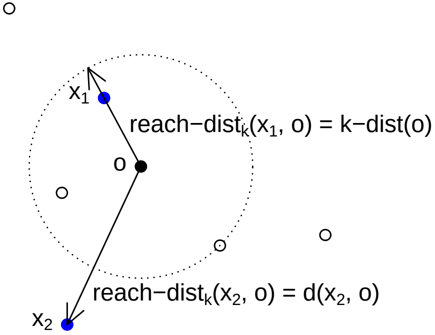
k-**neighborhood** of a point $\boldsymbol{x}$:
Given the $k\text{-}dist(\boldsymbol{x})$, the k-neighborhood of $\boldsymbol{x}$, denoted $N_k(\boldsymbol{x})$, contains every point whose distance from $\boldsymbol{x}$ is not greater than the $k\text{-}dist(\boldsymbol{x})$, *i.e.*:

$$N_k(\boldsymbol{x}) = \left\{ \boldsymbol{q} \in \boldsymbol{X} \setminus \{\boldsymbol{x}\} \,|\, d(\boldsymbol{x}, \boldsymbol{q}) \leq k\text{-}dist(\boldsymbol{x}) \right\}.$$

Local outlier factor (Breunig, Kriegel, Ng, Sander; 2000)

Reachability distance of order k of point $\boldsymbol{x}$ w.r.t. point $\boldsymbol{o}$:
For $k \in \mathbb{N}$, the reachability distance of order k of point $\boldsymbol{x}$ with respect to point $\boldsymbol{o}$ is defined as:

$$\textit{reach-dist}_k(\boldsymbol{x}, \boldsymbol{o}) = \max\{\textit{k-dist}(\boldsymbol{o}), d(\boldsymbol{x}, \boldsymbol{o})\}\,.$$



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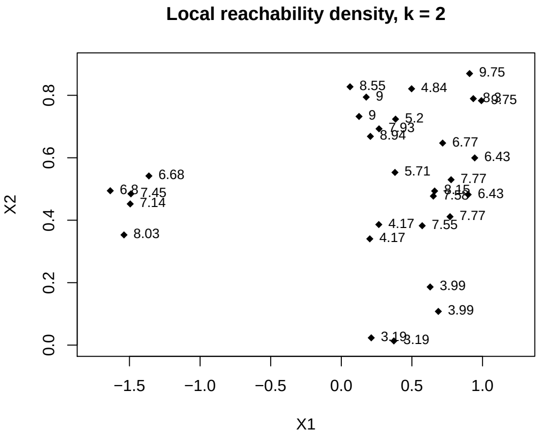

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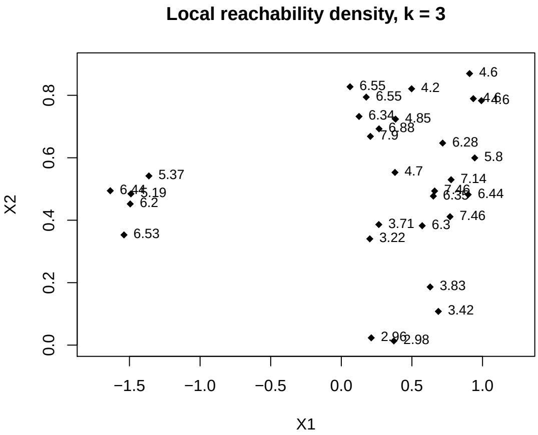

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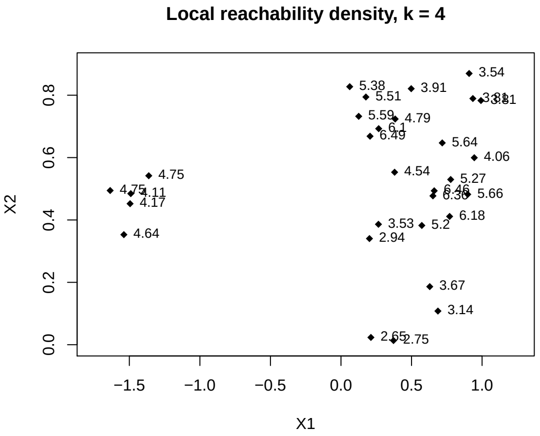

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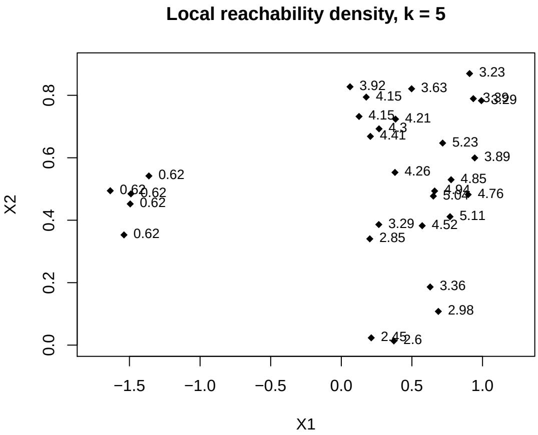

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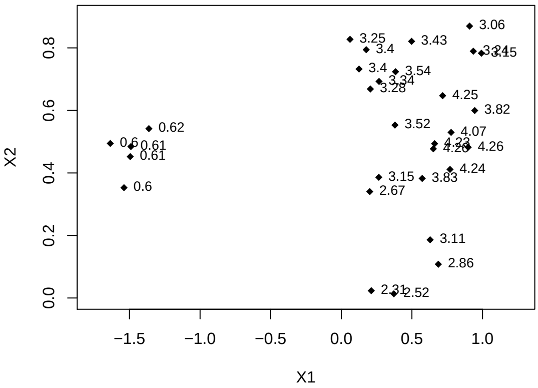
Local reachability density, k = 6



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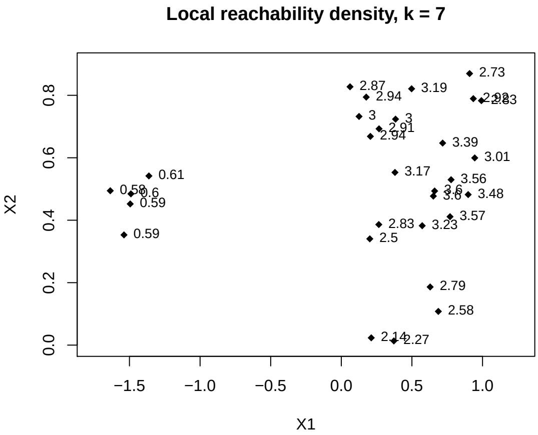

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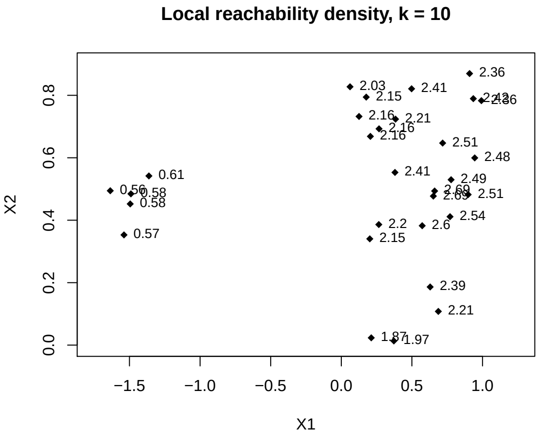

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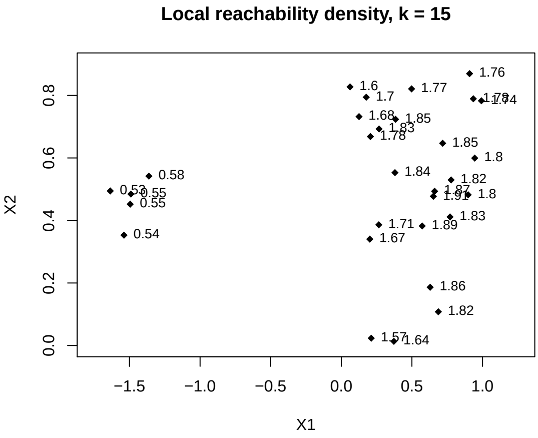

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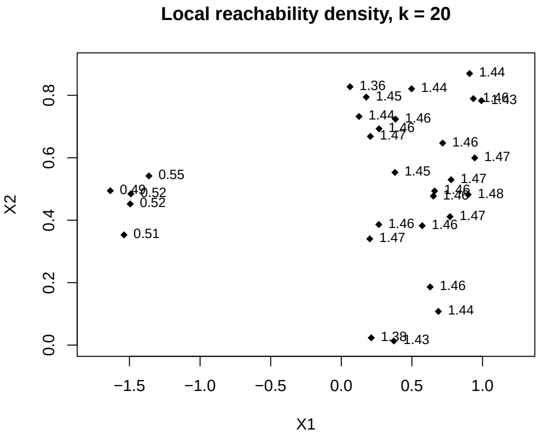

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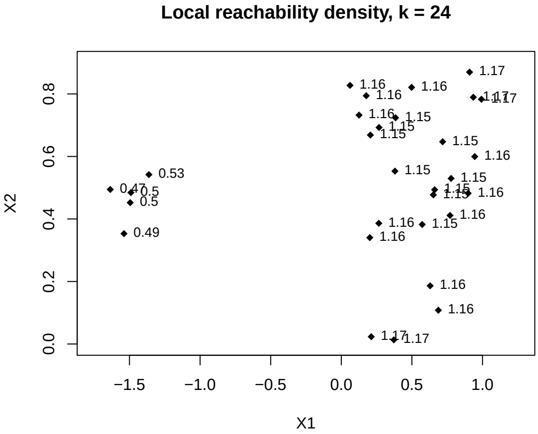

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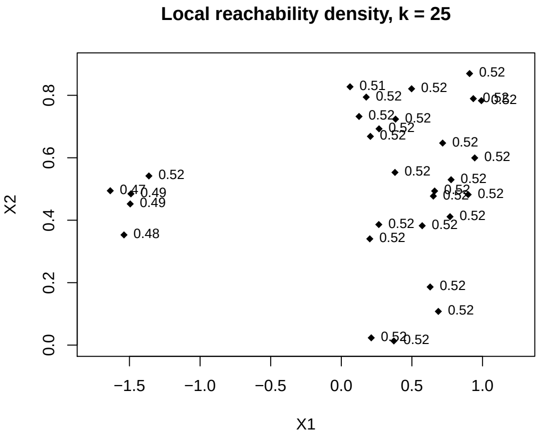

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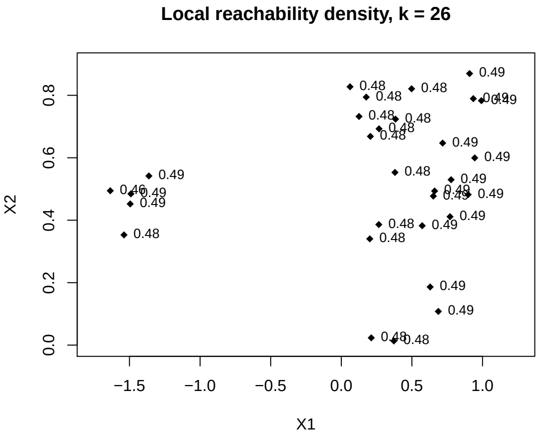

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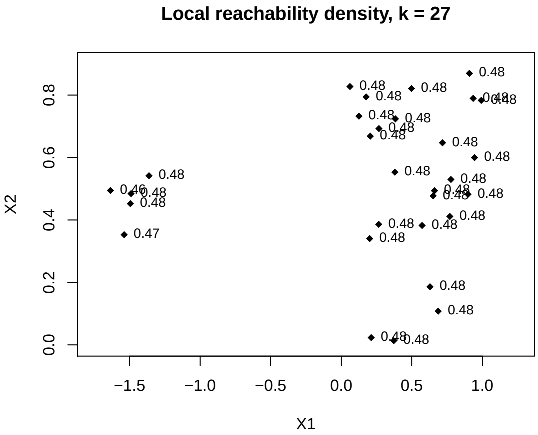

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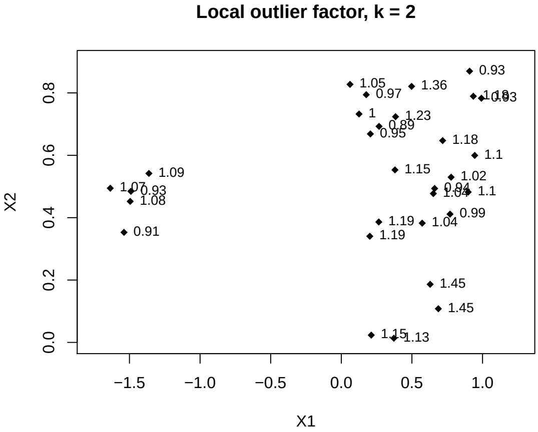

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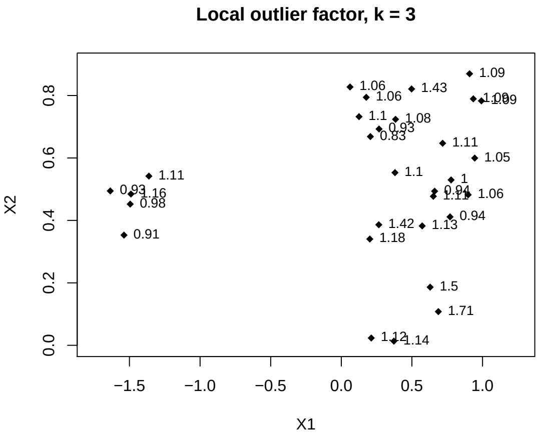

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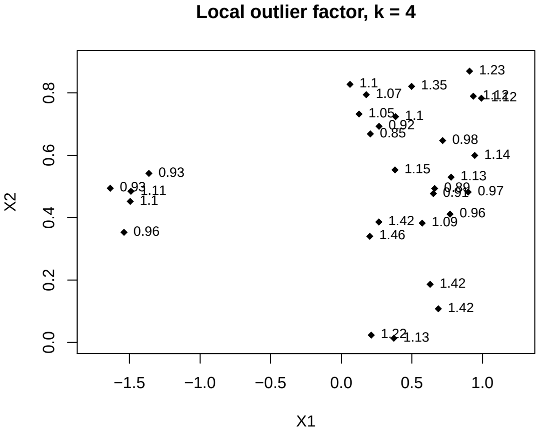

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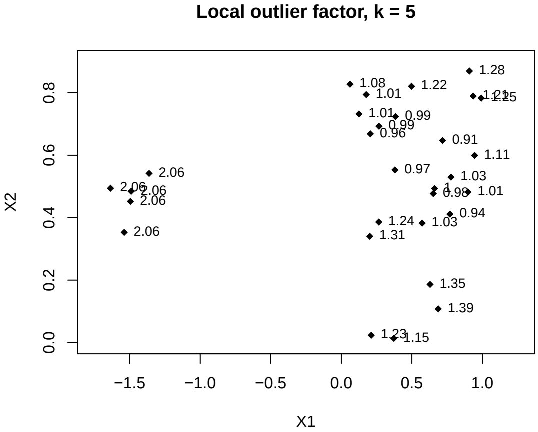

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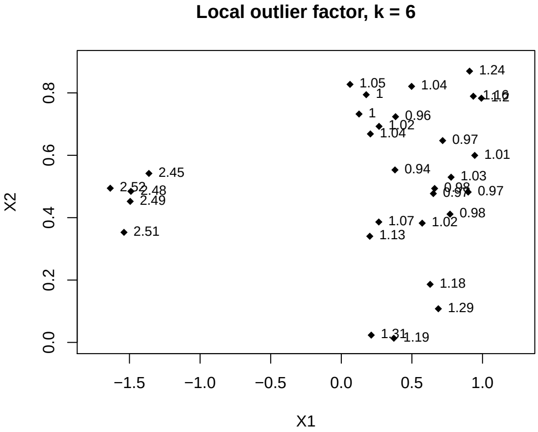

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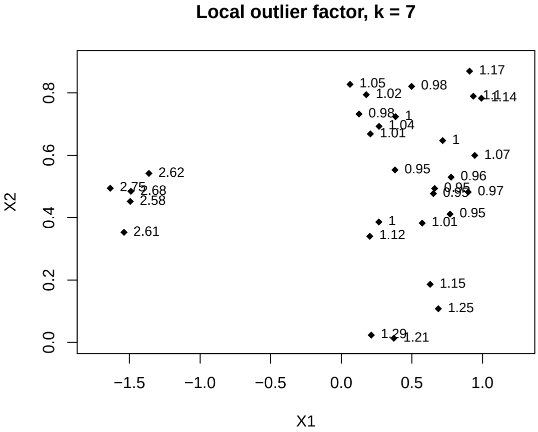

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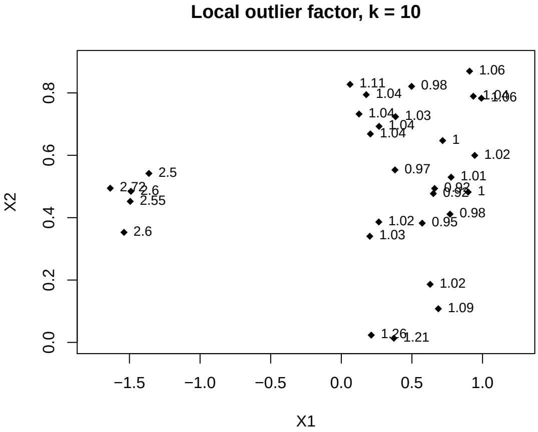

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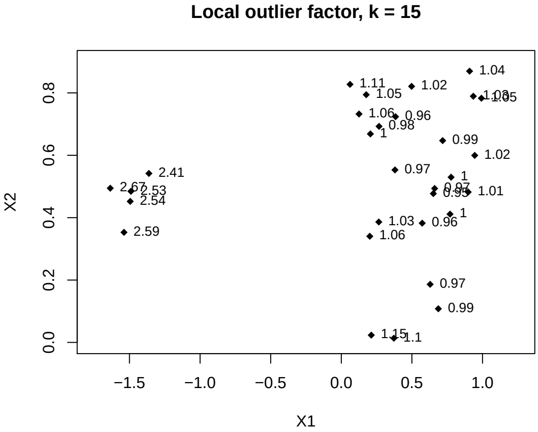

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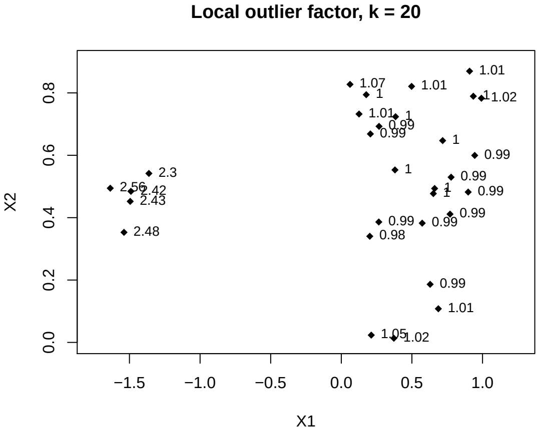

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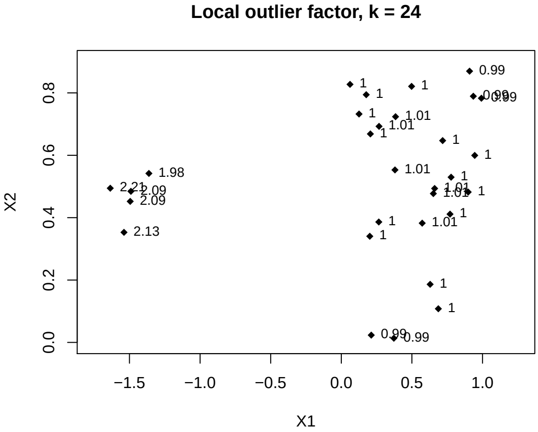

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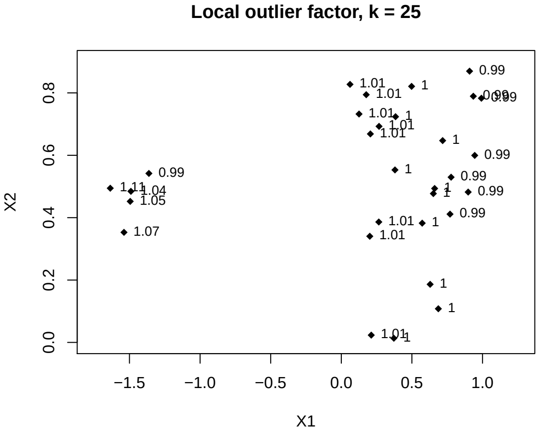

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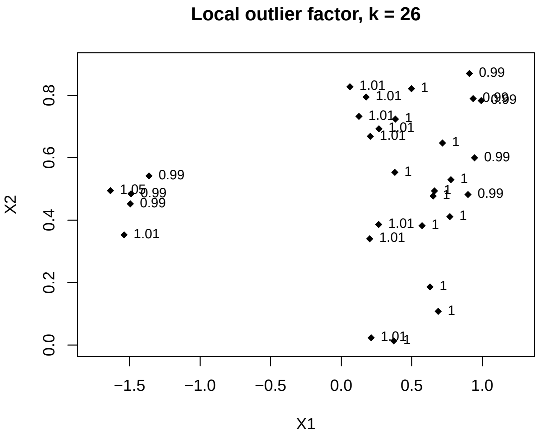

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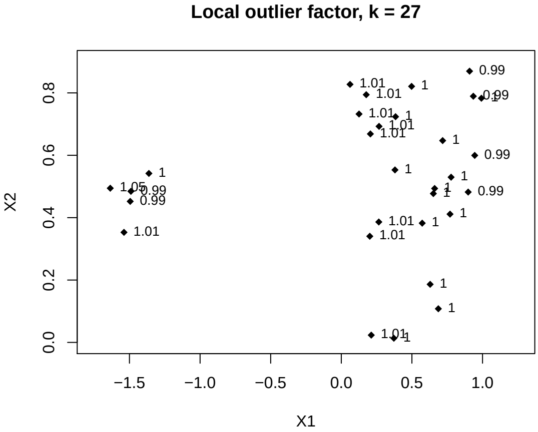

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- Since no supervised feedback is given, isolation forest is based on purely random (uniform) variable-based partitioning.
- **Main idea:** Outlying observations are isolated faster.
- Tree-kind partitioning is done until "full isolation": outlying observations will have smaller depth (on an average) in the isolation tree.
- A monotone transform is usually applied to the aggregated estimate.
- To reduce both masking effect and computation cost, small-size sub-sampling is used instead of bootstrap.

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Stop when only one observation is in each node: isolation.

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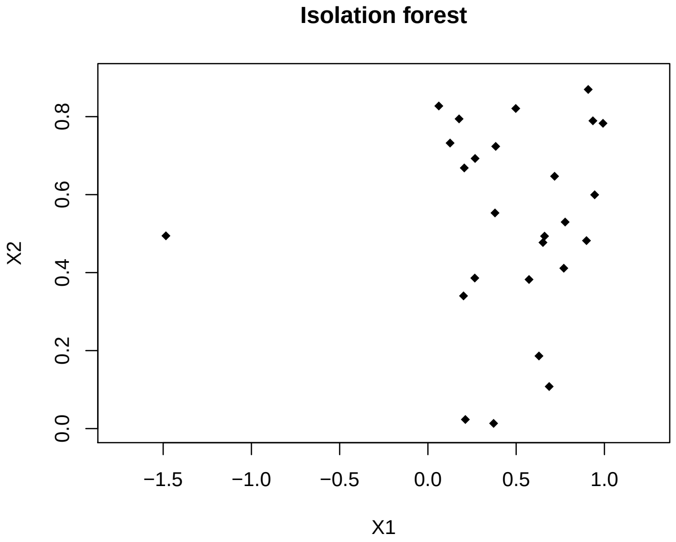
Illustration: Isolation tree



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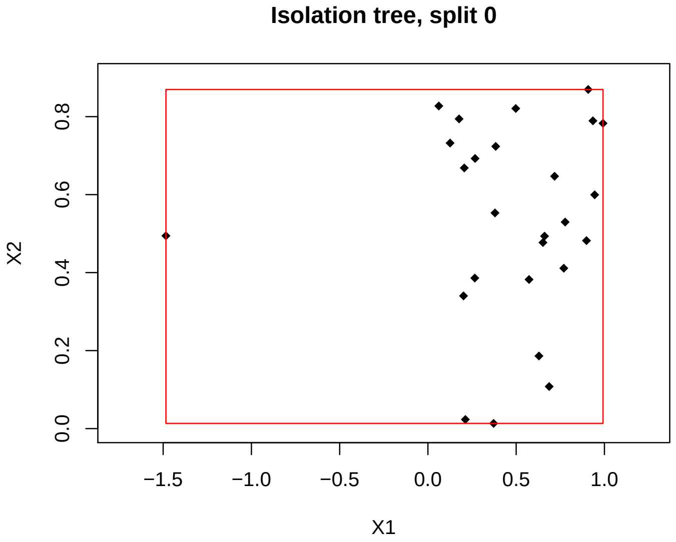

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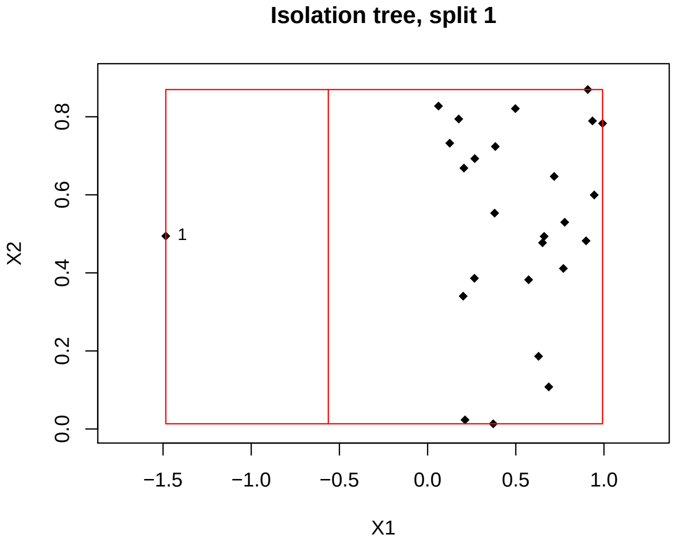

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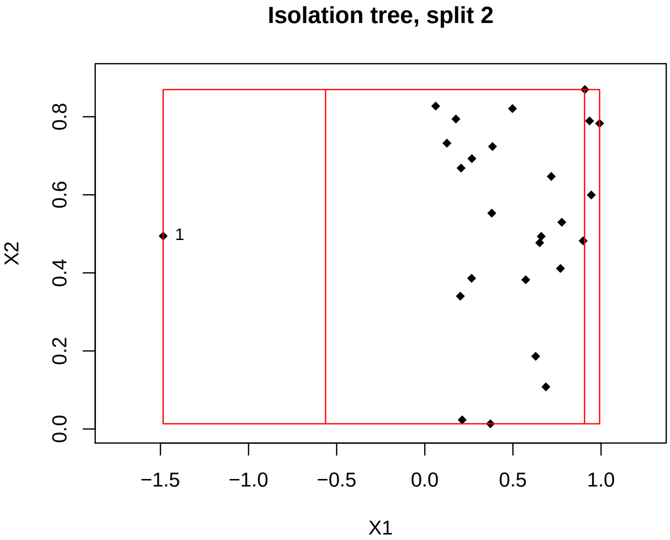

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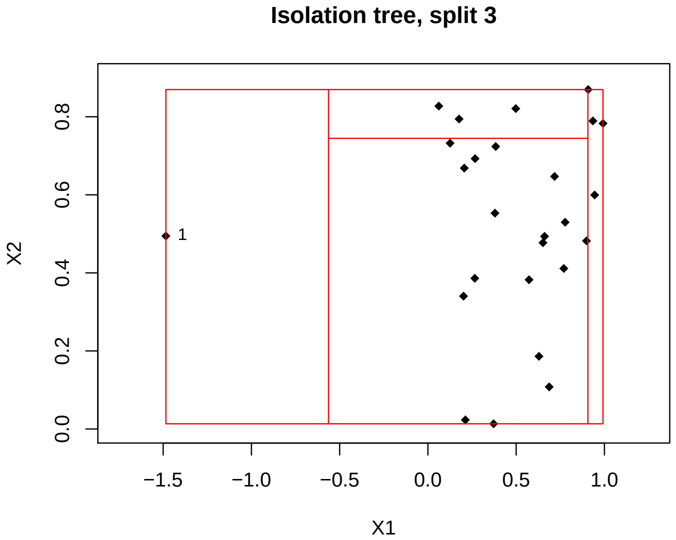

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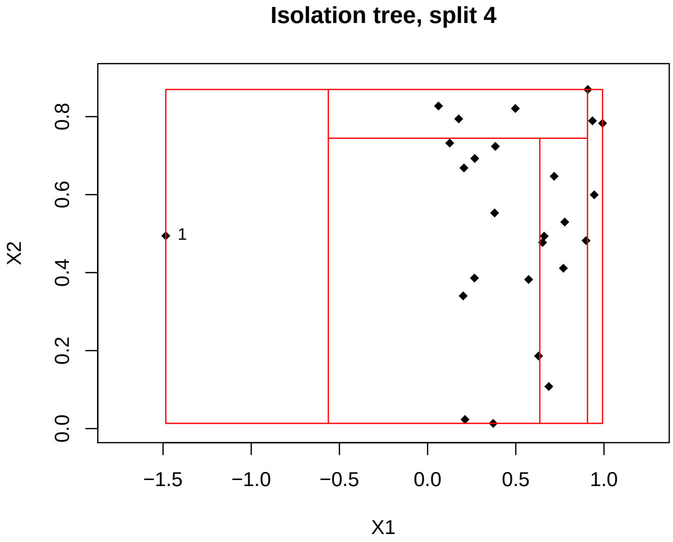

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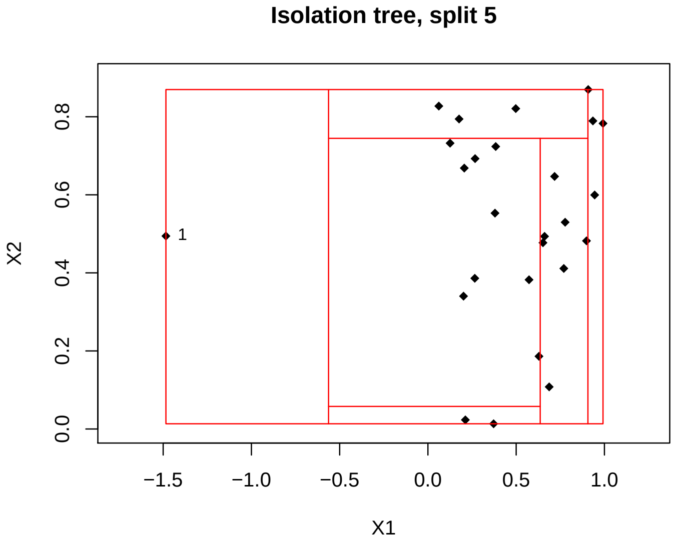

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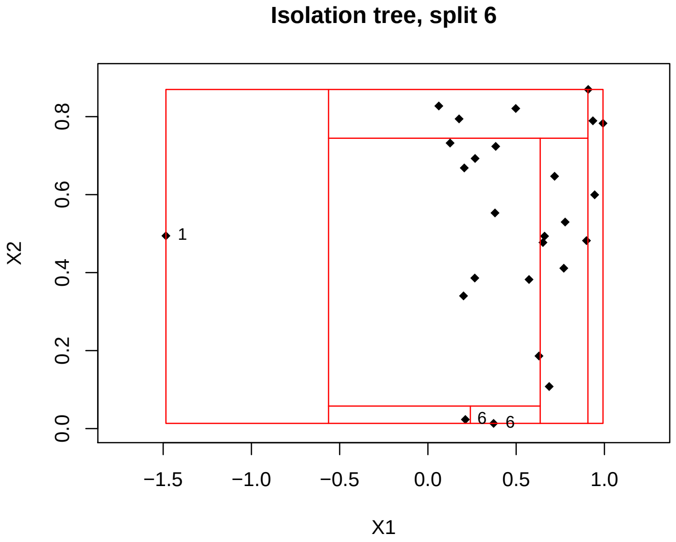

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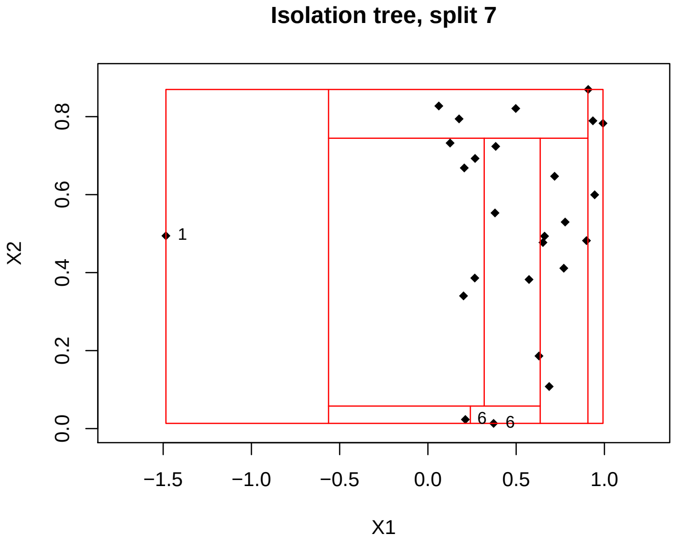

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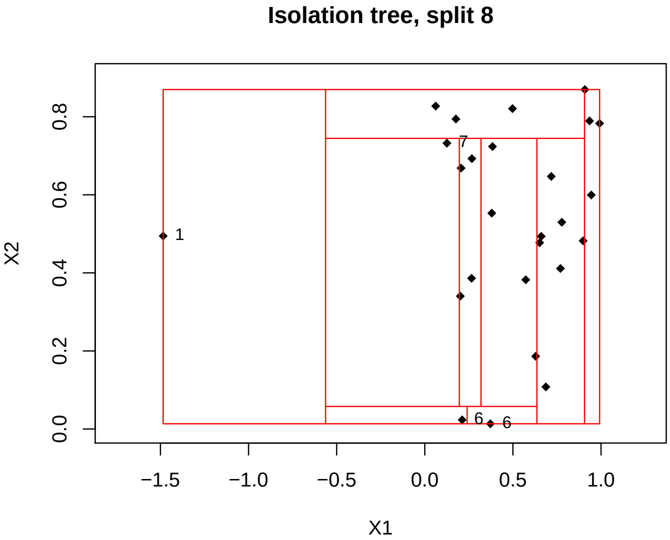

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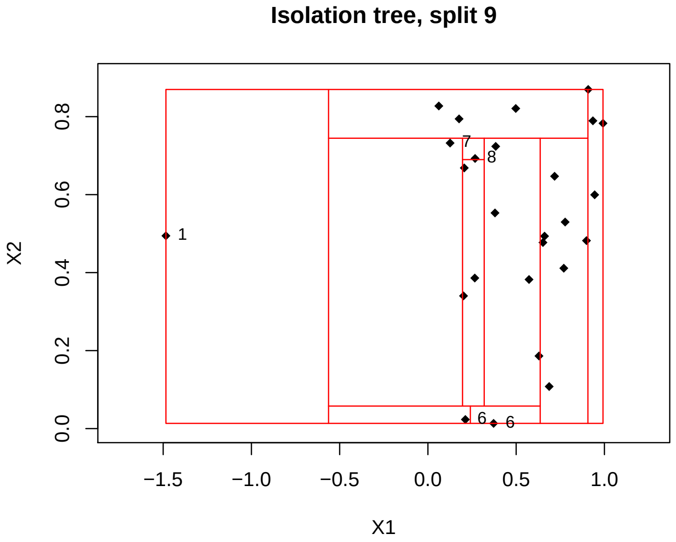

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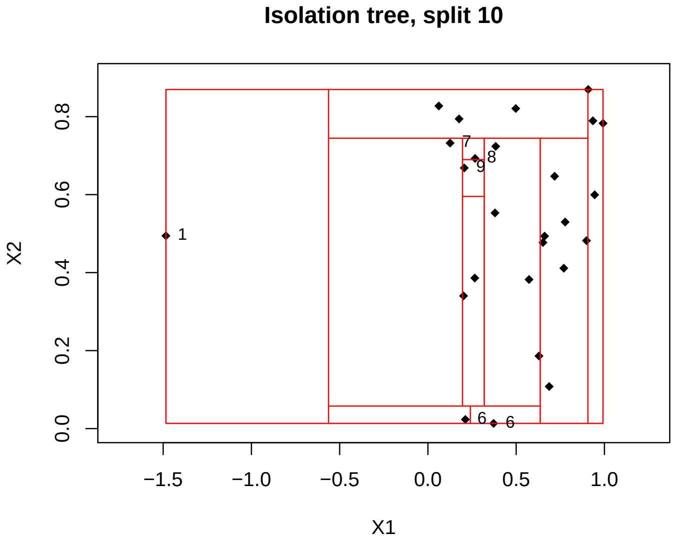

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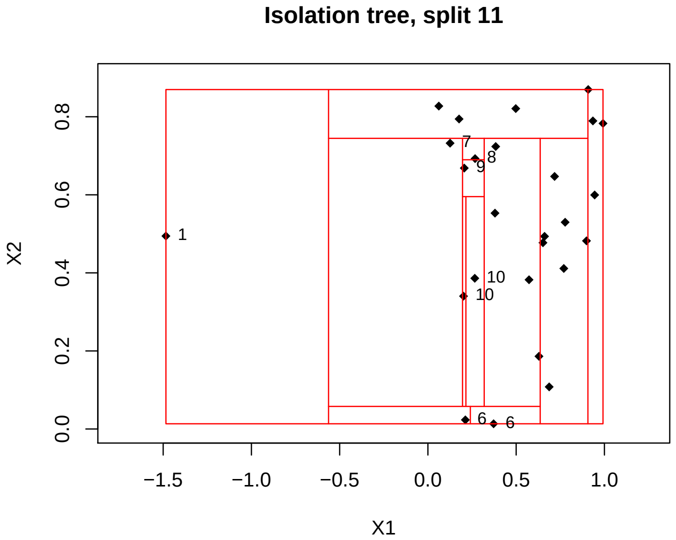

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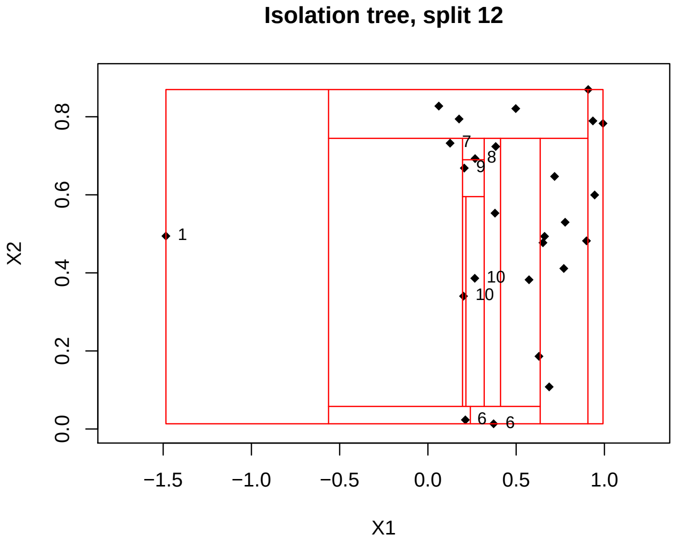

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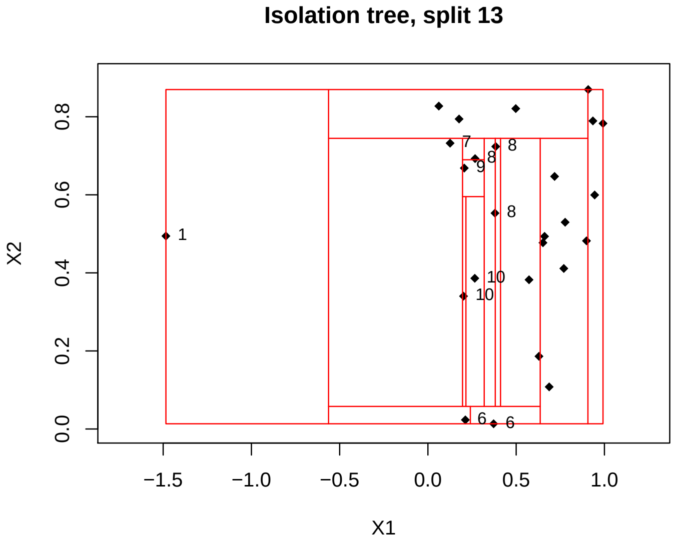

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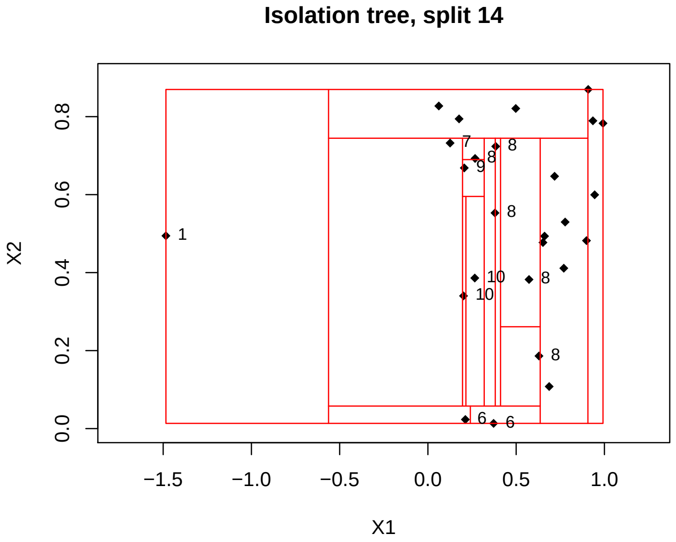

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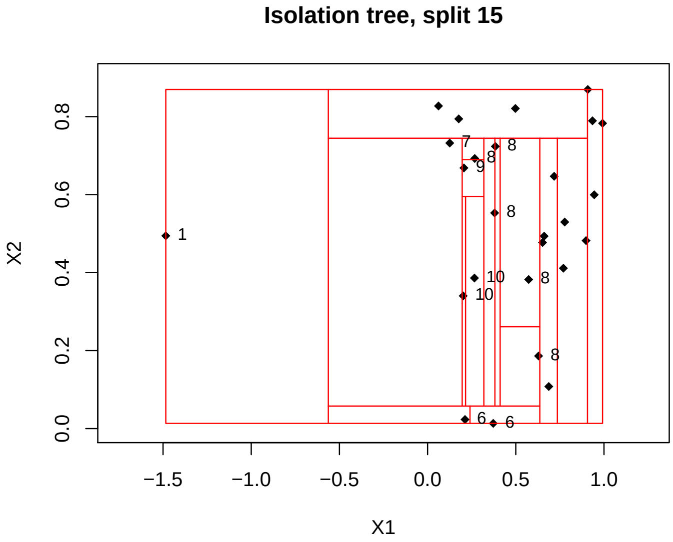

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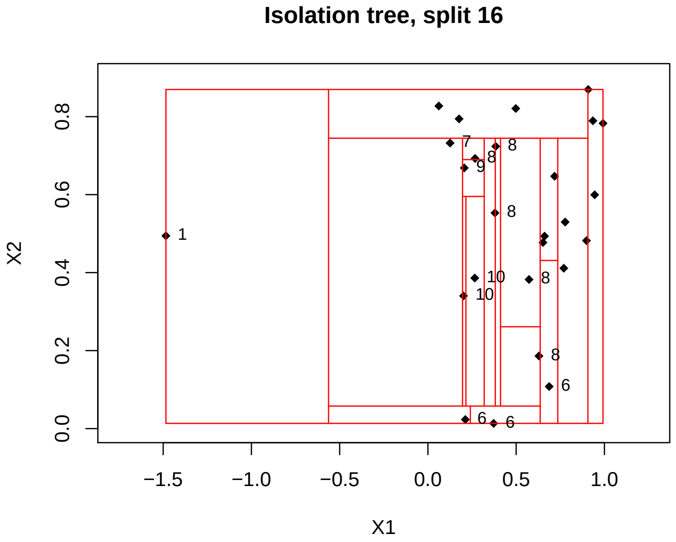

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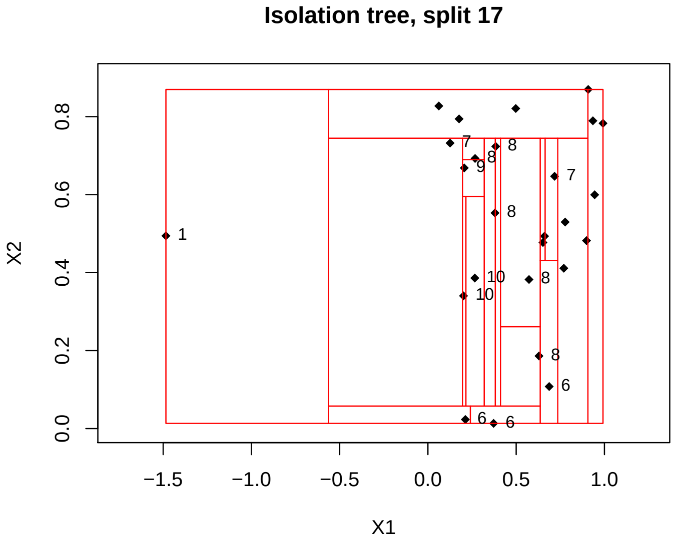

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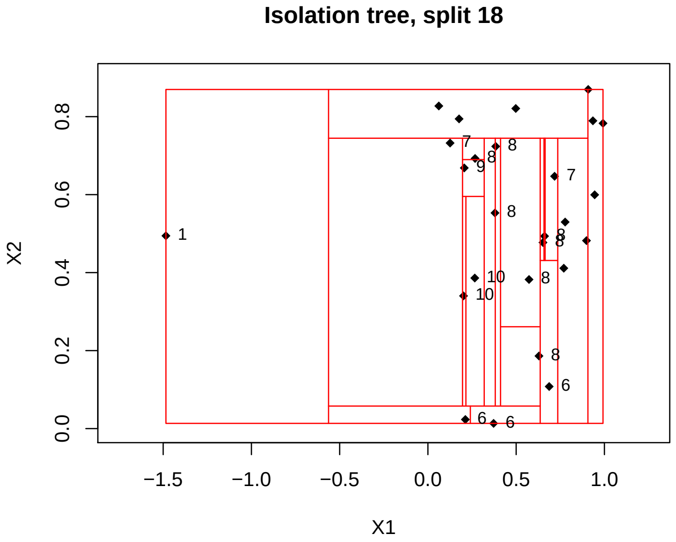

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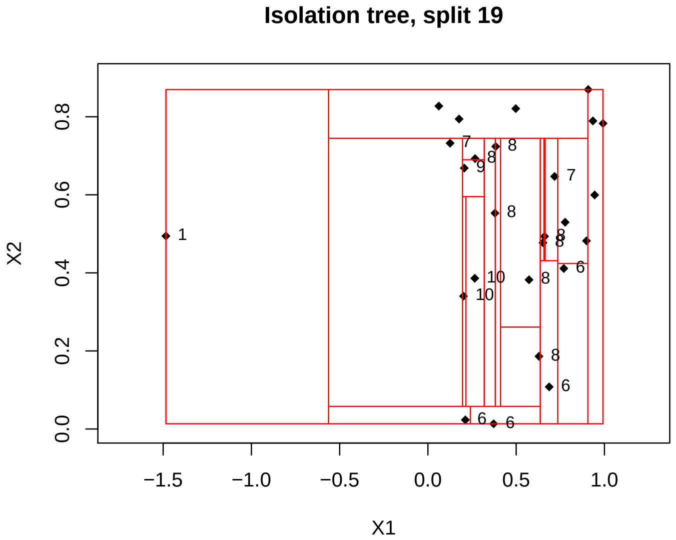

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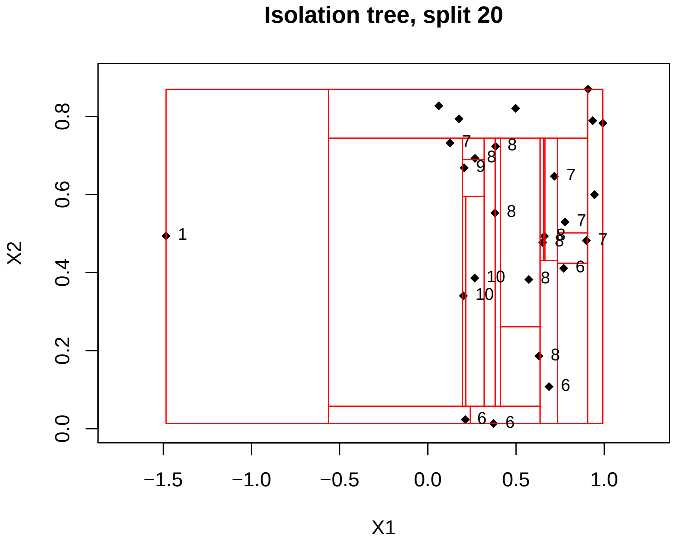

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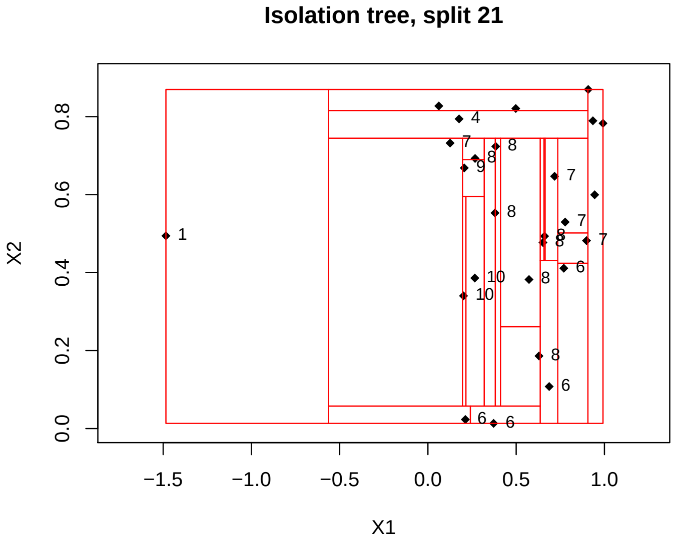

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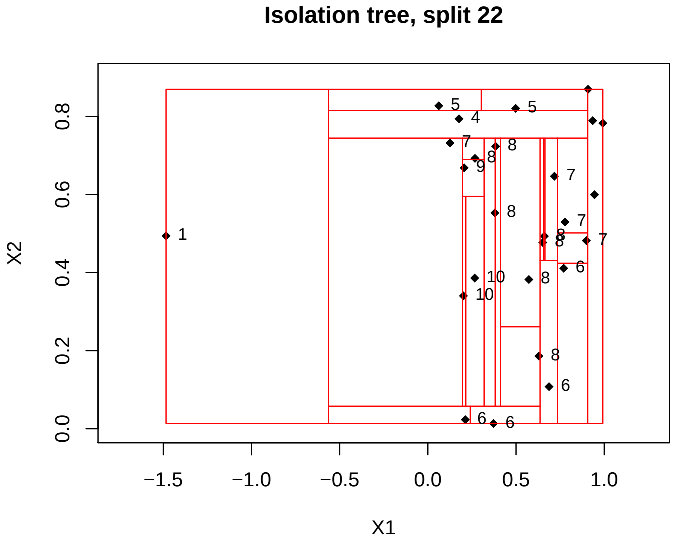

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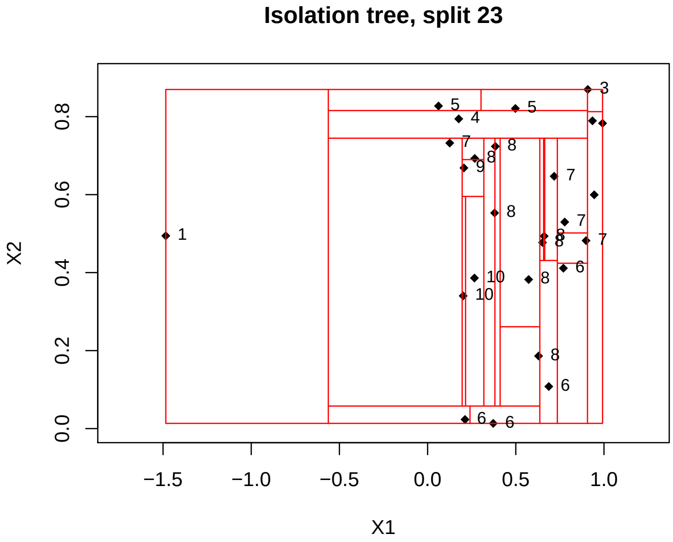

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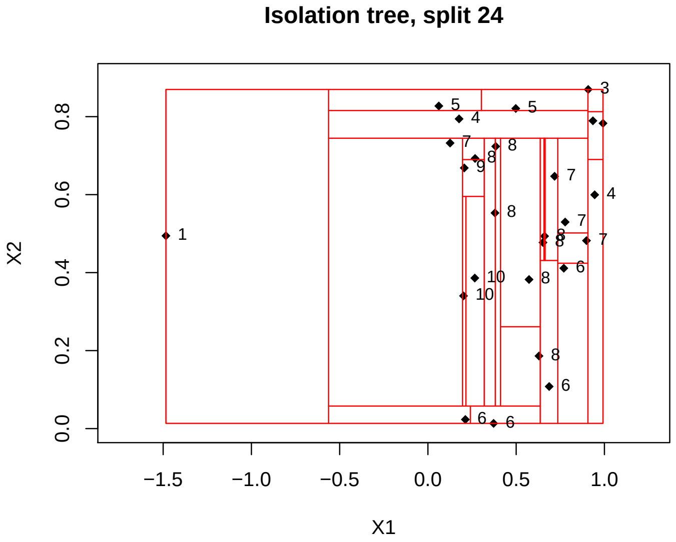

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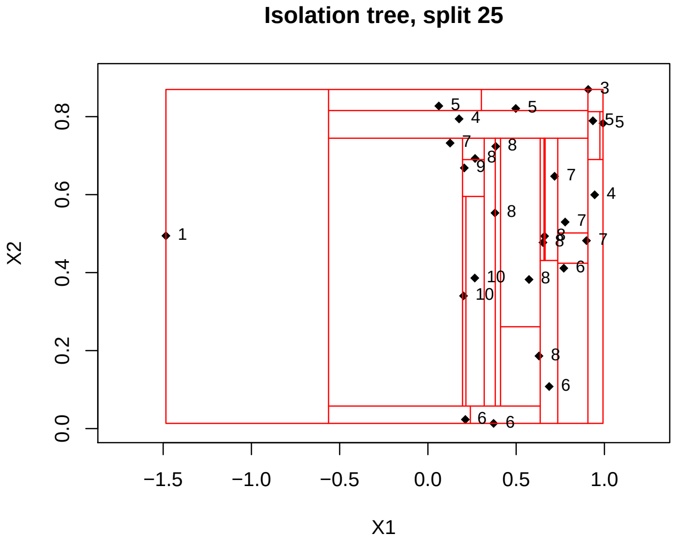

Isolation forest (Liu, Ting, Zhou; 2008)

Anomaly score calculation for observation $\boldsymbol{x}$:

1. For each isolation tree $i \in \{1, ..., T\}$, locate $\boldsymbol{x}$ in a terminal node and calculate the depth of this node $h_i(\boldsymbol{x})$.
2. Attribute the anomaly score:

$$s(\boldsymbol{x}) = 2^{-\frac{\frac{1}{n}\sum_{i=1}^{T} h_i(\boldsymbol{x})}{c(n)}},$$

with $c(n) = 2H(n-1) - \frac{2(n-1)}{n}$ where $H(k)$ is the harmonic number and can be estimated by $\ln(k) + 0.5772156649$.

Score behavior:

- when $\frac{1}{n}\sum_{i=1}^{T} h_i(\boldsymbol{x}) \to c(n)$, $s(\boldsymbol{x}) \to 0.5$,
- when $\frac{1}{n}\sum_{i=1}^{T} h_i(\boldsymbol{x}) \to 0$, $s(\boldsymbol{x}) \to 1$,
- when $\frac{1}{n}\sum_{i=1}^{T} h_i(\boldsymbol{x}) \to n-1$, $s(\boldsymbol{x}) \to 0$.

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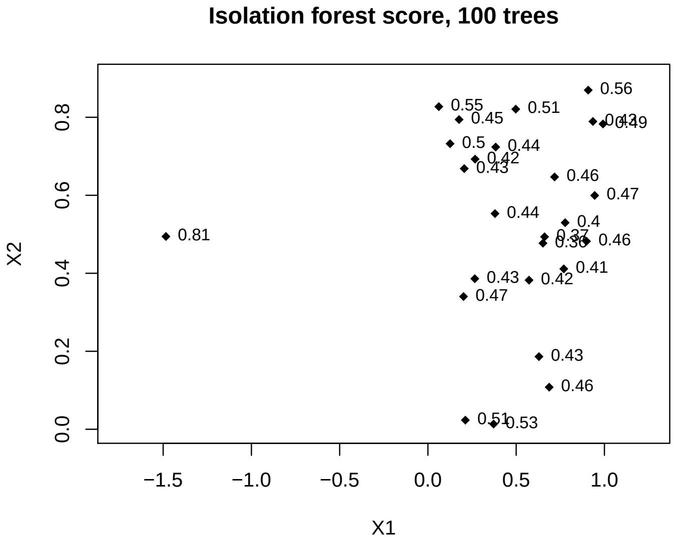
Illustration: Anomaly score



Contents

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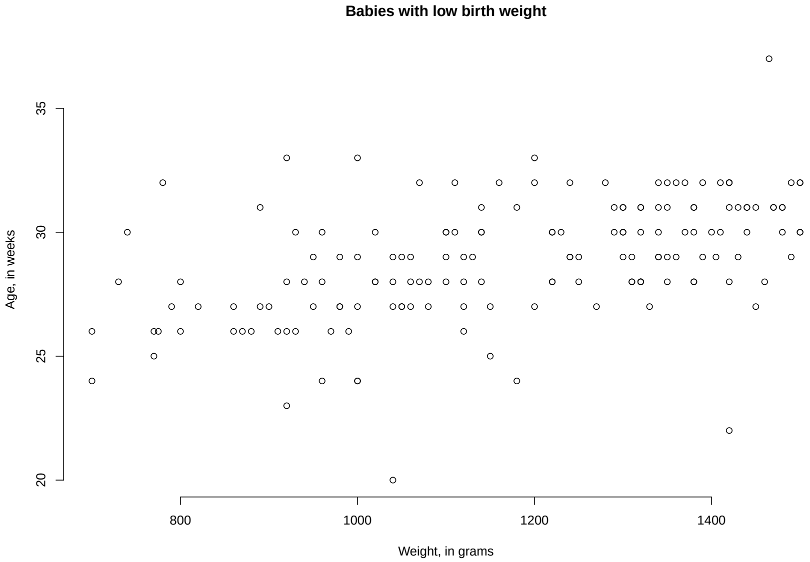
Data depth



Babies with low birth weight
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30
25
20
Age, in weeks
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1200
1400
Weight, in grams

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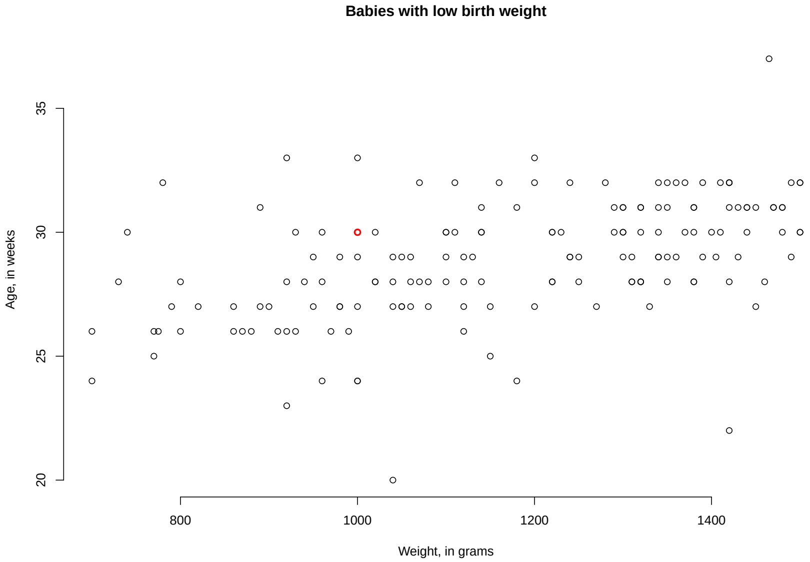
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Statistical data depth

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D5 **upper semicontinuous in $\boldsymbol{x}$:** the upper-level sets $D_\alpha(X) = \{\boldsymbol{x} \in \mathbb{R}^p : D(\boldsymbol{x}|X) \geq \alpha\}$ are closed for all α.

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Depth notions: **Mahalanobis** ('36), **projection** (Stahel, '81; Donoho, '82), **simplicial volume** (Oja, '83), **simplicial** (Liu, '90), **zonoid** (Koshevoy, Mosler, '97), **spatial** (Vardi, Zhang, '00; Serfling, '02), **lens** (Liu, Modarres, '11), ... depth.

Applications of data depth:

- **Multivariate data analysis** (Liu, Parelius, Singh '99);
- **Statistical quality control** (Liu, Singh '93);
- **Cluster analysis and classification** (Mosler, Hoberg '06; Li, Cuesta-Albertos, Liu '12; M., Mosler, Lange '15);
- **Tests for multivariate location, scale, symmetry** (Liu '92; Dyckerhoff '02; Dyckerhoff, Ley, Paindaveine '15);
- **Outlier detection** (Hubert, Rousseeuw, Segaert '15);
- **Multivariate risk measurement** (Cascos, Mochalov '07);
- **Robust linear programming** (Bazovkin, Mosler '15);
- **Missing data imputation** (M., Josse, Husson '20);
- etc.

R-package **ddalpha** (Pokotylo, M., Dyckerhoff, Nagy):
calculates a number of depths; performs depth-based classification of multivariate and functional data; contains 50 multivariate and 5 functional data sets.

Tukey (=halfspace, location) depth

Tukey (1975) — "Mathematics and the picturing of data"

Tukey depth of $\boldsymbol{x} \in \mathbb{R}^p$ w.r.t. a d-variate random vector X distributed as P is defined as the smallest probability mass of a closed halfspace containing **x**:

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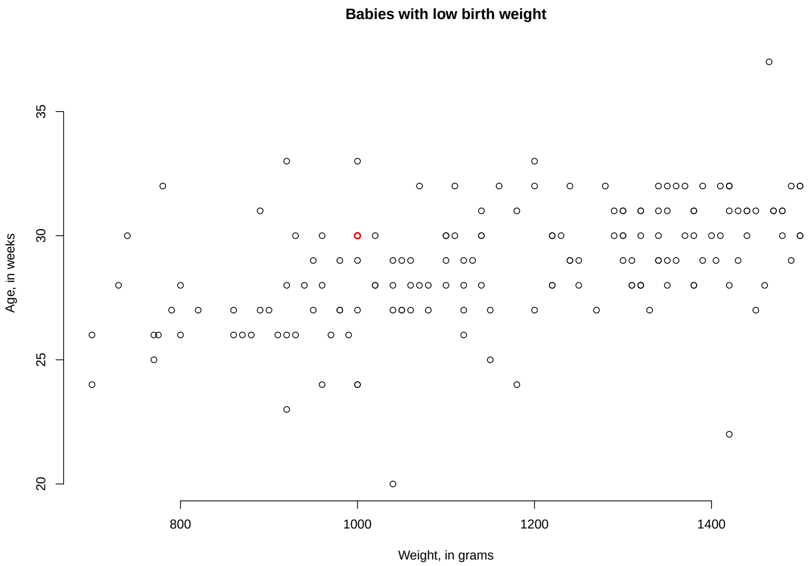
Tukey depth

- satisfies all the above postulates,
- is purely non-parametric and robust,
- has direct connection to quantiles and many applications.

Tukey (=halfspace, location) data depth



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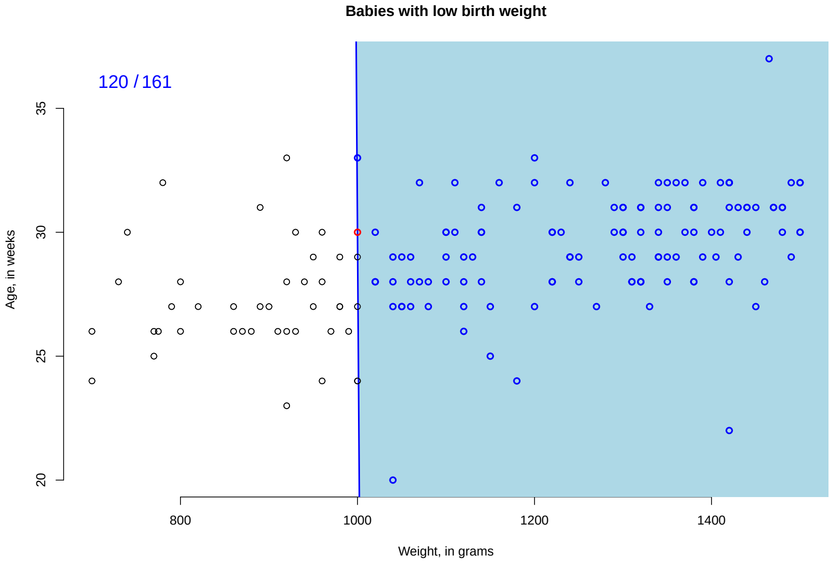

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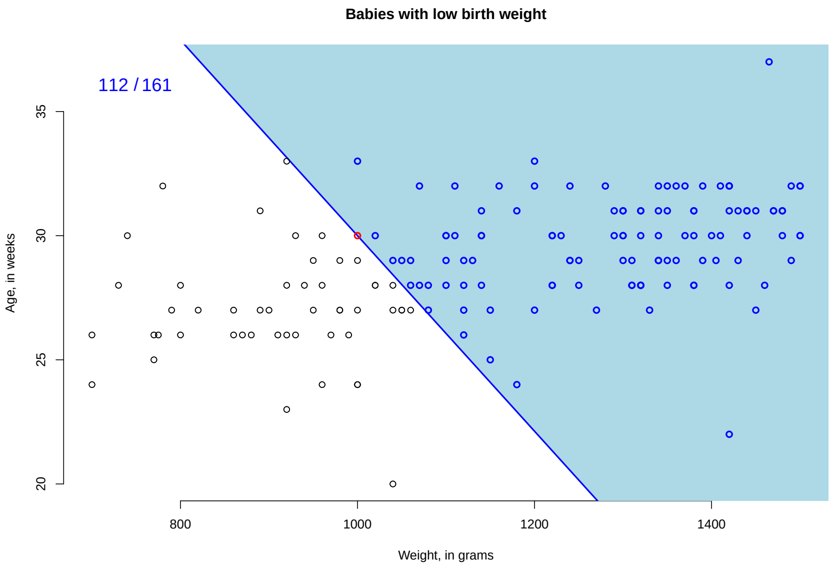

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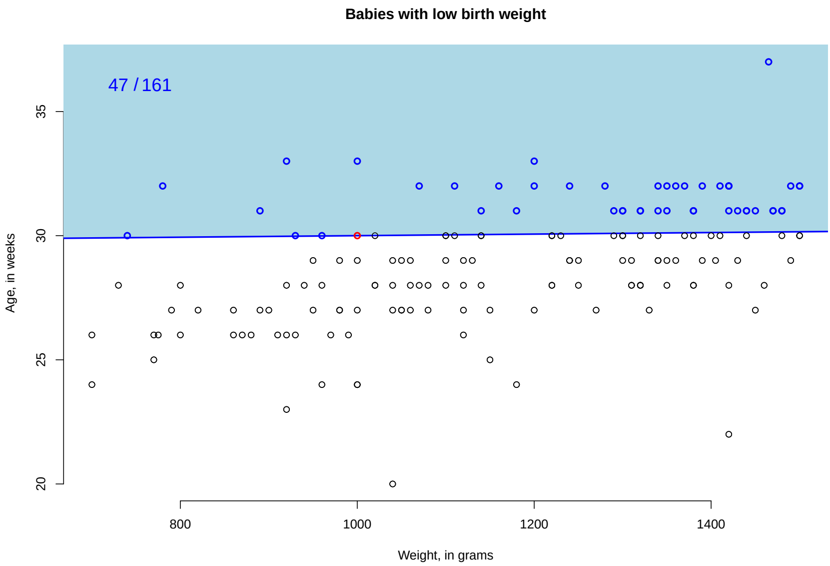

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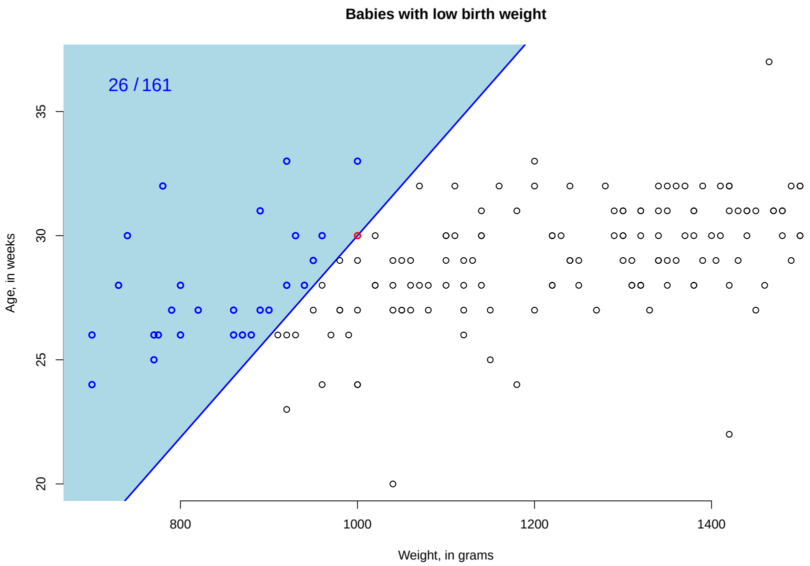

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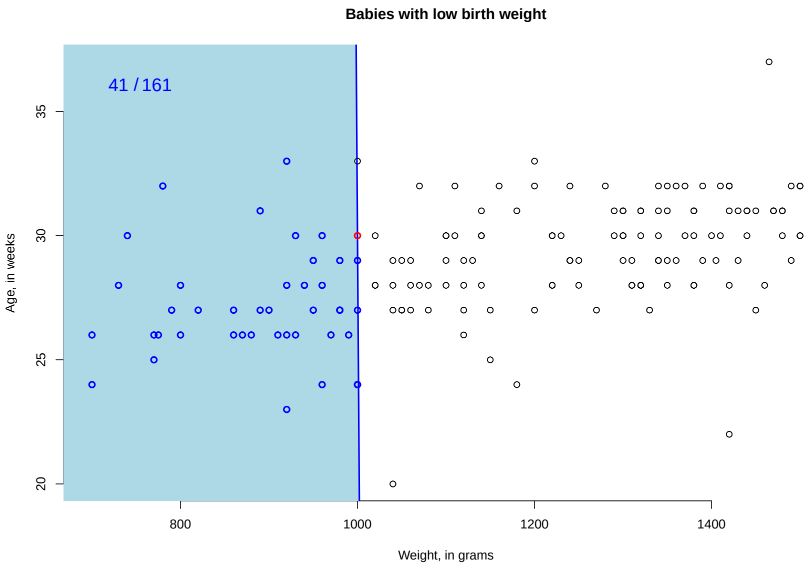

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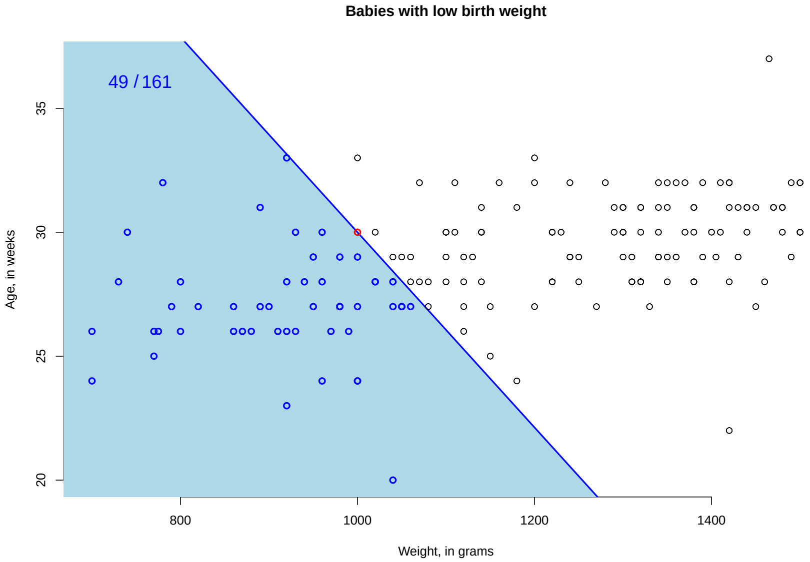

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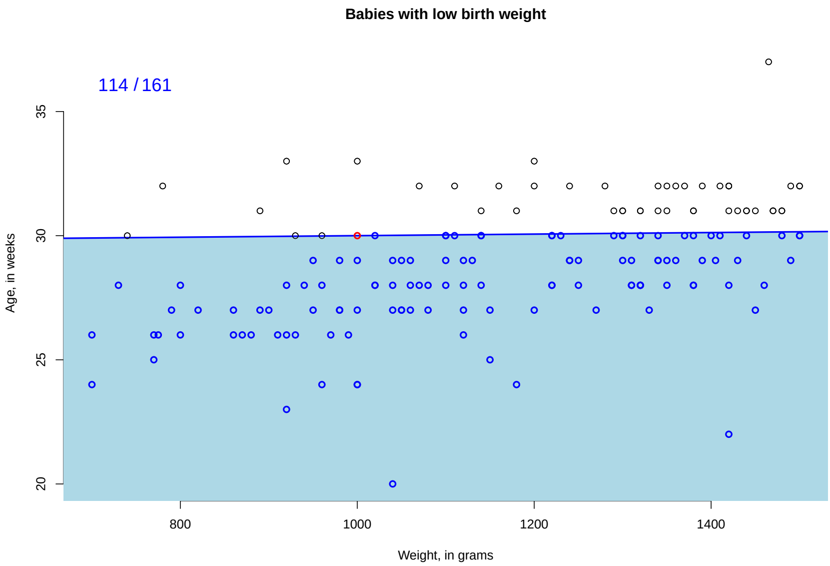

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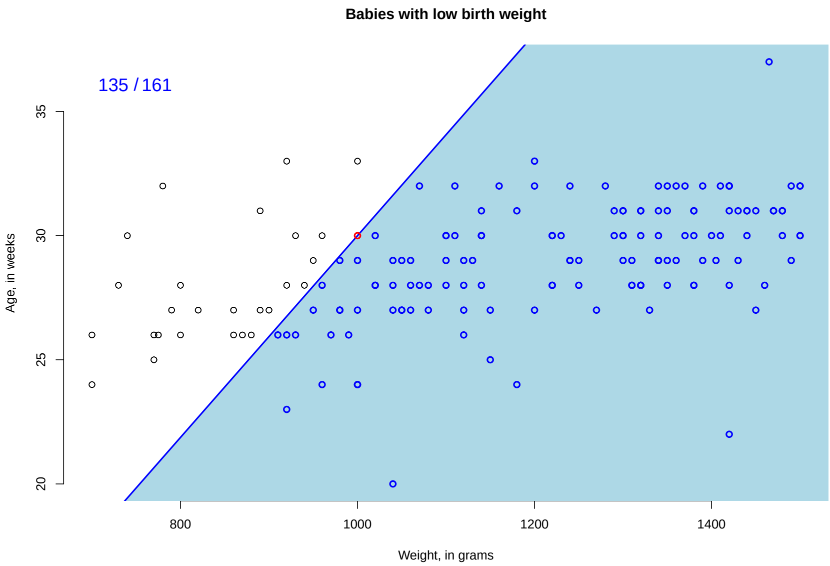

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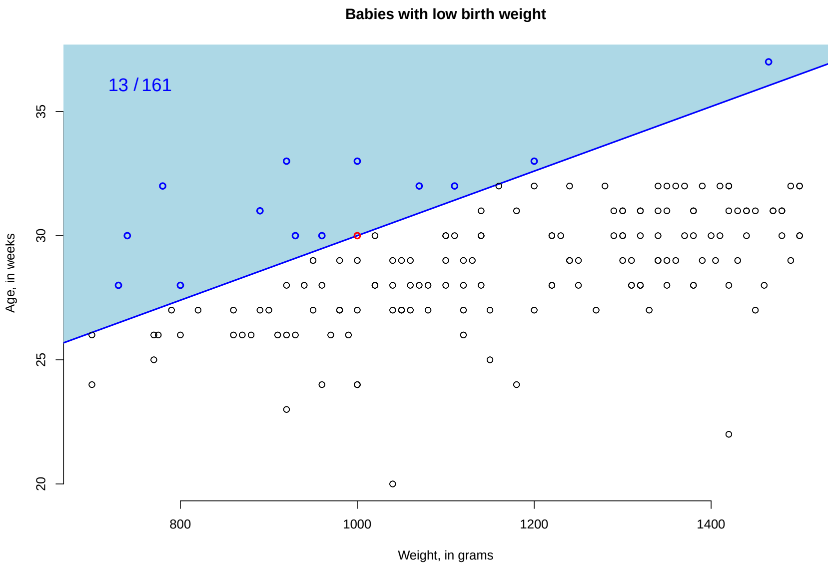

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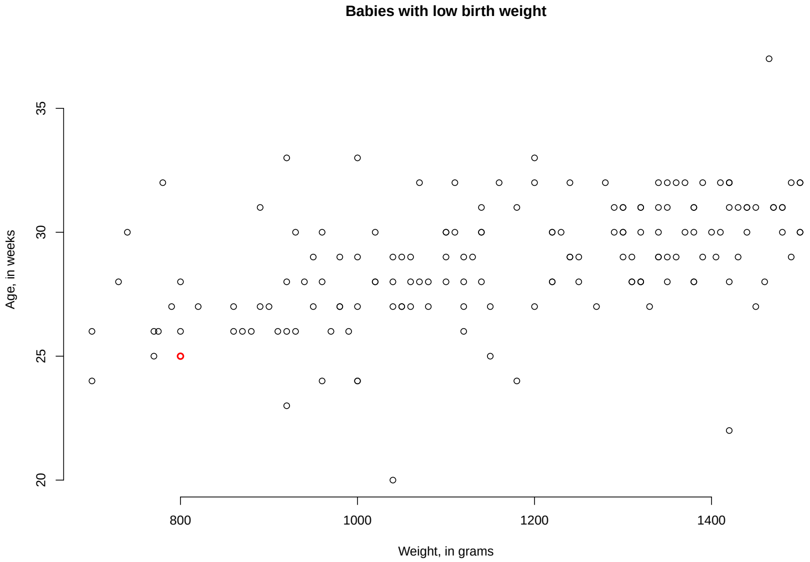

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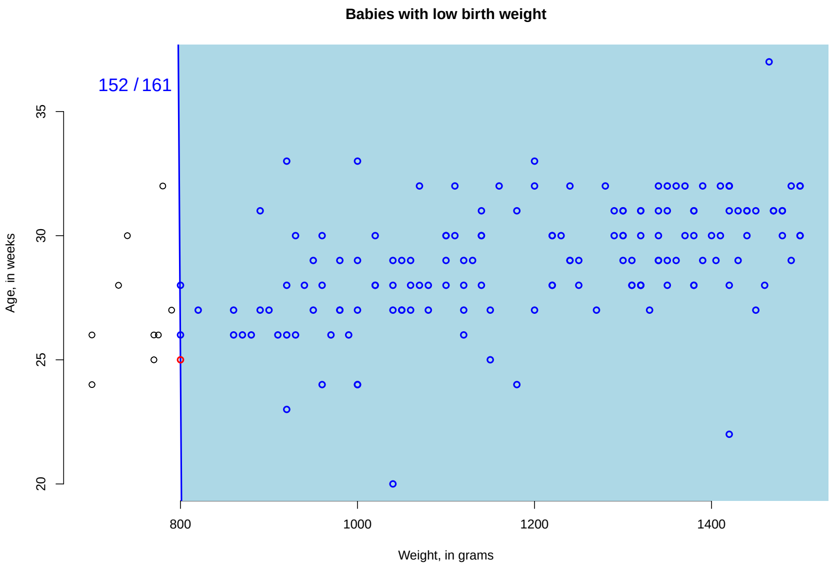

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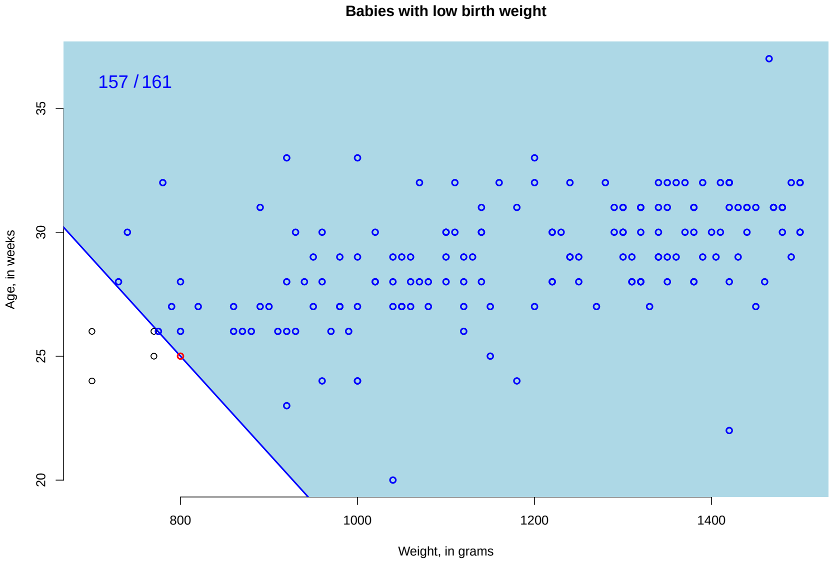

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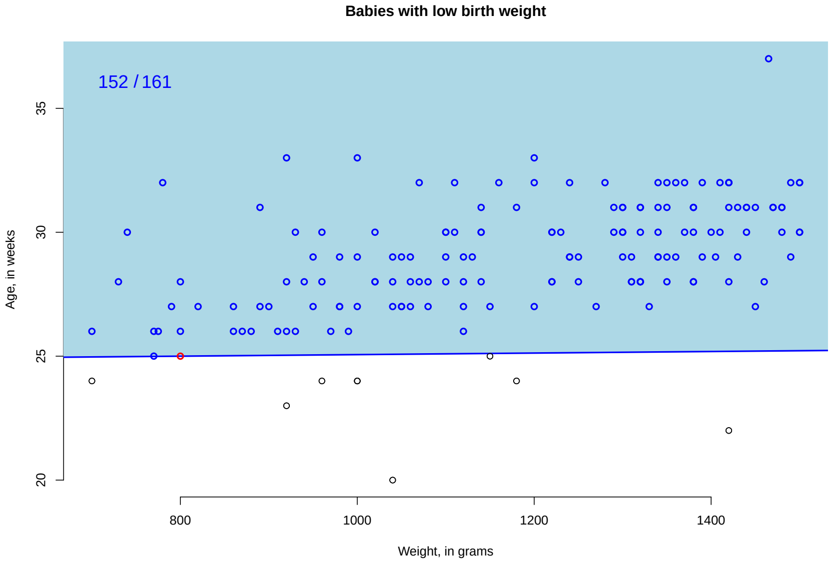

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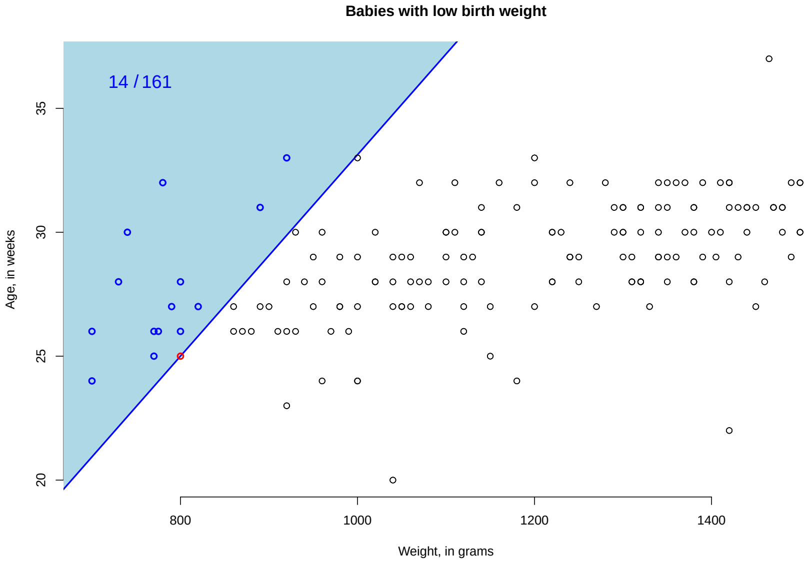

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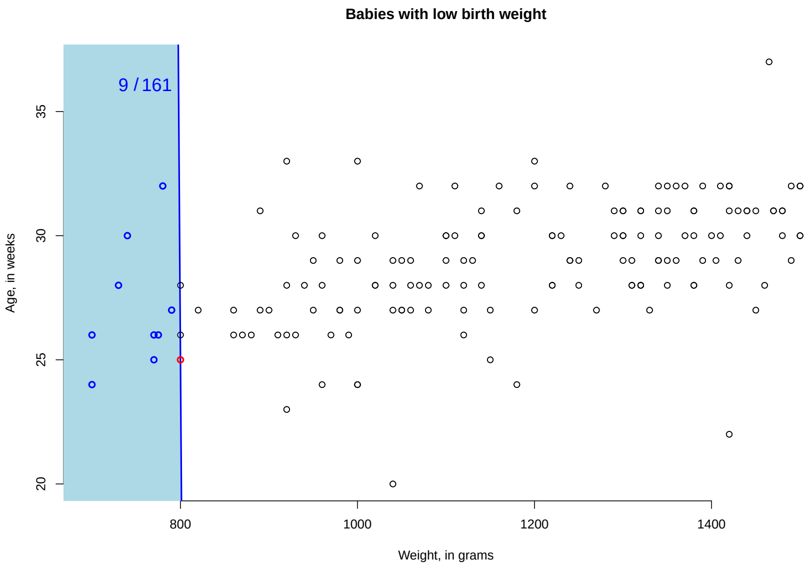

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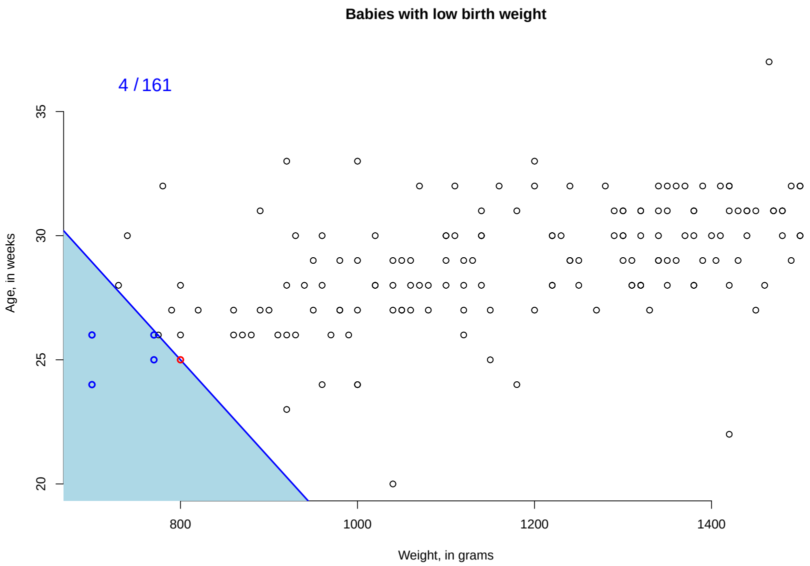

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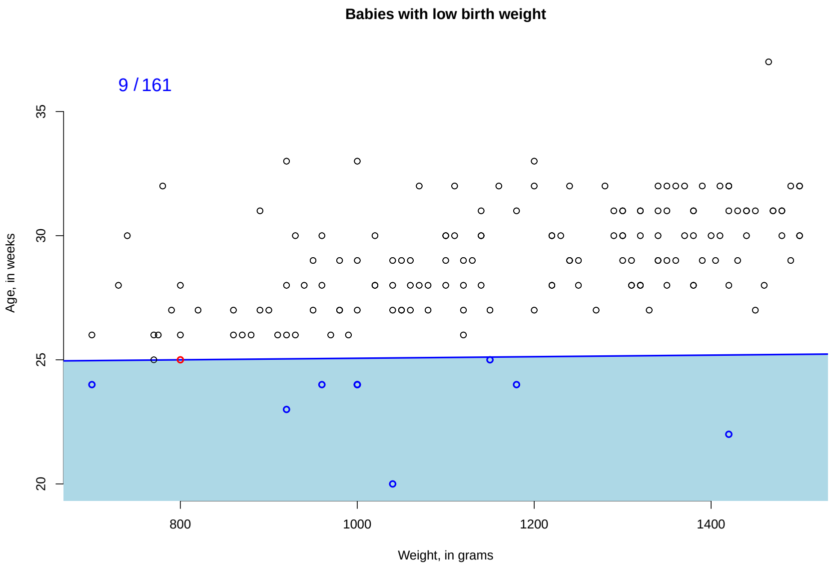

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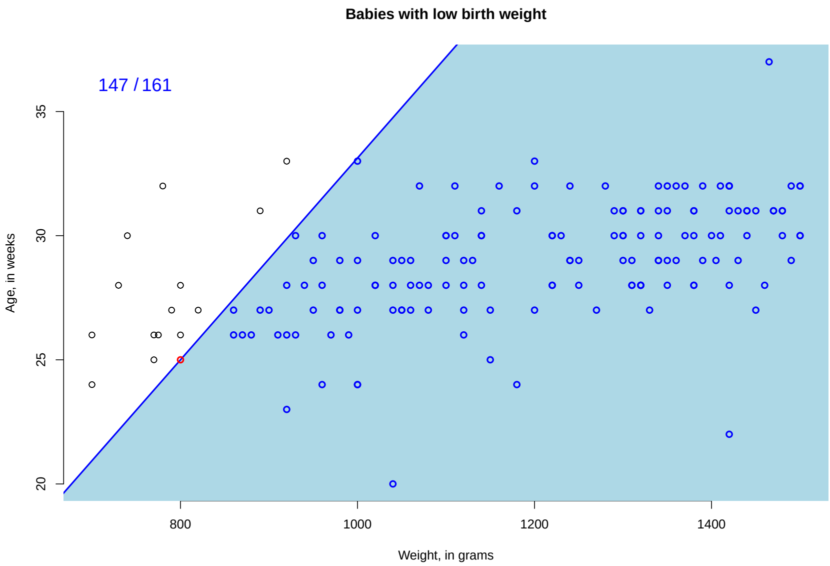

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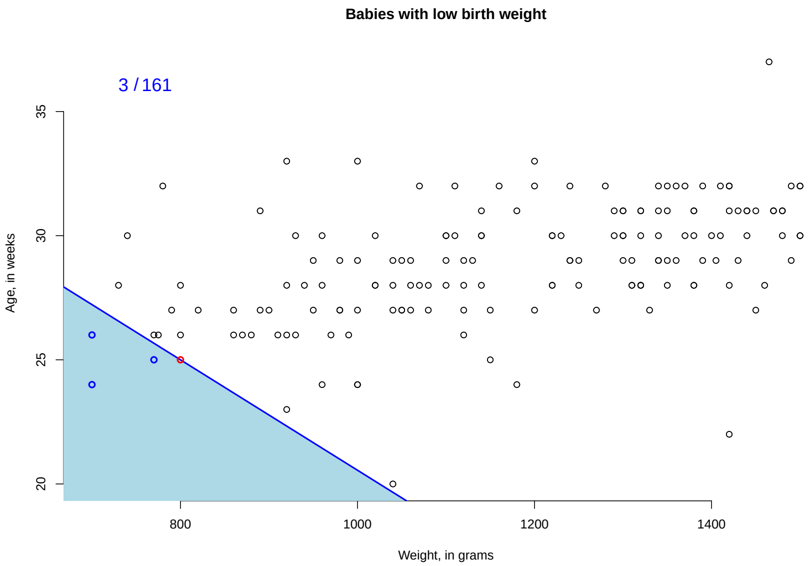

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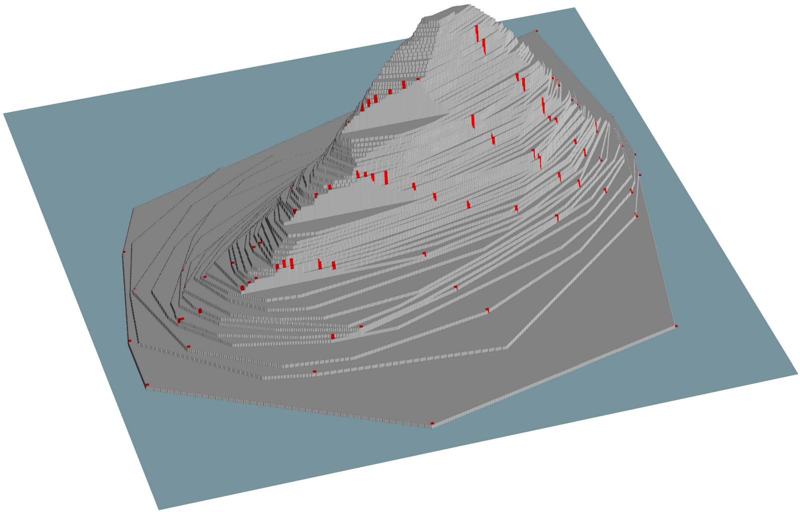

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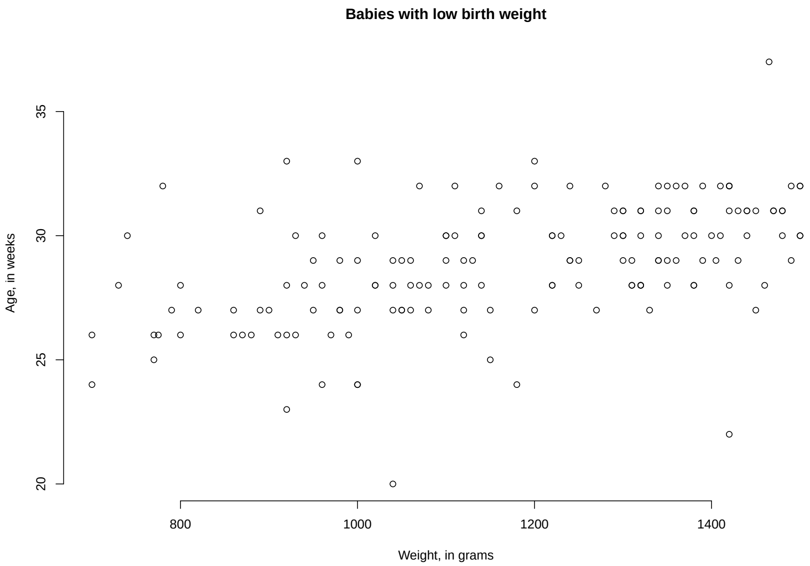

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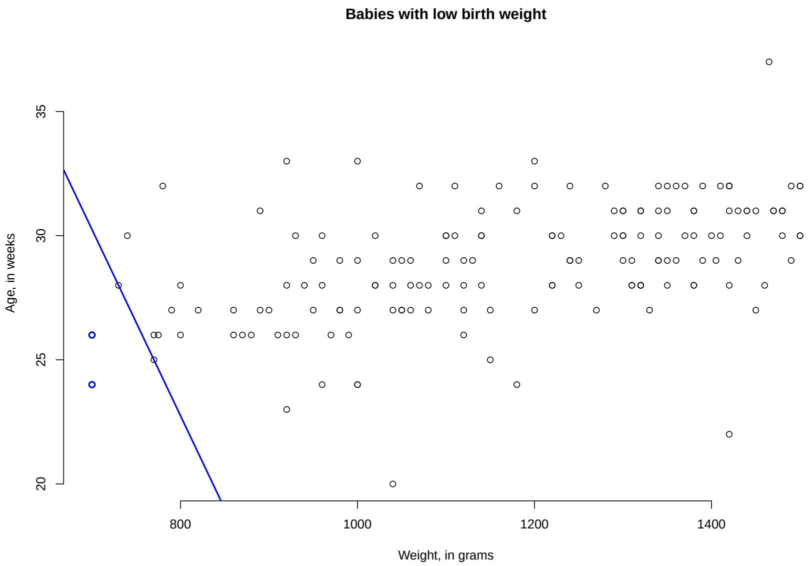

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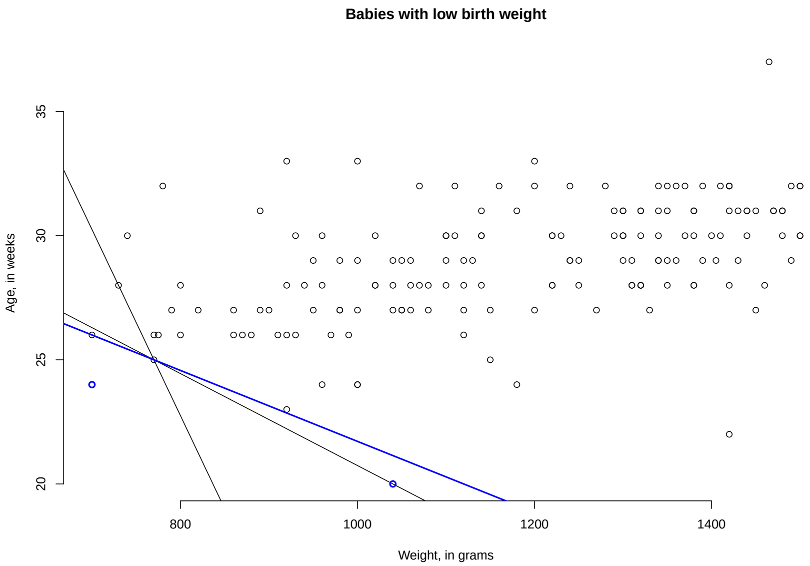

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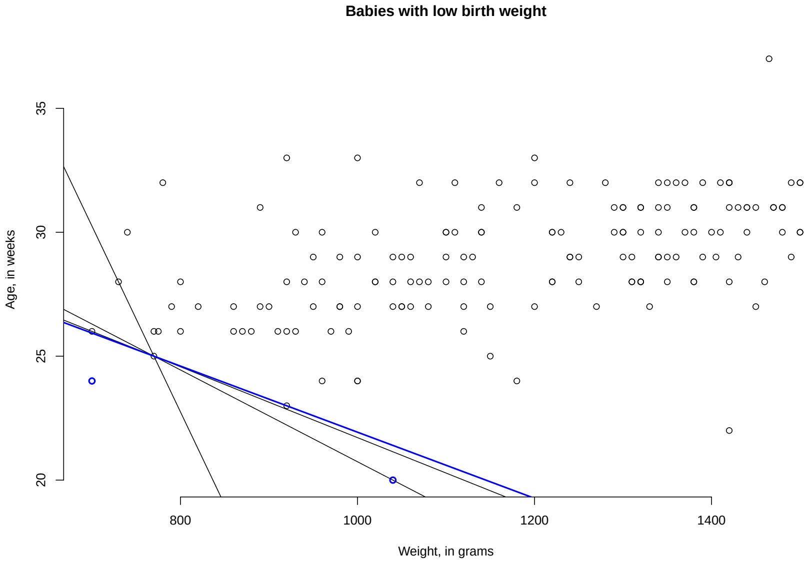

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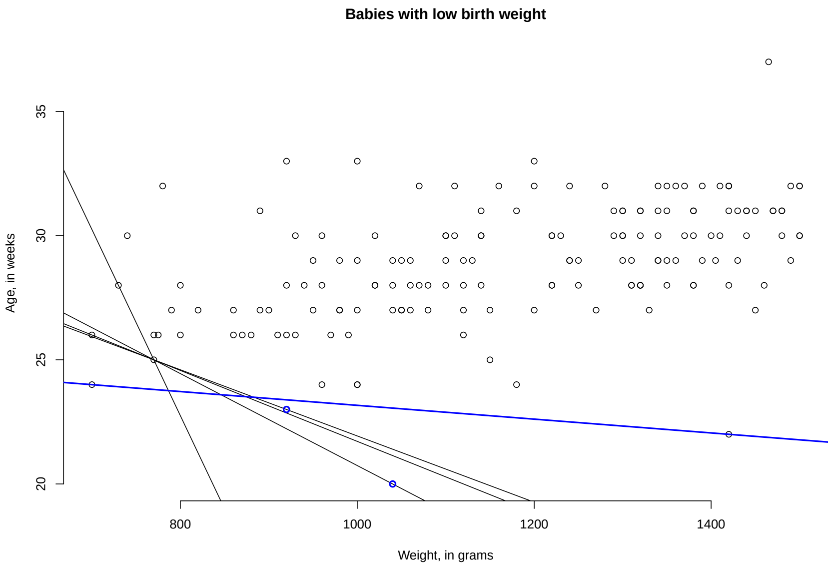

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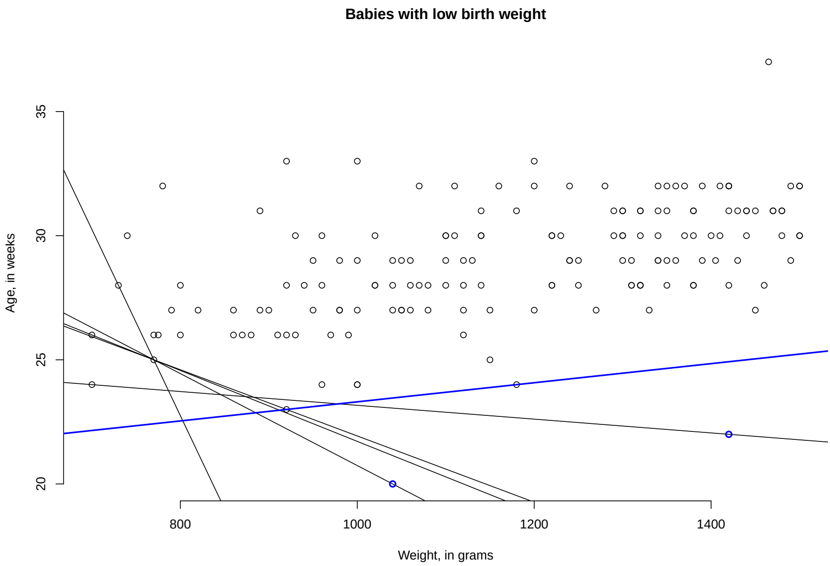

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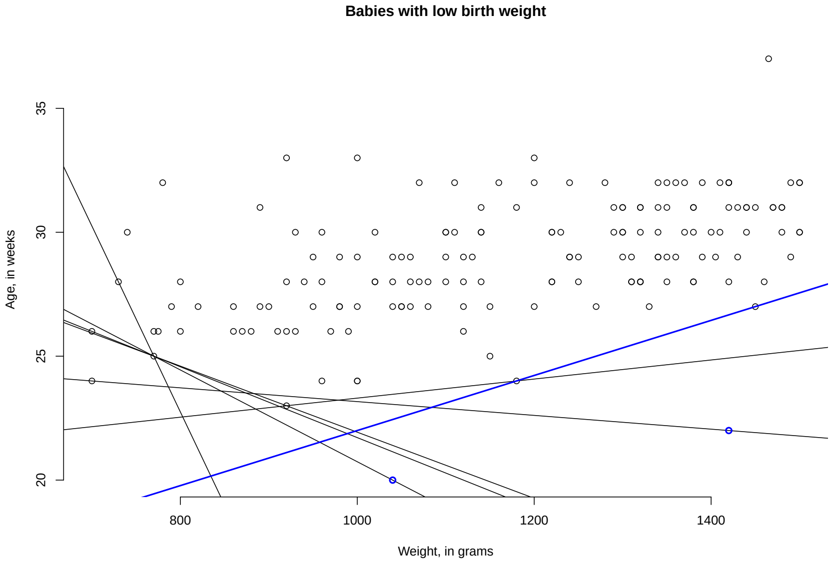

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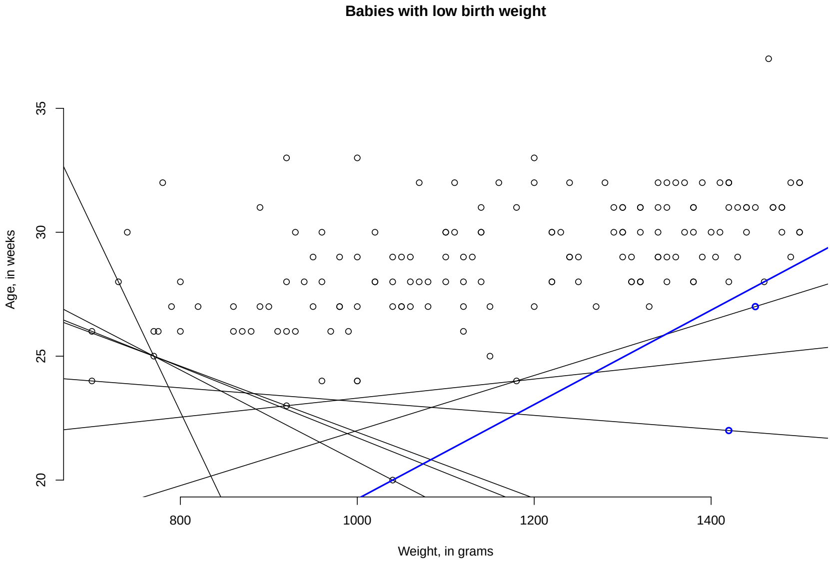

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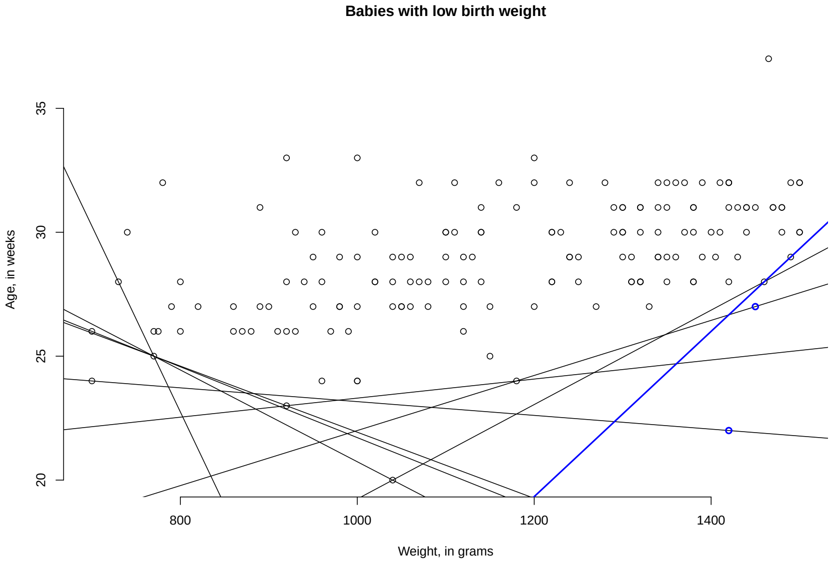

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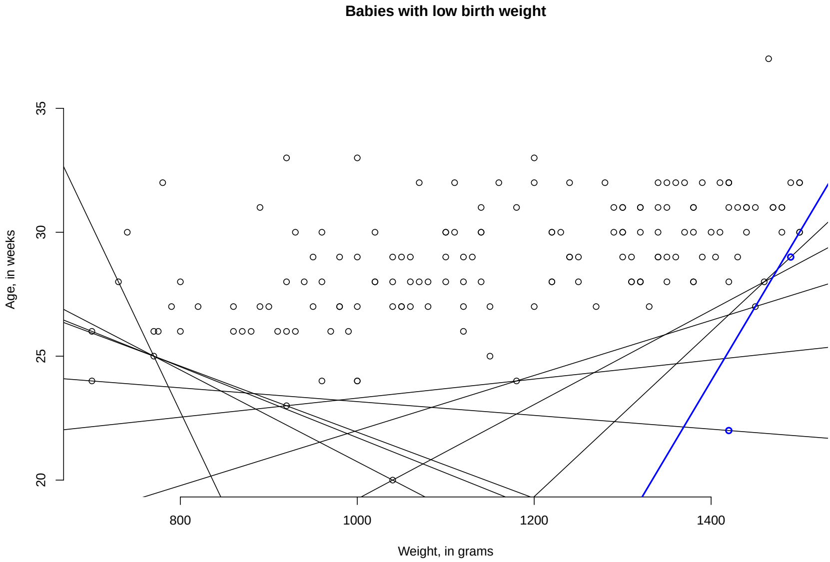

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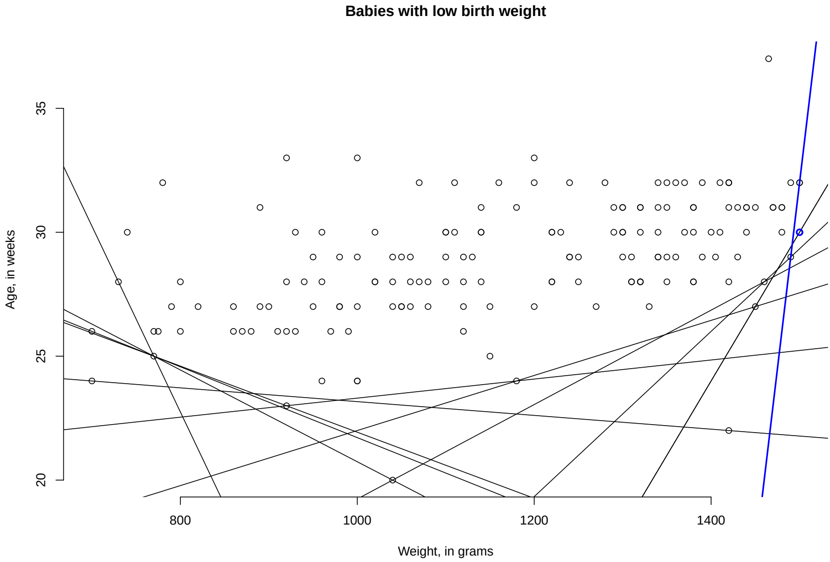

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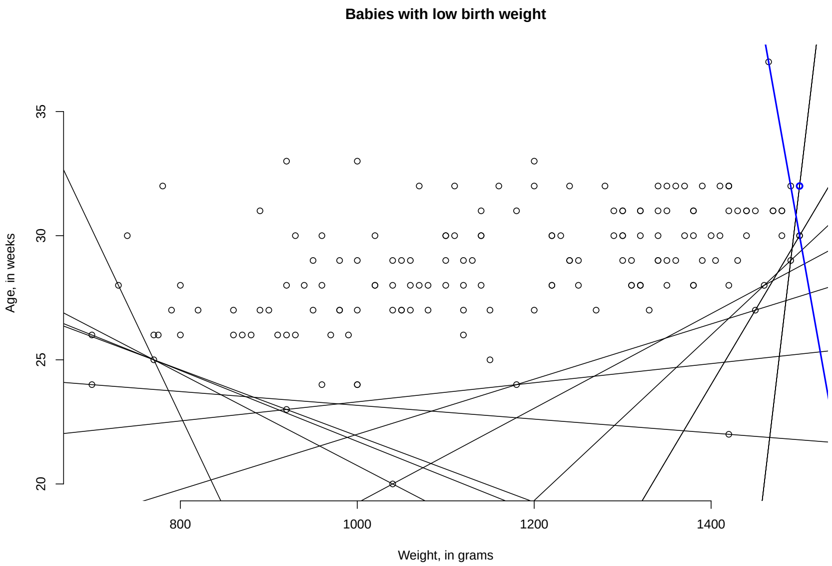

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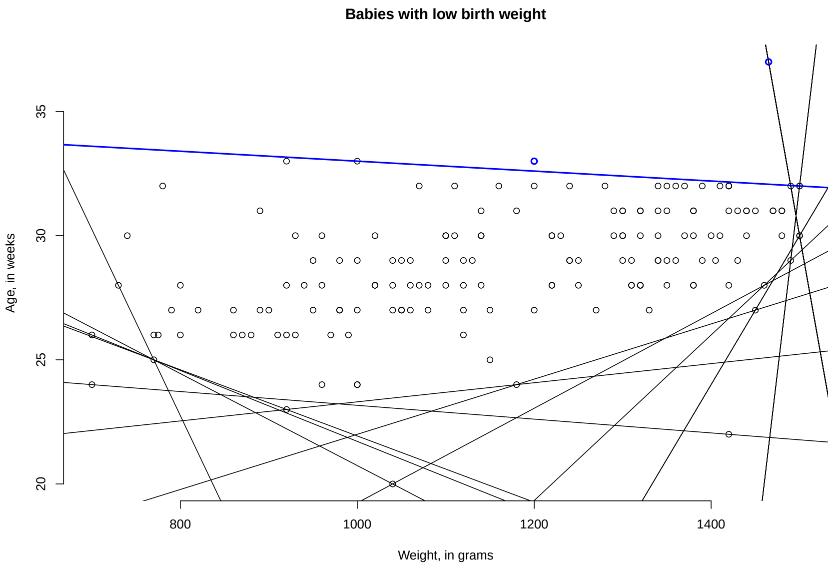

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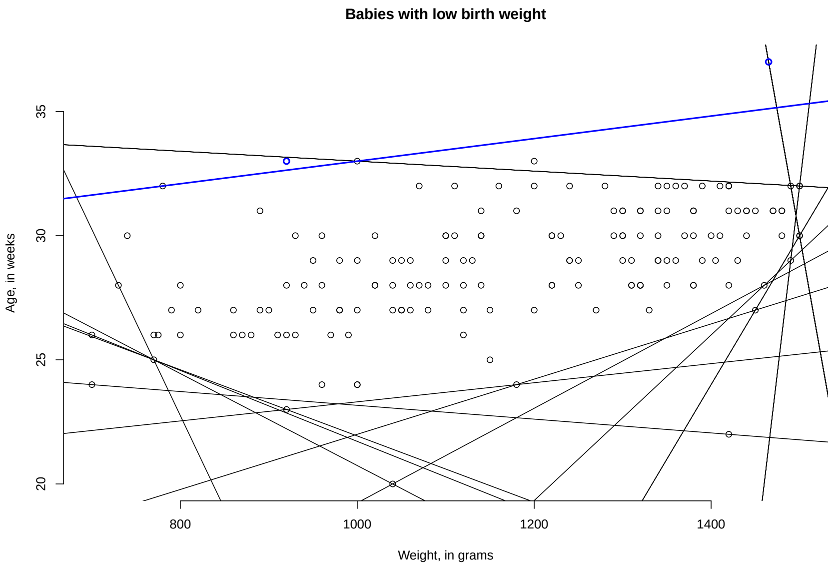

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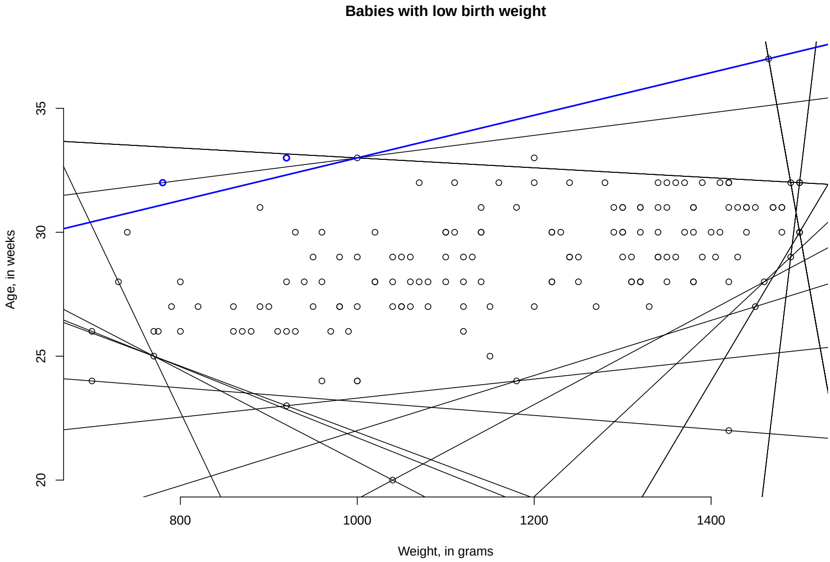

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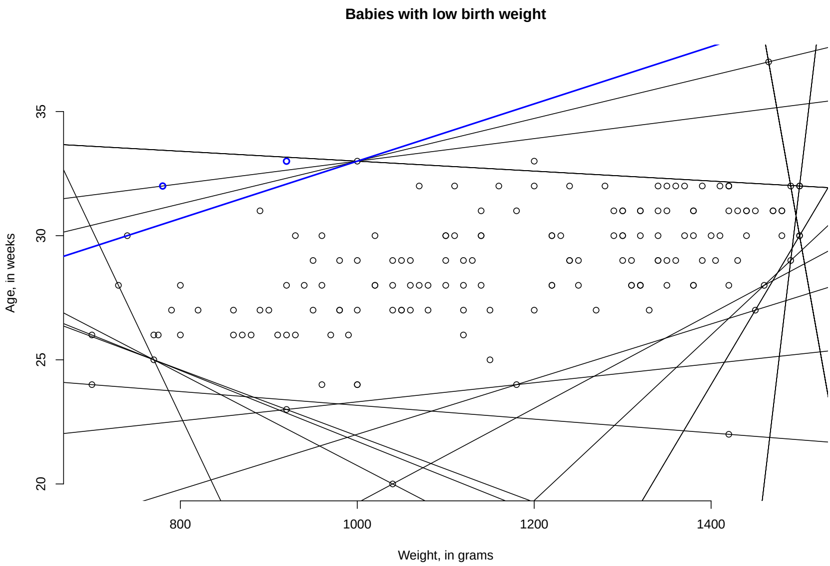

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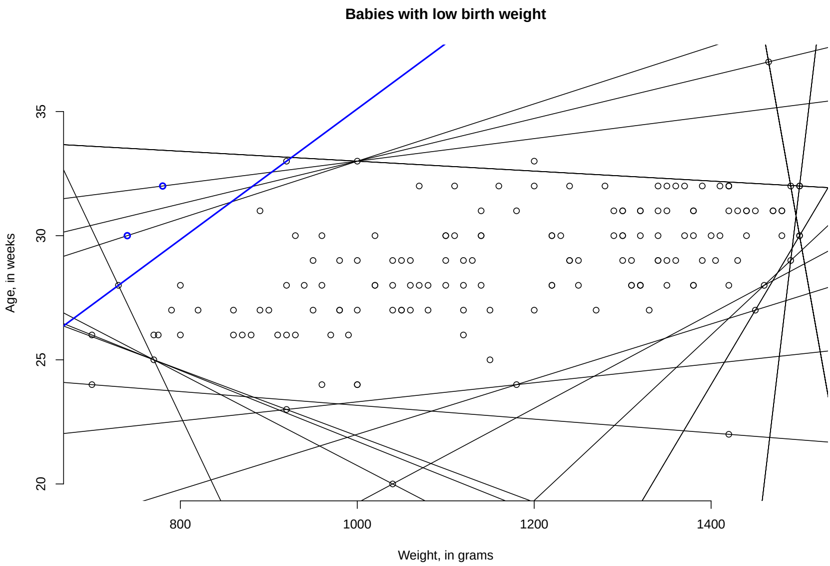

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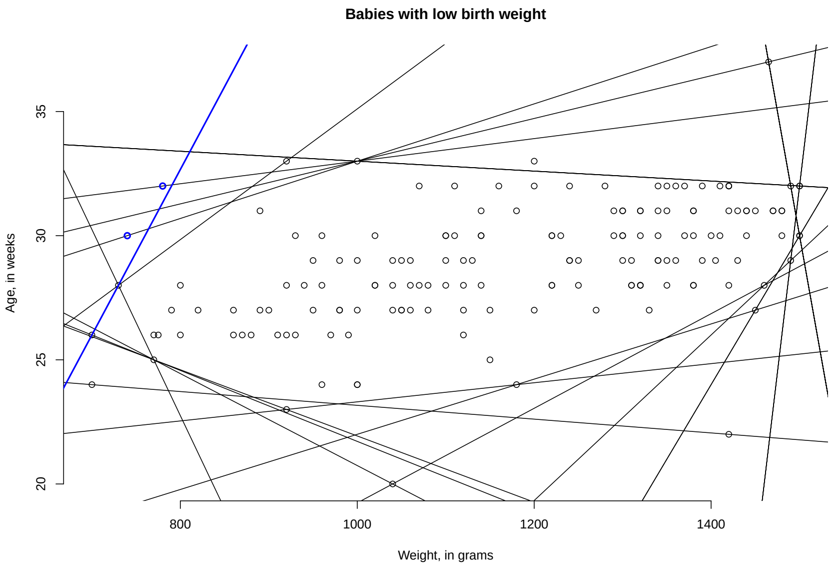

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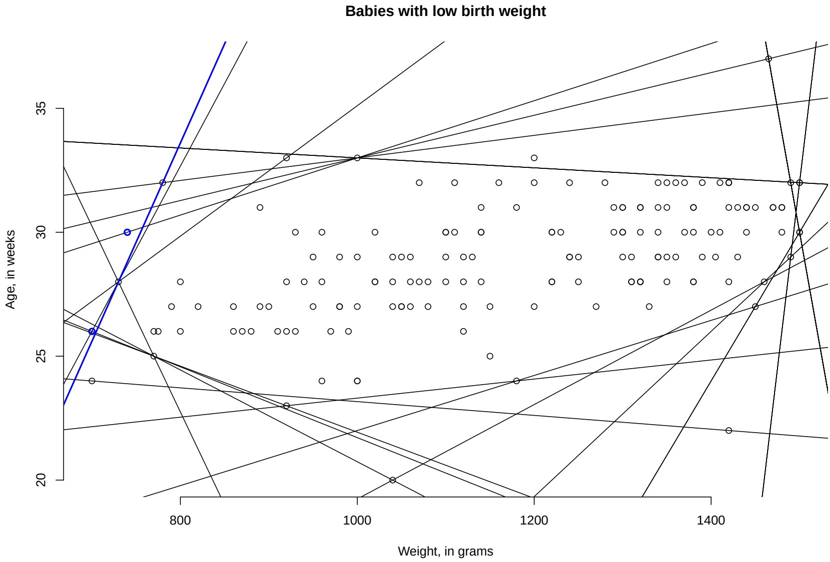

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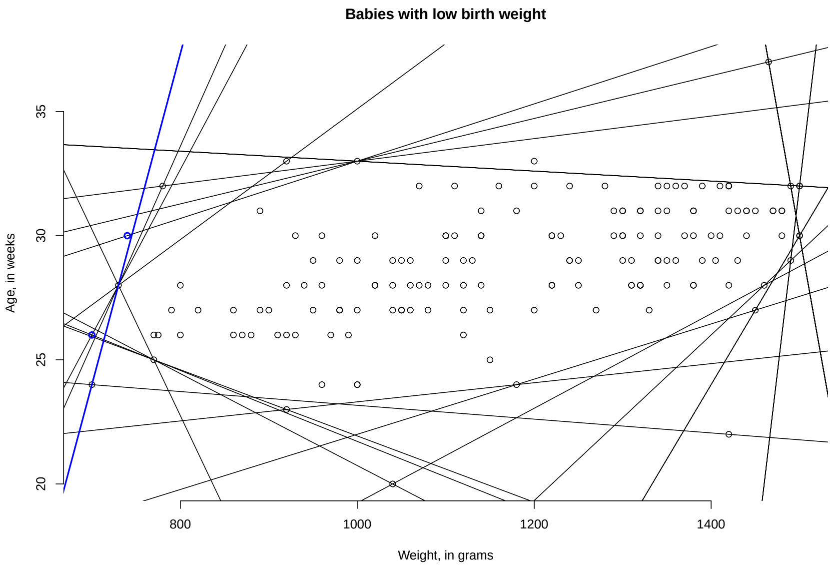

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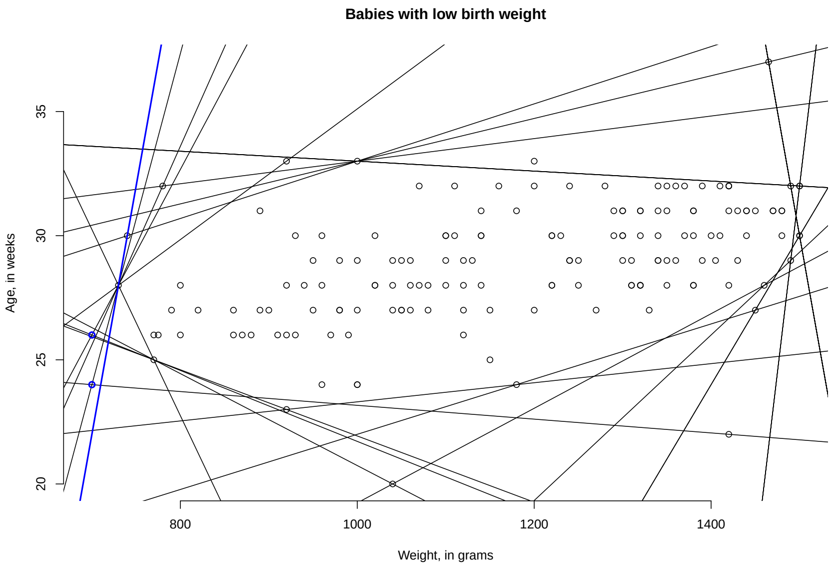

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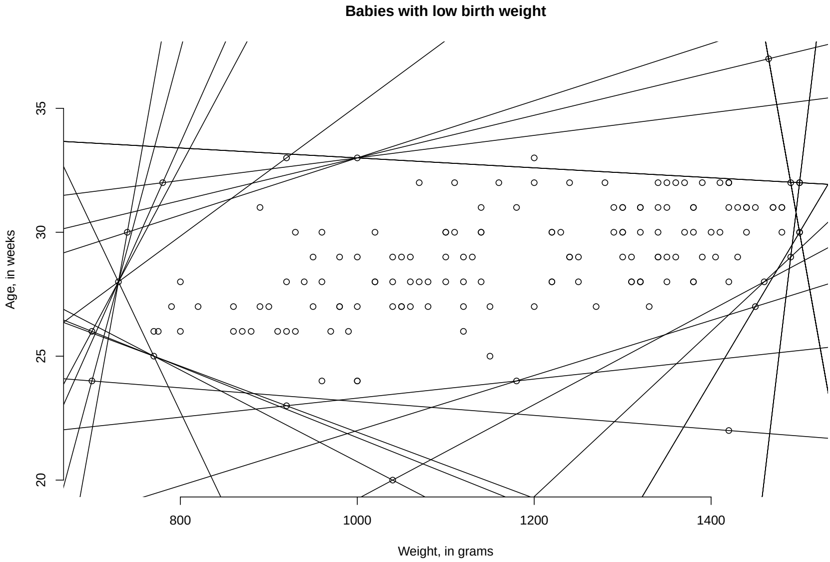

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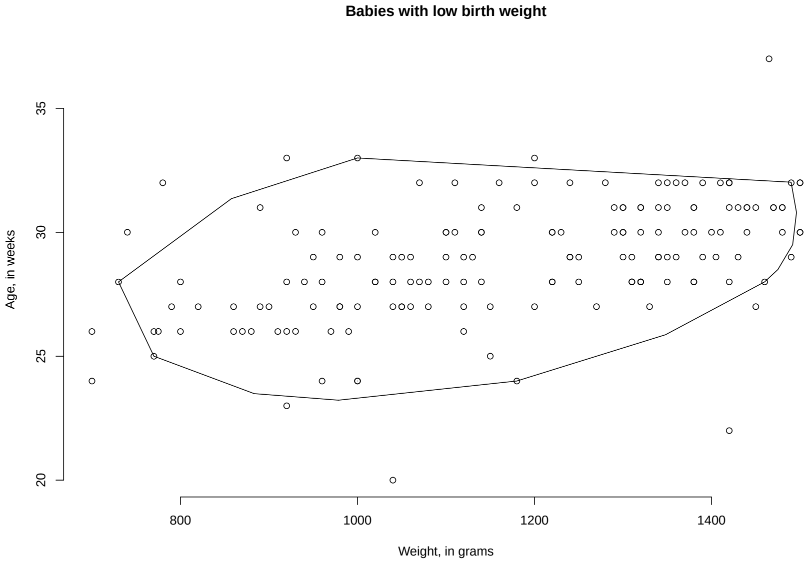

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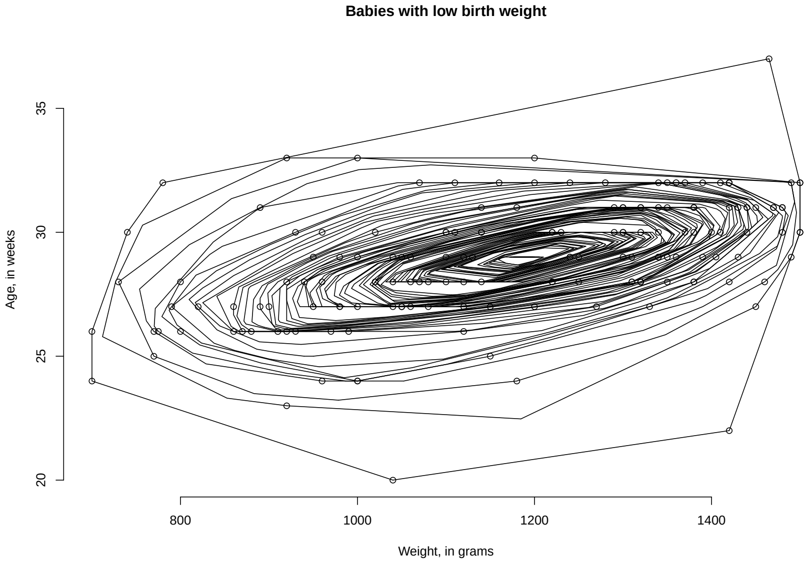

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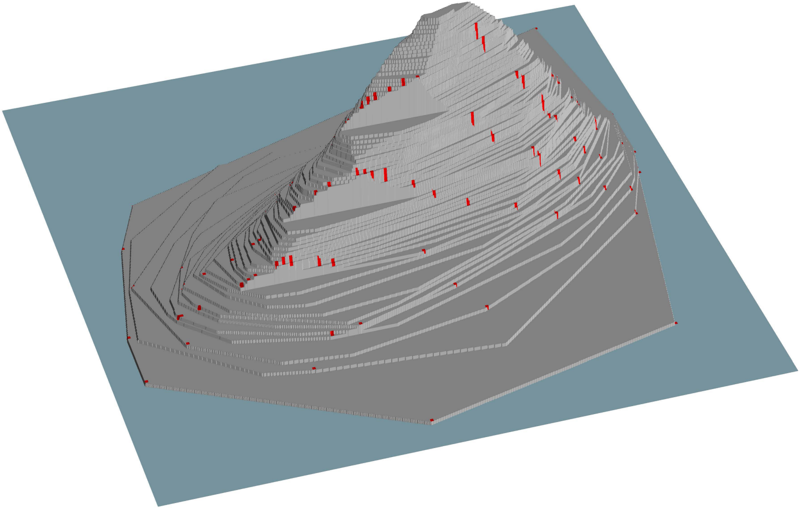

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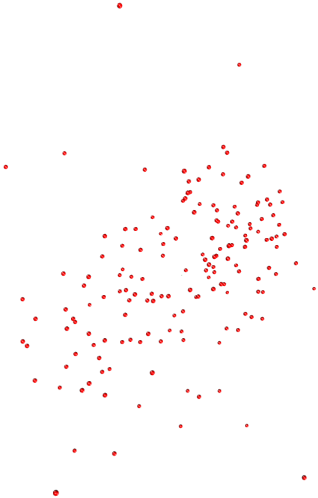
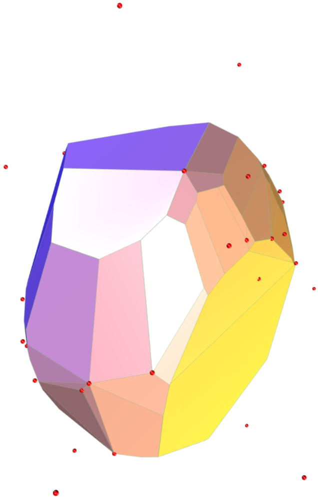


Tukey (=halfspace, location) depth region: $\tau = 2/161$



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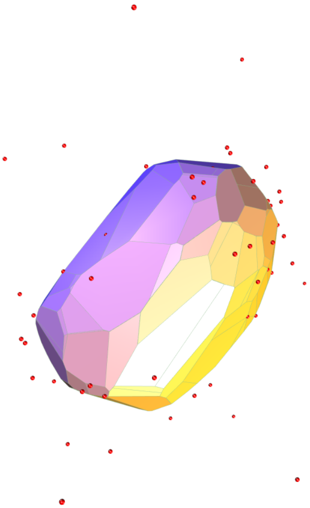

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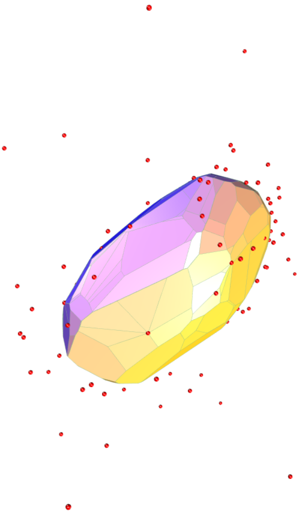

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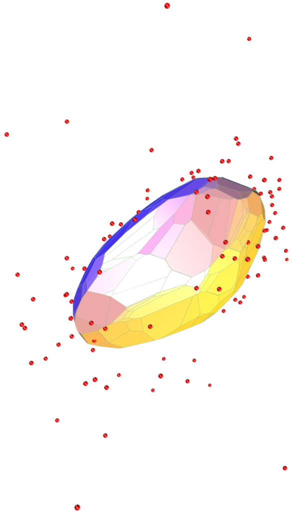

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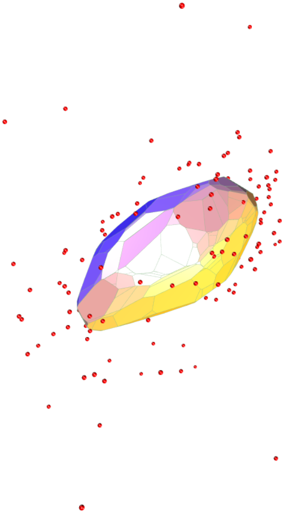
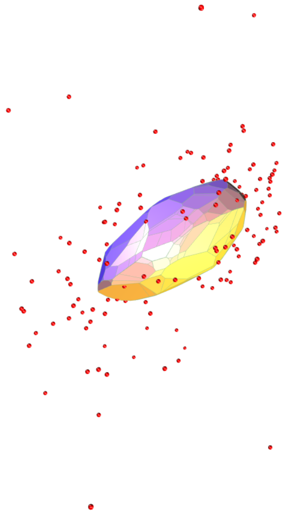


Tukey (=halfspace, location) depth region: $\tau = 25/161$



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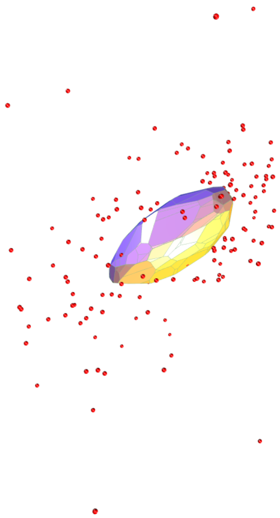

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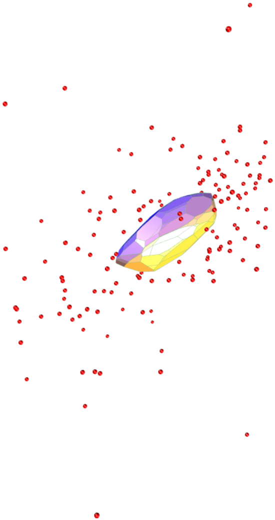
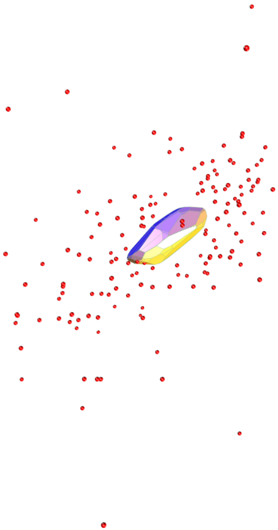


Tukey (=halfspace, location) depth region: $\tau = 49/161$



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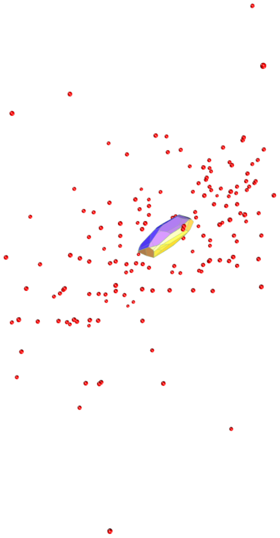
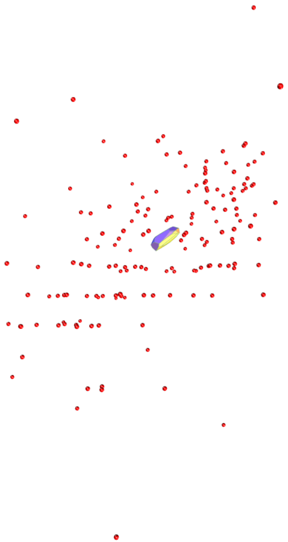


Tukey (=halfspace, location) depth region: $\tau = 65/161$



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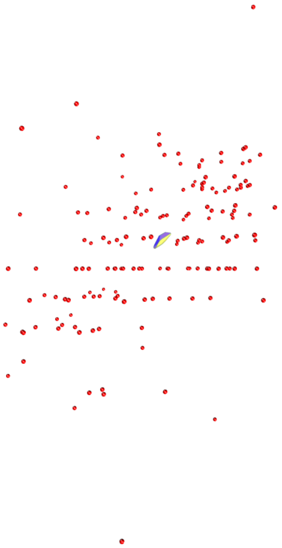

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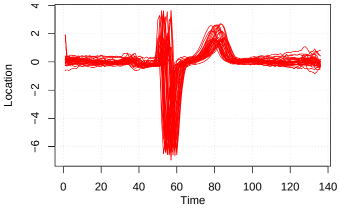
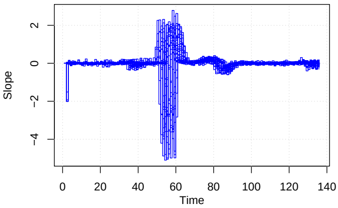
ECG five days data





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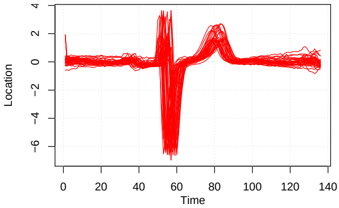

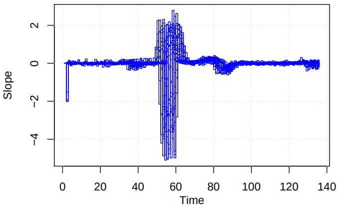

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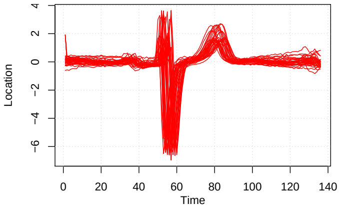

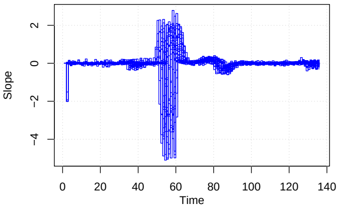

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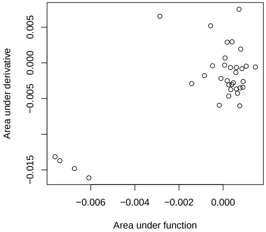

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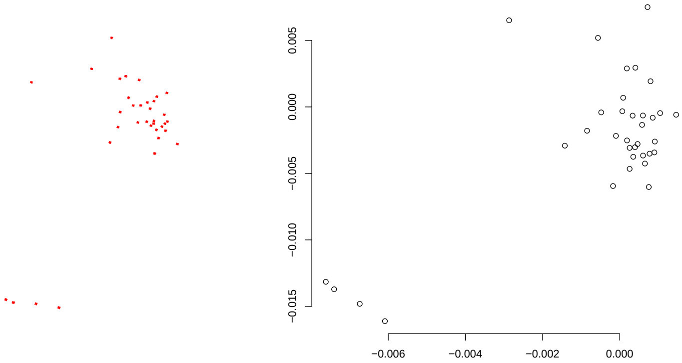

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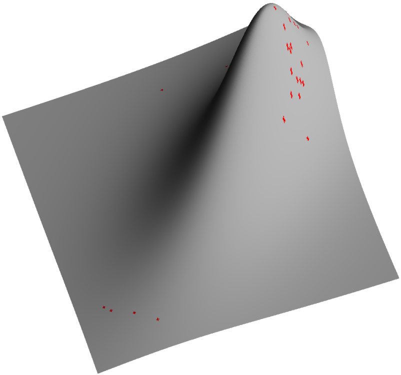
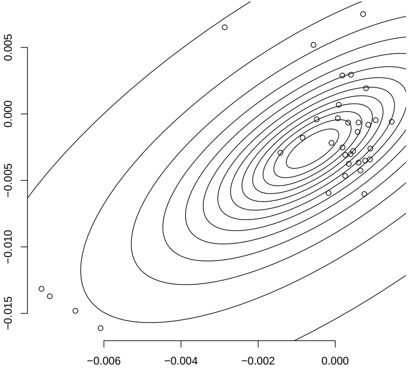
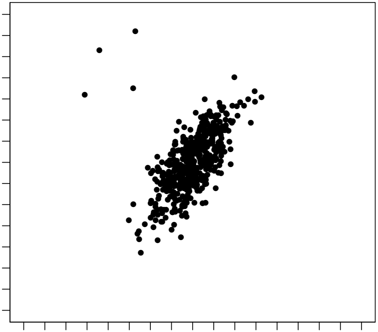




Multivariate anomaly detection: an example



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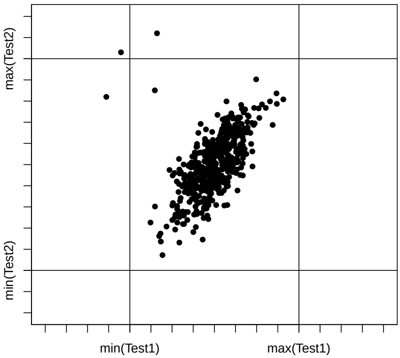

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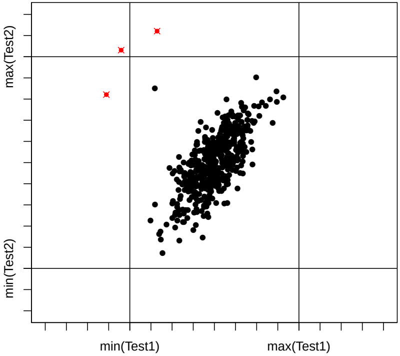

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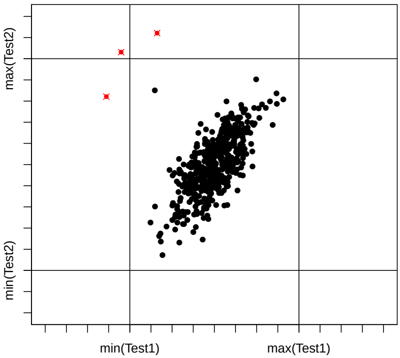

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- **Not all** anomalies can be detected.

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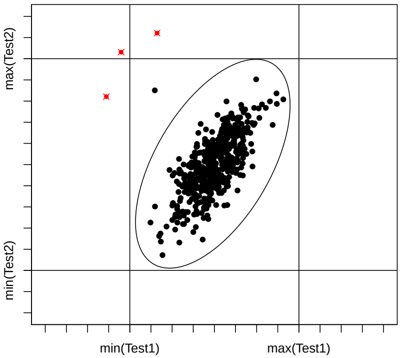

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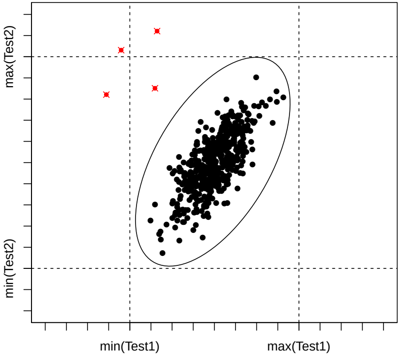

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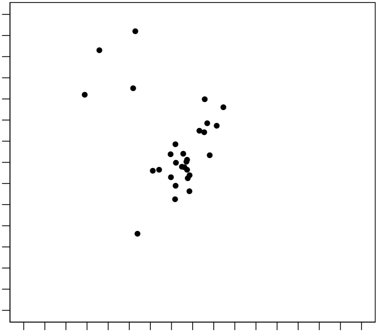

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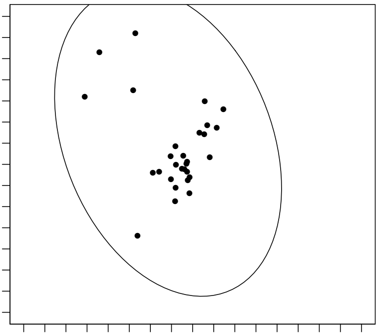

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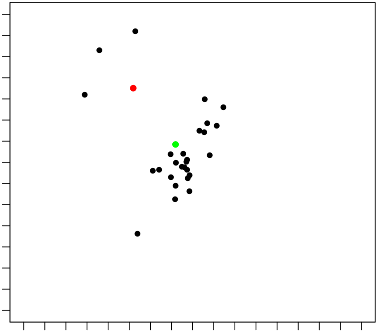

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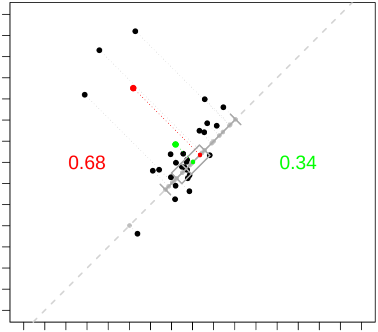

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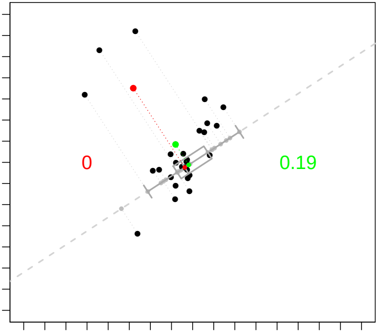

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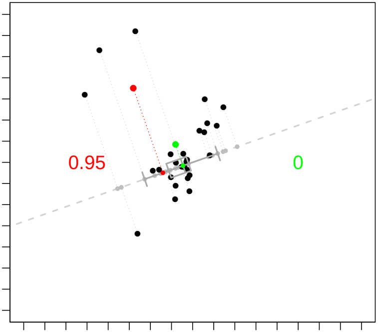

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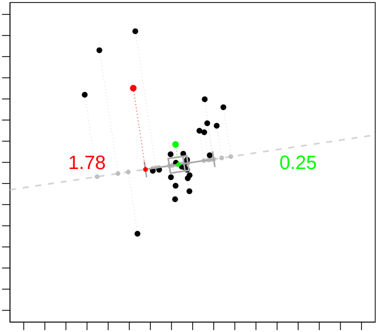

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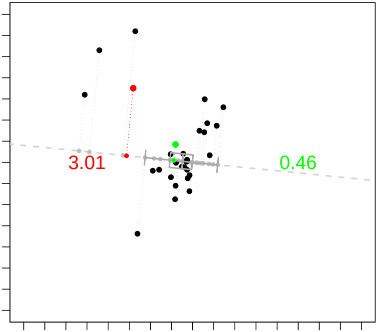

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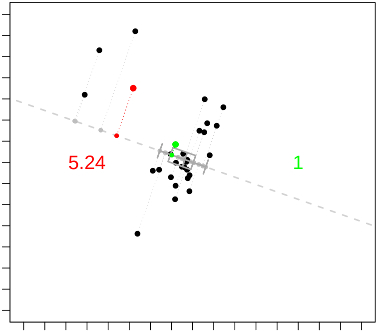

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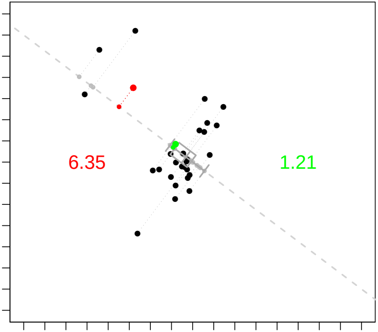

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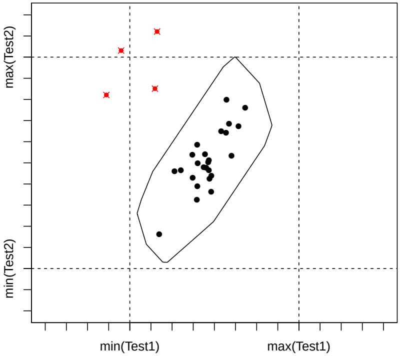

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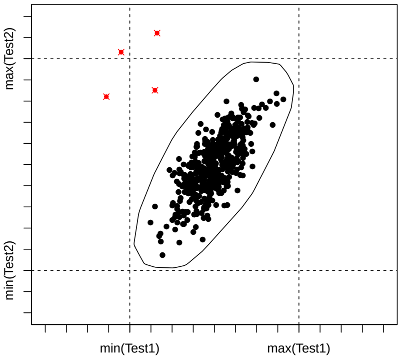

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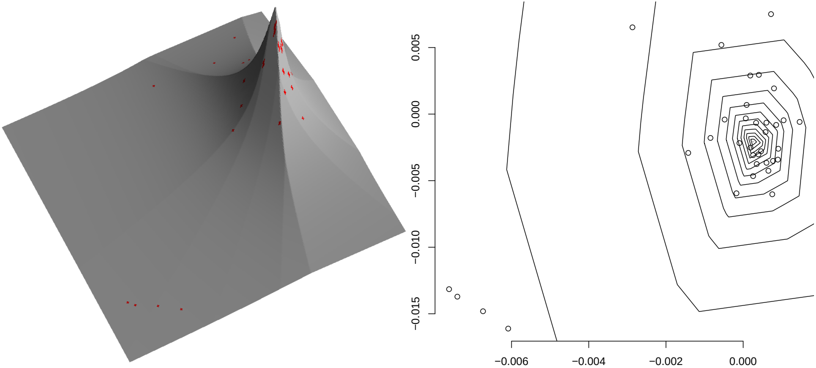

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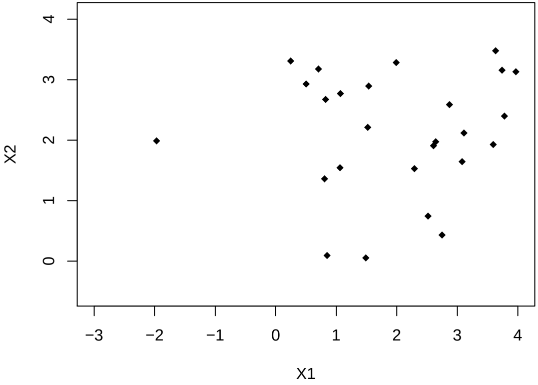

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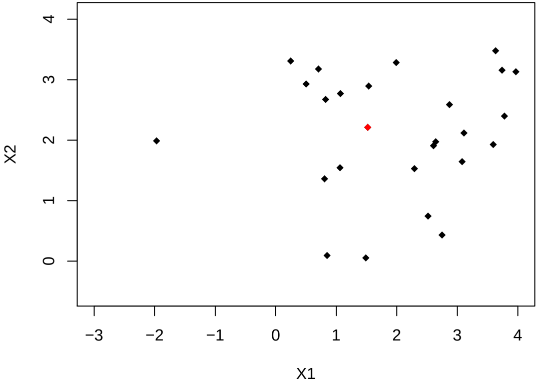

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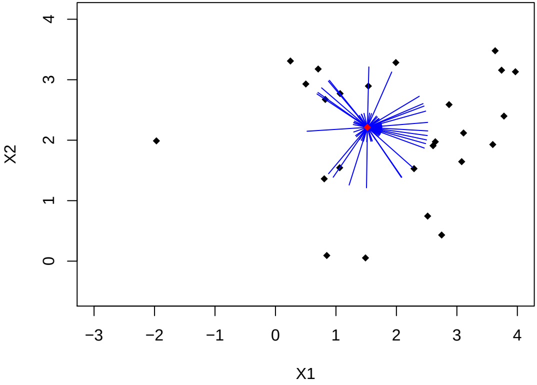

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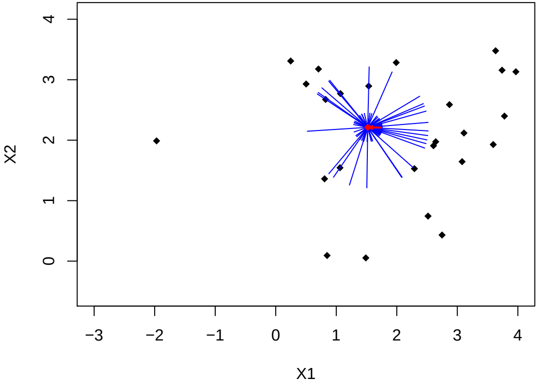

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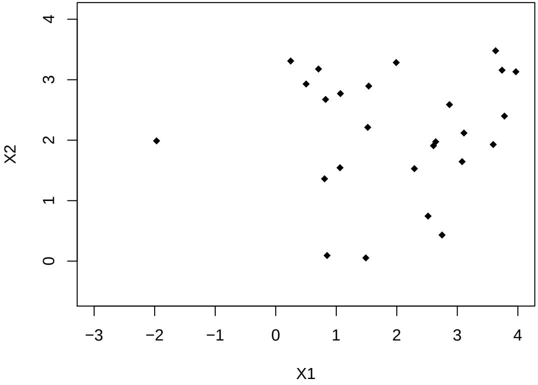

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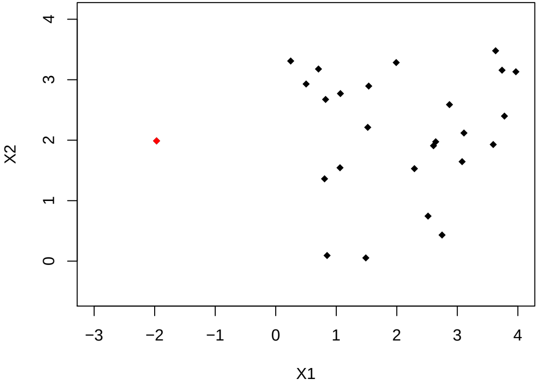

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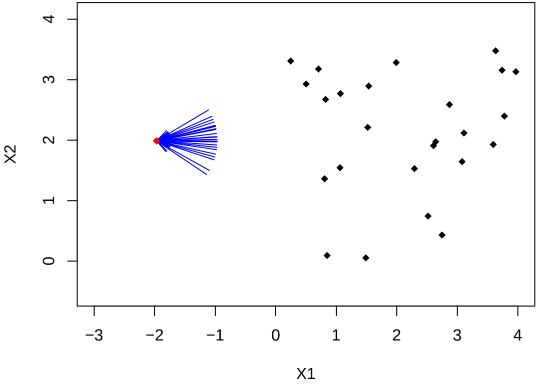

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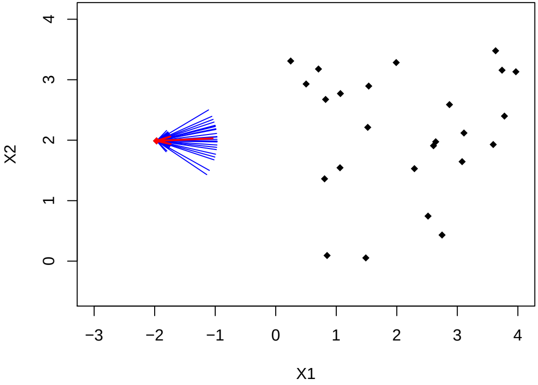

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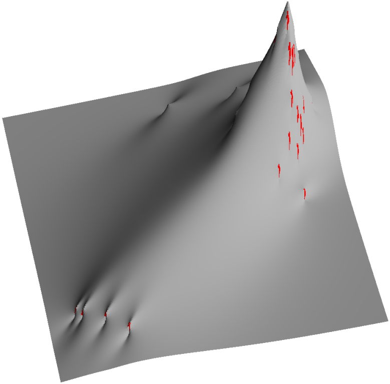

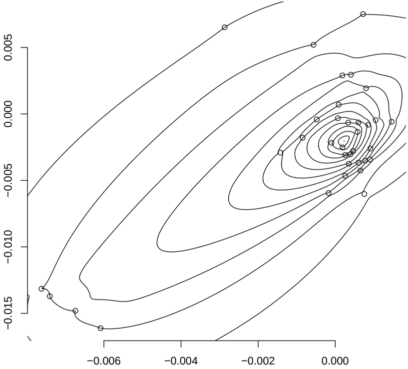

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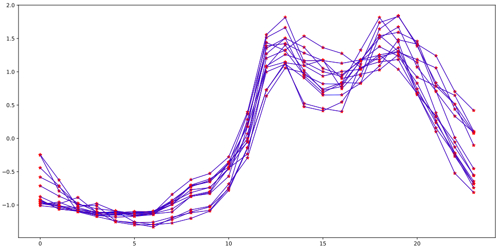
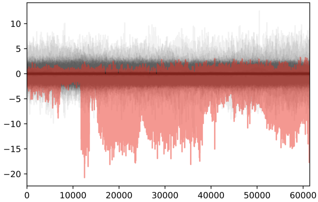
- The first step: reconstruct **functional object** from partial observations (time-series) with interpolation or basis decomposition.

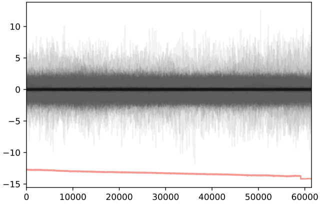


Taxonomy of functional anomalies

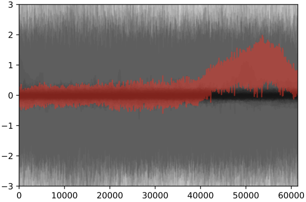
A non-complete taxonomy of functional abnormalities:

Magnitude (=location, shift) anomalies

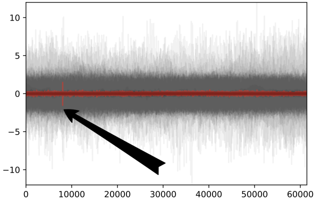




Shape anomalies



Isolated anomalies



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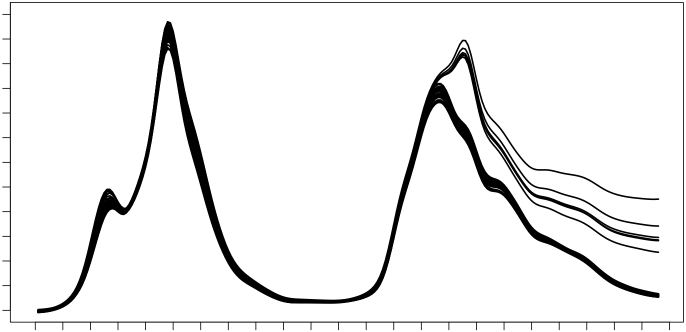

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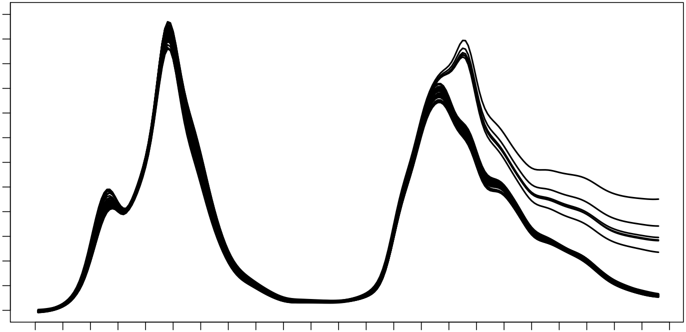

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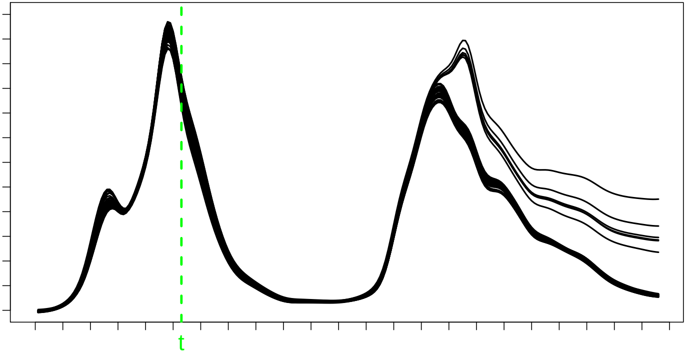

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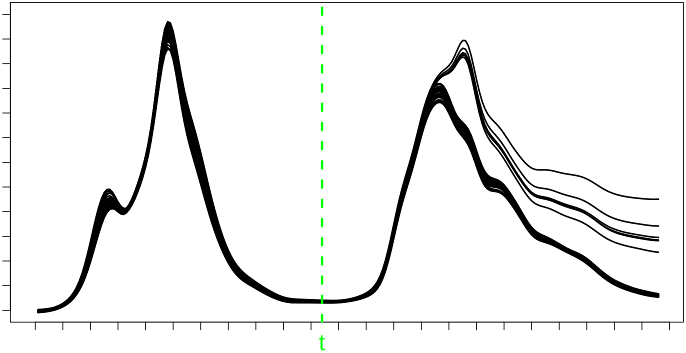

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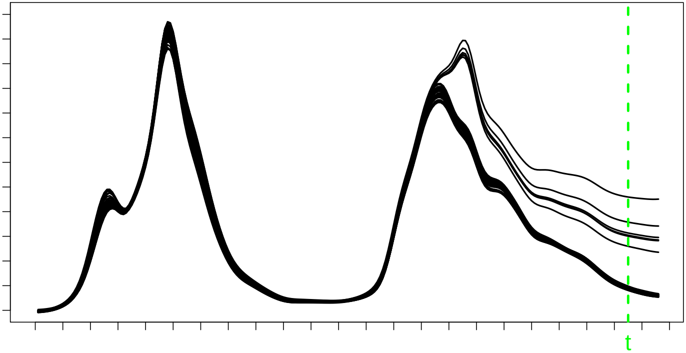

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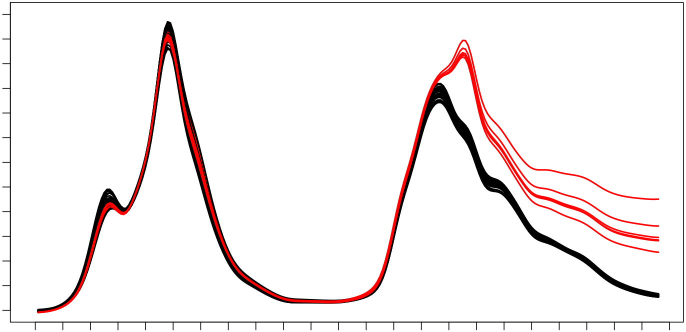

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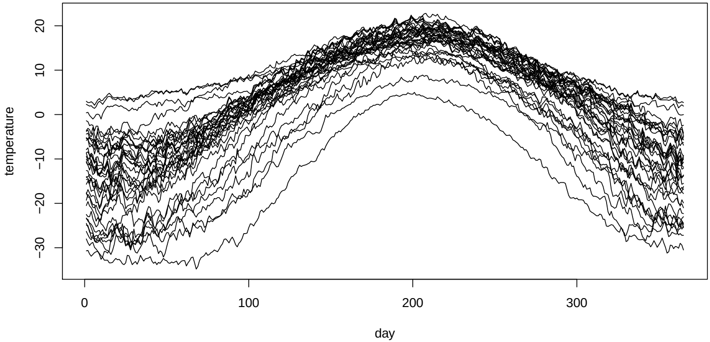
- Label $\boldsymbol{f}$ as anomaly if $D(\boldsymbol{f}|\mathcal{F}) < \min(D)$.

Integrated depth for functional data



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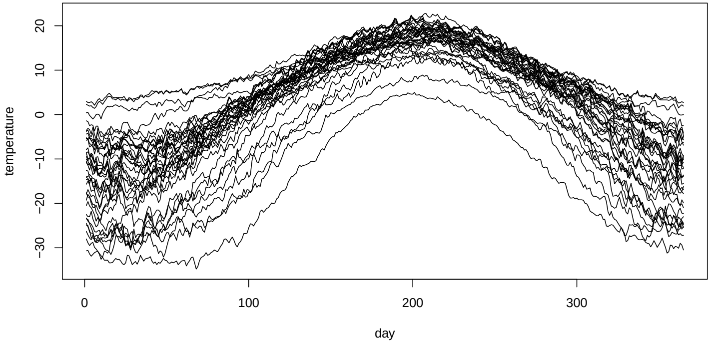

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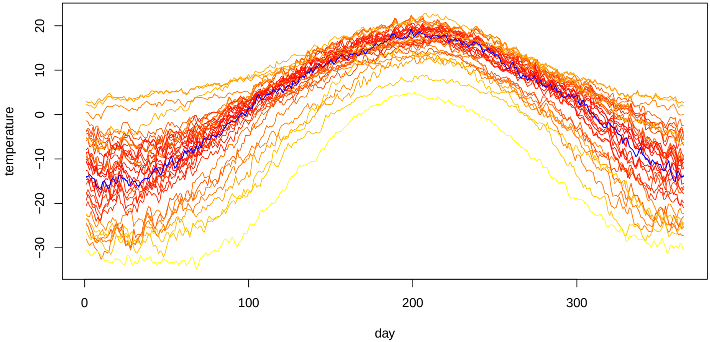

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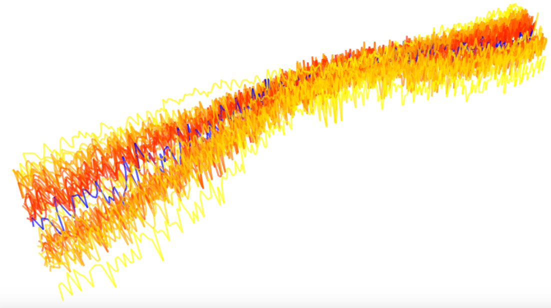
Multivariate functional halfspace depth



Let $\boldsymbol{F}$ be a d-real-valued stochastic process with continuous paths defined on $[0, 1]$, and $\boldsymbol{f}$ its realization. Then:

$$MFD(\boldsymbol{f}|\boldsymbol{F}) = \int_0^1 D\big(\boldsymbol{f}(t)|\boldsymbol{F}(t)\big) \cdot w(t)dt,$$

$$w(t) = w_\alpha\big(t, \boldsymbol{F}(t)\big) = \frac{\text{vol}\big\{D_\alpha\big(\boldsymbol{F}(t)\big)\big\}}{\int_0^1 \text{vol}\big\{D_\alpha\big(\boldsymbol{F}(u)\big)\big\}du}.$$

see Claeskens, Hubert, Slaets, Vakili, 2014.

Contents

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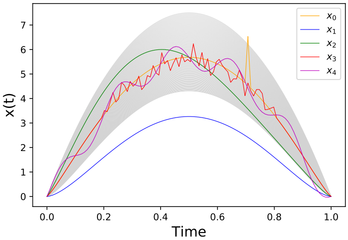
- To account for both location and shape anomalies, we suggest the following **scalar product** that provides a compromise between the both (for $\lambda = 0.5$, Sobolev $W_{1,2}$ scalar product):

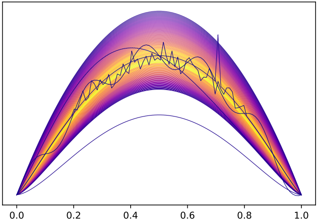
$$\langle \mathbf{f}, \mathbf{d} \rangle := \lambda \times \frac{\langle \mathbf{f}, \mathbf{d} \rangle_{L_2}}{||\mathbf{f}||\,||\mathbf{d}||} + (1-\lambda) \times \frac{\langle \mathbf{f}', \mathbf{d}' \rangle_{L_2}}{||\mathbf{f}'||\,||\mathbf{d}'||}, \quad \lambda \in [0,1]\,.$$

Functional isolation forest: an example

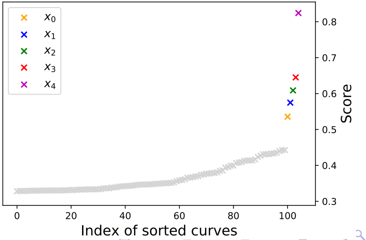
Functional **data set** with **anomalies**



Color-indicated anomaly score



Anomaly score



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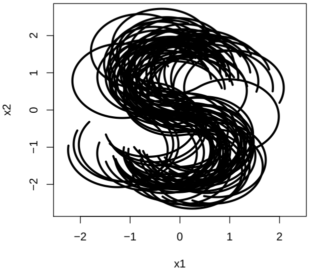
Functional depth: Motivation 1



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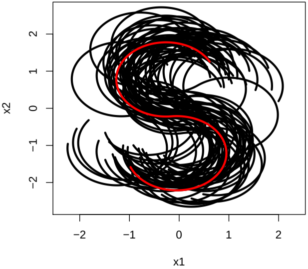

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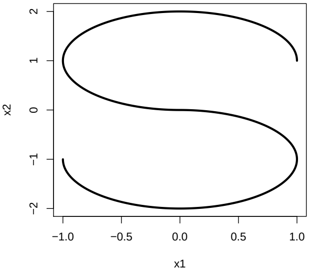

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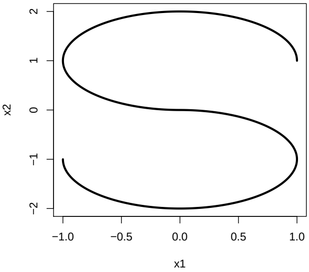

Regard the following different parametrizations of a curve:

Parametrization A:

$$x_1(t) = -\big(\cos(t)+1\big)\mathbb{1}\{t < \tfrac{3\pi}{2}\} - \big(\cos(3t-3\pi)\ +1\big)\mathbb{1}\{t \geq \tfrac{3\pi}{2}\}+1$$
$$x_2(t) = \quad \big(sin(t)+1\big)\mathbb{1}\{t < \tfrac{3\pi}{2}\} - \big(sin(3t-3\pi)+1\big)\mathbb{1}\{t \geq \tfrac{3\pi}{2}\}$$

Parametrization B:

$$x_1(t) = -\big(cos(3t)+1\big)\mathbb{1}\{t < \tfrac{\pi}{2}\} - \big(cos(t+\pi)+1\big)\mathbb{1}\{t \geq \tfrac{\pi}{2}\}+1$$
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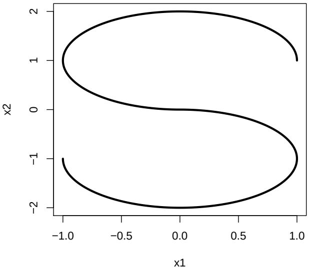
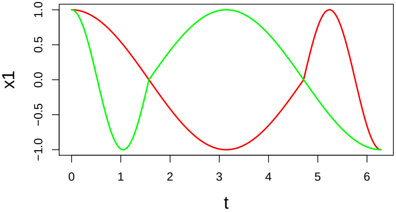
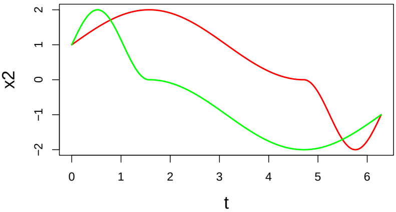
Functional depth: Motivation 1



Regard the following different parametrizations of a curve:





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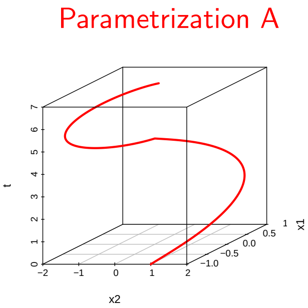
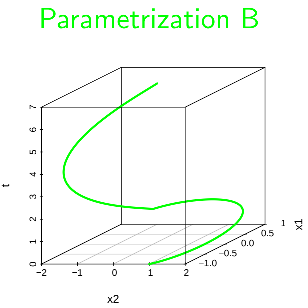
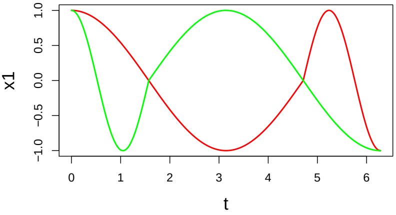
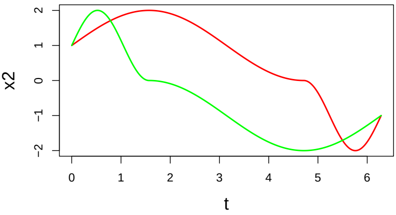




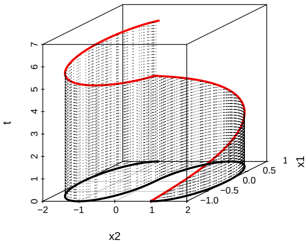
Parametrization:



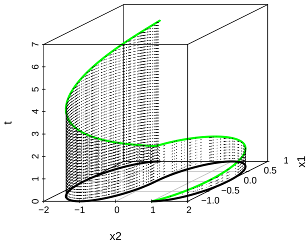


Functional depth: Motivation 1

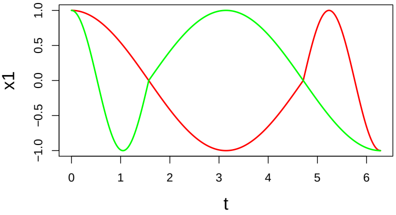
Parametrization A

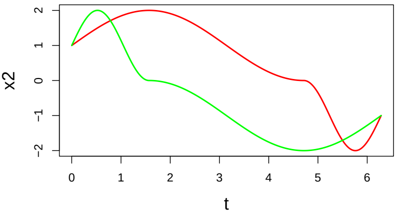


Parametrization B



Parametrization:





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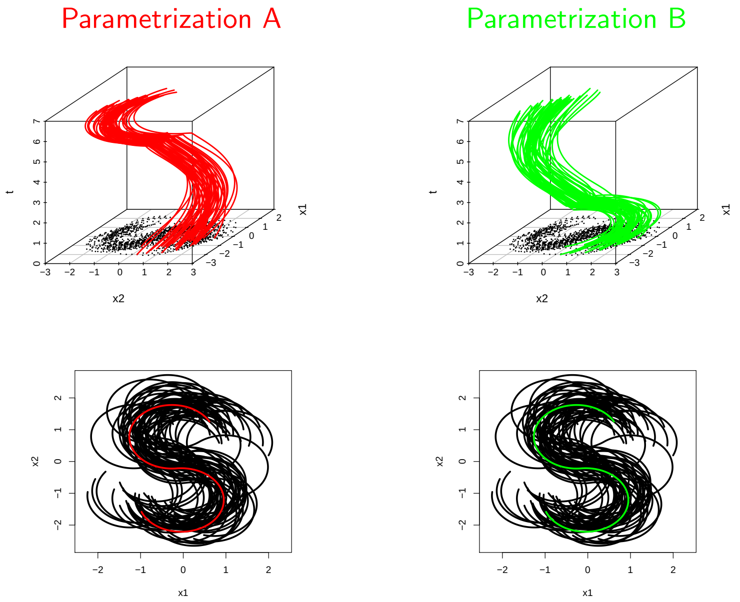

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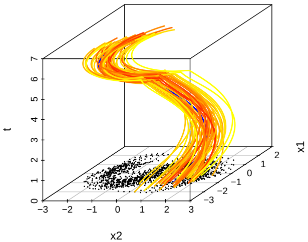
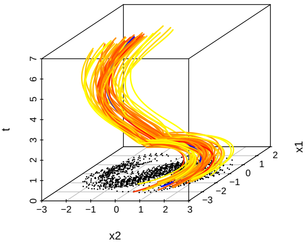
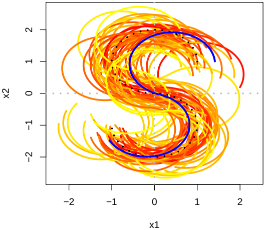
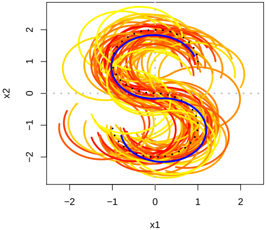


Parametrization B



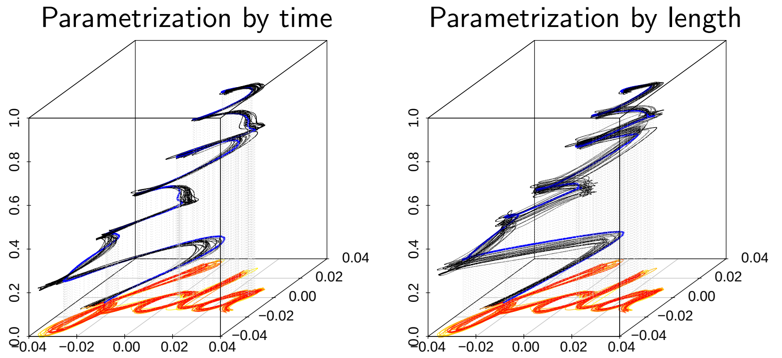
The depth-induced orders differ!





Functional depth: Motivation 2

Functional halfspace depth for the FDA-data



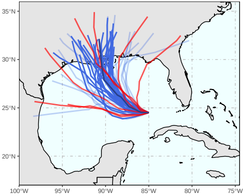
Depth-induced ranking for parametrizations by time and by length:

Time	2	3	13	12	4	8	**1**	17	11	9	7	19	15	20	18	16	14	5	6	10
Length	6	3	16	14	5	7	**13**	11	1	17	2	19	8	20	12	18	15	4	9	10

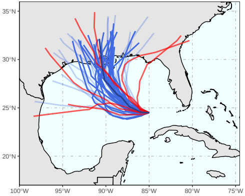
Functional depth: Motivation 3

Simulated hurricane tracks: **curve boxplot**

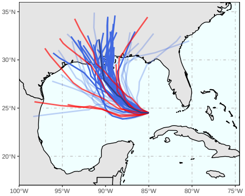
MFH depth – par. time



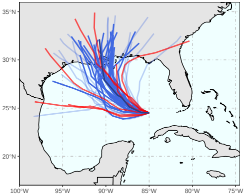
MFH depth - par. length



mSB depth – par. time



mSB depth – par. length



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- We endow $\mathfrak{B}$ with the Fréchet *metric*:

$$d_{\mathfrak{B}}\left(\mathcal{C}_1, \mathcal{C}_2\right) = \inf\left\{\|\beta_1 - \beta_2\|_\infty, \beta_1 \in \mathcal{C}_1,\ \beta_2 \in \mathcal{C}_2\right\}, \quad \mathcal{C}_1, \mathcal{C}_2 \in \mathfrak{B}\,.$$

Associated distribution and the sampling scheme

- Let $\mathcal{C}$ be an unparameterized curve. The *length of* $\mathcal{C}$:

$$L(\mathcal{C}) = \sup_{\tau} \left\{ \sum_{i=1}^{N} |\beta(\tau_i) - \beta(\tau_{i-1})|_2 \ : \tau \text{ is a partition of } [0,1] \right\},$$

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- An unparametrized curve $\mathcal{C}$ is called *rectifiable* if $L(\mathcal{C})$ is finite. The length $L : \mathfrak{B} \to \mathbb{R} + \cup \{\infty\}$ is measurable:

$$\mathcal{P} = \left\{ P \text{ prob. measure on } (\mathfrak{B}, d_{\mathfrak{B}}) \ : \ P(\{\mathcal{C} \in \mathfrak{B}; 0 < L(\mathcal{C}) < \infty\}) = 1 \right\}.$$

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- We derive the probability distribution Q_P on $\mathbb{R}^d$ as follows: if $X \sim Q_P$, then distribution of $X \,|\, \mathcal{X} = \mathcal{C}$ is the (uniform on $\mathcal{C}$) probability distribution $\mu_{\mathcal{C}}$:

$$\mu_{\mathcal{C}}(A) = \int_{\mathcal{C}} \mathbb{1}_A(x) dx.$$

Associated distribution and the sampling scheme

The statistical model:

$$\mathcal{X}_1, \ldots, \mathcal{X}_n \text{ i.i.d. from } P.$$

For Monte-Carlo estimation, we can consider the following **sampling scheme**:

$$\begin{cases} \mathcal{X}_1, \ldots, \mathcal{X}_n \text{ i.i.d. from } P, \\ \text{for all } i = 1, \ldots, n \\ \qquad X_{i,1}, \ldots, X_{i,m} \text{ i.i.d. from } \mu_{\mathcal{X}_i}. \end{cases}$$

Data depth for an unparametrized curve

Definition

The **Tukey curve depth** of $\mathcal{C} \in \mathfrak{B}$ w.r.t. Q_P is defined as:

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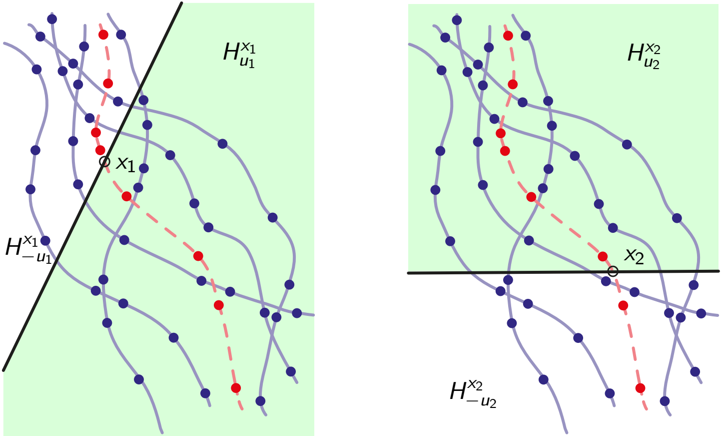
Data depth for an unparametrized curve: intuition



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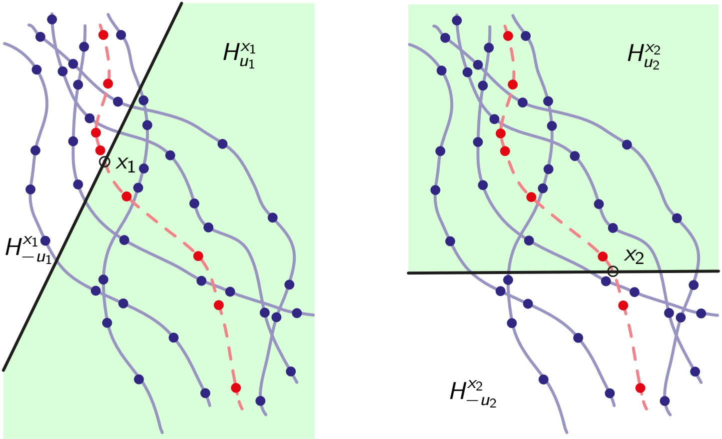

Traditional reasoning:

$\widehat{Q}_P(H^{x_1}_{u_1}) = \frac{25}{40}$, $\widehat{\mu}_{\mathcal{C}}(H^{x_1}_{u_1}) = \frac{4}{8}$

$\widehat{Q}_P(H^{x_1}_{-u_1}) = \frac{15}{40}$, $\widehat{\mu}_{\mathcal{C}}(H^{x_1}_{-u_1}) = \frac{4}{8}$

Curve-based reasoning:

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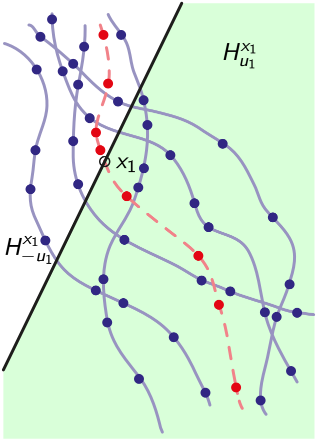

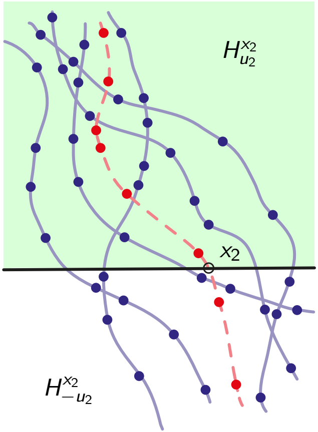

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Data depth for an unparametrized curve: empirical version

- Let a chosen curve consist of two (independently drawn on $\mathcal{C}$) parts $\mathbb{Y}_{1,m} = (Y_{1,1}, \ldots, Y_{1,m})$ and $\mathbb{Y}_{2,m} = (Y_{2,1}, \ldots, Y_{2,m})$ with empirical distribution

$$\widehat{\mu}_m = \frac{1}{m} \sum_{i=1}^{m} \delta_{Y_{1,i}} \,,$$

where $\delta_{\boldsymbol{x}}$ is the Dirac measure in $\boldsymbol{x} \in \mathbb{R}^d$.

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- Let $\widehat{Q}_{n,m}$ be the empirical distribution (observed sample) $\mathbb{X}_{n,m} = \{X_{i,j},\ i = 1, ..., n,\ j = 1, ..., m\}$

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- To compute the sample Tukey curve depth, a Monte Carlo approximation is used.

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- Let H be a closed halfspace in $\mathbb{R}^d$ and $\mathcal{H}_{\Delta}^{n,m}$ denote a collection of such halfspaces such that for all $H \in \mathcal{H}_{\Delta}^{n,m}$ either $\widehat{Q}_{n,m}(H) = 0$ or $\widehat{\mu}_m(H) > \Delta$, almost surely, for $\Delta \in (0, \frac{1}{2})$.

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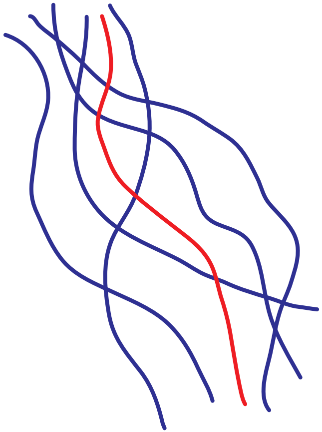
$$\widehat{D}_{n,m,\Delta}(\mathcal{C}|\mathcal{X}_1, ..., \mathcal{X}_n) = \frac{1}{m} \sum_{i=1}^{m} \widehat{D}(Y_{2,i}|\widehat{Q}_{n,m}, \widehat{\mu}_m, \mathcal{H}_{\Delta}^{n,m}) \,,$$

with the depth of an arbitrary point $\boldsymbol{x} \in \mathbb{R}^d$ w.r.t. $\widehat{Q}_{n,m}$ being:

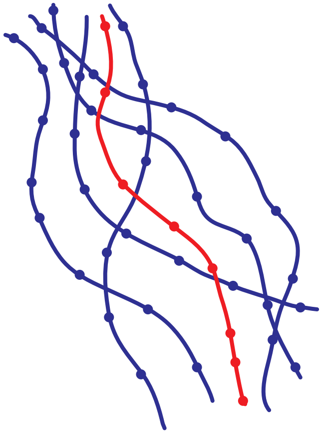
$$\widehat{D}(\boldsymbol{x}|\widehat{Q}_{n,m}, \widehat{\mu}_m, \mathcal{H}_{\Delta}^{n,m}) \;=\; \inf\Big\{\frac{\widehat{Q}_{n,m}(H)}{\widehat{\mu}_m(H)} \;:\; H \in \mathcal{H}_{\Delta}^{n,m}, \, \boldsymbol{x} \in \partial H\Big\}$$

and $\frac{0}{0} = 0$ as before.

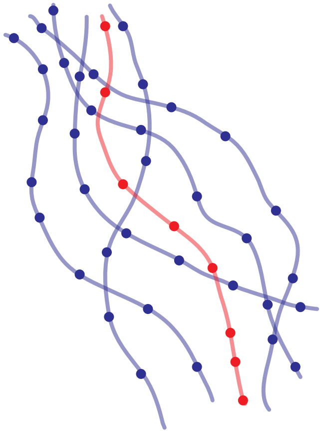
Calculation of the Tukey curve depth



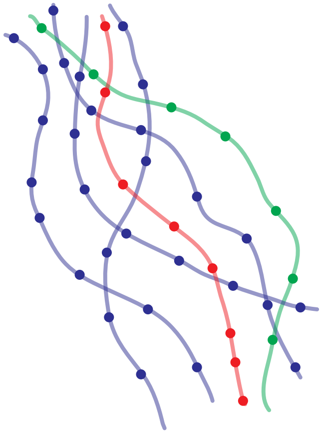
Calculation of the Tukey curve depth



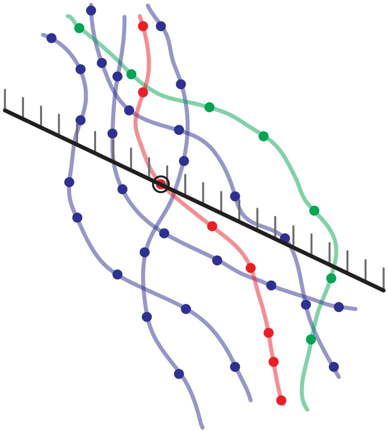
Calculation of the Tukey curve depth



Calculation of the Tukey curve depth



Calculation of the Tukey curve depth



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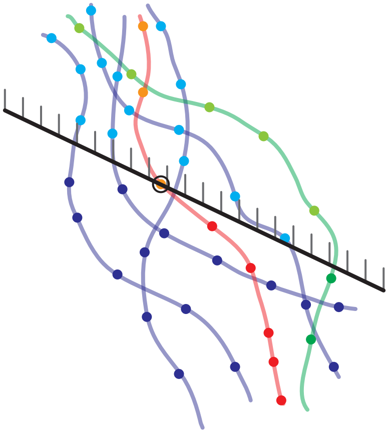
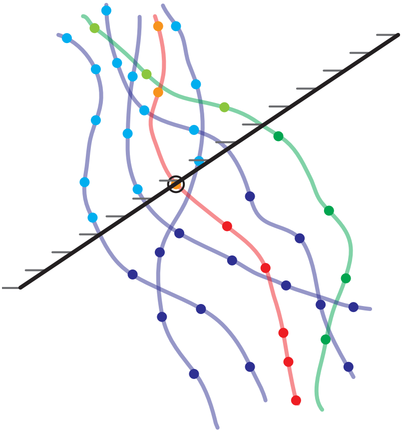


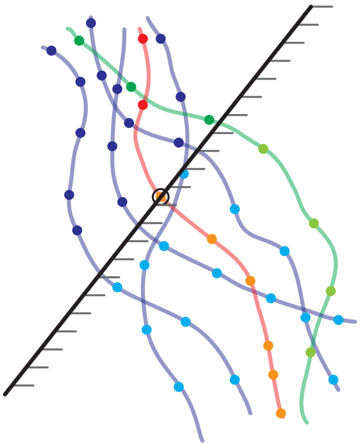
$$D(\mathbb{Y}_{2,c}|Q_m, \mathcal{H}_{m,b}) = \frac{\frac{1}{5}\left(\frac{5}{7}+\frac{3}{8}+\frac{6}{8}+\frac{2}{7}+\frac{3}{6}\right)}{\frac{2}{8}} = 2.1$$

Calculation of the Tukey curve depth



$$D(\mathbb{Y}_{2,c}|Q_m, \mathcal{H}_{m,b}) = \frac{\frac{1}{5}\left(\frac{3}{7}+\frac{5}{8}+\frac{4}{8}+\frac{3}{7}+\frac{3}{6}\right)}{\frac{2}{8}} = 1.9857$$

Calculation of the Tukey curve depth



$$D(\mathbb{Y}_{2,c}|Q_m, \mathcal{H}_{m,b}) = \frac{\frac{1}{5}\left(\frac{4}{7}+\frac{3}{8}+\frac{4}{8}+\frac{4}{7}+\frac{4}{6}\right)}{\frac{5}{8}} = 0.7159$$

Data depth for an unparametrized curve: properties

Restrict to $\mathfrak{B}_\ell$, the subset of unparametrized curves of positive length bounded by $\ell > 0$. Then the Tukey curve depth satisfies the following properties:

- **Nonnegativity and boundedness by one:**

$$D(\mathcal{C}|Q_P) \in [0, 1] \,.$$

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- **Similarity invariance:** Let $f : \mathbb{R}^d \to \mathbb{R}^d f$ be a similarity, i.e. there exists an orthogonal matrix A, a factor $r > 0$ and a vector $\boldsymbol{b} \in \mathbb{R}^d$ such that for all $\boldsymbol{x} \in \mathbb{R}^d$, $f(\boldsymbol{x}) = rA\boldsymbol{x} + \boldsymbol{b}$. In particular for all $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{R}^d$, $|f(\boldsymbol{x}) - f(\boldsymbol{y})|_2 = r|\boldsymbol{x} - \boldsymbol{y}|_2$. Denote by P_f the distribution of the image through f of a stochastic process having a distribution P. Then

$$D(f \circ \mathcal{C}|Q_{P_f}) = D(\mathcal{C}|Q_P)\,.$$

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$$D(f \circ \mathcal{C}|Q_{P_f}) = D(\mathcal{C}|Q_P)\,.$$

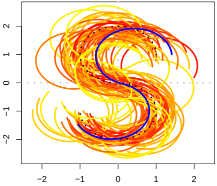
- **Vanishing at infinity:**

$$\lim_{d_{\mathbb{G}}(\mathcal{C},\mathbf{0}) \to \infty, \mathcal{C} \in \mathfrak{B}_\ell} D(\mathcal{C}, Q_P) = \inf_{\mathcal{C} \in \mathfrak{B}_\ell} D(\mathcal{C}, Q_P) = 0\,.$$

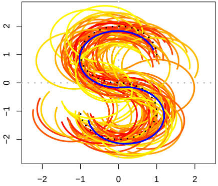
Comparison with functional depth: Example 1

Simulated S letters: **depth-induced ranking**

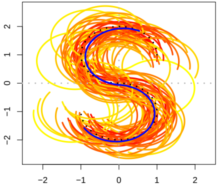
MFHD – time



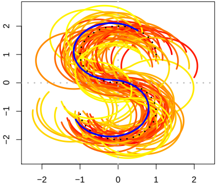
MFHD - length



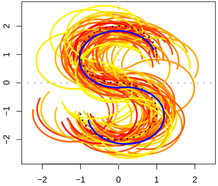
Curve depth



mSBD – time



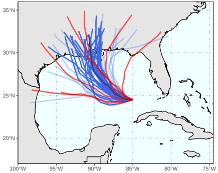
mSBD – length



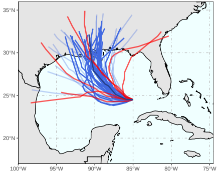
Comparison with functional depth: Example 2

Simulated hurricane tracks: **curve boxplot**

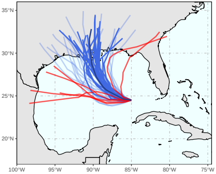
MFHD – time



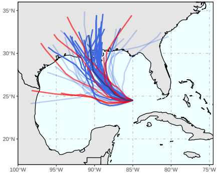
MFHD - length



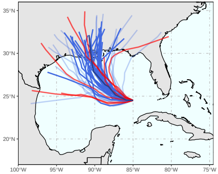
Curve depth



mSBD – time

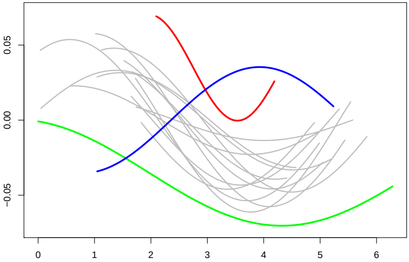


mSBD – length



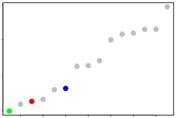
Comparison with functional depth: Anomaly detection 1

Data set 1 with introduced anomalies:

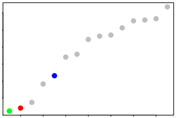


Ordered depth values:

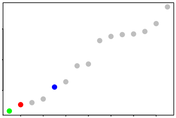
mSBD



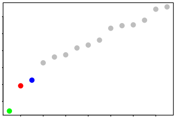
saPRJ



MFHD

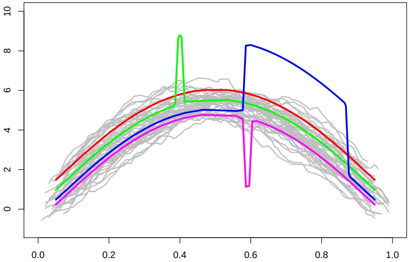


Curve depth



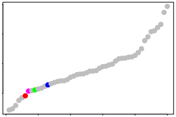
Comparison with functional depth: Anomaly detection 2

Data set 2 with introduced anomalies:

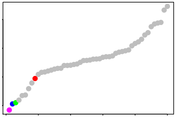


Ordered depth values:

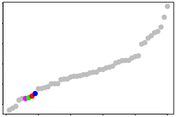
mSBD



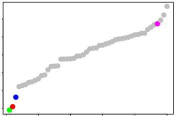
saPRJ



MFHD



Curve depth



Contents

Thank you for attention! (and a short list of literature)

- Chandola, V., Banerjee, A., and Kumar, V. (2009). Anomaly detection: A survey. ACM *Computing Surveys (CSUR)*, 41(3):15, 1–58.
- Breunig, M.M., Kriegel, H.-P., Ng, R.T., and Sander, J. (2000). LOF: Identifying density-based local outliers. In: *Proceedings of the 2000 ACM SIGMOD International Conference on Management of Data*, 29, 93–104.
- Schölkopf, B., Platt, J.C., Shawe-Taylor, J., Smola, A., and Williamson, R. (2001). Estimating the support of a high-dimensional distribution. *Neural Computation*, 13(7), 1443–1471.
- Liu, F.T., Ting, K.M., and Zhou, Z. (2008). Isolation forest. In: *Proceedings of the 2008 Eighth IEEE International Conference on Data Mining*, 413–422.
- Mosler, K. (2013). Depth statistics. In: *Robustness and Complex Data Structures: Festschrift in Honour of Ursula Gather*, 17-34.
- Hubert, M., Rousseeuw, P.J., and Segaert, P. (2015). Multivariate functional outlier detection. *Statistical Methods & Applications*, 24(2), 177-202.

Practical session

Notebooks:

- anomdet_simulation1.Rmd,
- anomdet_hurricanes.Rmd,
- anomdet_brainimaging.Rmd,
- anomdet_cars.ipynb,
- anomdet_airbus.ipynb.

Data sets:

- carsanom.csv: Data set on anomaly detection for cars.
- airbus_data.csv: Data set from Airbus.
- hurdat2-1851-2019-052520.txt: Historical hurricane data.
- 101_1_dwi_fa.nii: Anatomical brain volume data.
- 101_1_dwi.voxelcoordsL.txt: Left brain fiber's bundle.
- 101_1_dwi.voxelcoordsR.txt: Right brain fiber's bundle.

Supplementary scripts:

- depth_routines.py: Routines for data depth calculation.
- FIF.py: Implementation of the functional isolation forest.
- depth_routines.R: Routines for curves' parametrization.
- DTI.R: Routines for input of brain imaging data.

Literature (mentioned in the tutorial) (1)

- Boser, B.E., Guyon, I., and Vapnik, V.N. (1992). A training algorithm for optimal margin classifiers. In: *Proceedings of the Fifth Annual Workshop of Computational Learning Theory*, Pittsburgh, ACM, 5, 144–152.
- Breunig, M.M., Kriegel, H.-P., Ng, R.T., and Sander, J. (2000). LOF: Identifying density-based local outliers. In: *Proceedings of the 2000 ACM SIGMOD International Conference on Management of Data*, 29, 93–104.
- Chandola, V., Banerjee, A., and Kumar, V. (2009). Anomaly detection: A survey. ACM *Computing Surveys (CSUR)*, 41(3):15, 1–58.
- Chaudhuri P. (1996). On a geometric notion of quantiles for multivariate data. *Journal of the American Statistical Association*, 91, 862–872.
- Claeskens, G., Hubert, M., Slaets, L., and Vakili, K. (2014). Multivariate functional halfspace depth. *Journal of the American Statistical Association*, 109(505), 411-423.
- Cortes, C. and Vapnik, V. (1995). Support-vector networks. *Machine Learning*, 20, 273–297.
- Donoho D. (1982). *Breakdown Properties of Multivariate Location Estimators.* Ph.D. thesis, Harvard University.
- Donoho D.L., Gasko M. (1992). Breakdown properties of location estimates based on halfspace depth and projected outlyingness. *The Annals of Statistics*, 20, 1803–1827.

Literature (mentioned in the tutorial) (2)

- Fraiman, R. and Muniz, G. (2001). Trimmed means for functional data. *TEST*, 10, 419-440.
- Hariri, S., Carrasco Kind, M., and Brunner, R.J. (2018). Extended isolation forest. `arXiv:1811.02141`.
- Hubert, M., Rousseeuw, P.J., and Segaert, P. (2015). Multivariate functional outlier detection. *Statistical Methods & Applications*, 24(2), 177-202.
- Koltchinskii V. (1997). M-estimation, convexity and quantiles. *The Annals of Statistics*, 25, 435–477.
- Koshevoy G., Mosler K. (1997). Zonoid trimming for multivariate distributions. *The Annals of Statistics*, 25, 1998–2017.
- Liu R.Y. (1990). On a notion of data depth based on random simplices. *The Annals of Statistics*, 18, 405–414.
- Liu, Z. and Modarres, R. (2011). Lens data depth and median. *Journal of Nonparametric Statistics*, 23, 1063–1074.
- Liu, F.T., Ting, K.M., and Zhou, Z. (2008). Isolation forest. In: *Proceedings of the 2008 Eighth IEEE International Conference on Data Mining*, 413–422.

Literature (mentioned in the tutorial) (3)

- López-Pintado, S. and Romo, J. (2009). On the concept of depth for functional data. *Journal of the American Statistical Association*, 104(486), 718–734.
- Mahalanobis P.C. (1936). On the generalized distance in statistics. *Proceedings of the National Institute of Sciences of India*, 12, 49–55.
- Markou, M. and Singh, S. (2003). Novelty detection: a review - Part 1: Statistical approaches. *Signal Processing*, 83(12), 2481–2497.
- Markou, M. and Singh, S. (2003). Novelty detection: a review - Part 2: Neural network based approaches. *Signal Processing*, 83(12), 2499–2521.
- Miljković, D. (2010). Review of novelty detection methods. *The 33rd International Convention MIPRO*, Opatija, 593–598.
- Mosler, K. (2013). Depth statistics. In: *Robustness and Complex Data Structures: Festschrift in Honour of Ursula Gather*, 17-34.
- Oja, H. (1983). Descriptive statistics for multivariate distributions. *Statistics and Probability Letters*, 1, 327–332.
- Pimentel, M.A.F., Clifton, D.A., Clifton, L., and Tarassenko, L. (2014). A review of novelty detection. *Signal Processing*, 99, 215–249.
- Schölkopf, B., Platt, J.C., Shawe-Taylor, J., Smola, A., and Williamson, R. (2001). Estimating the support of a high-dimensional distribution. *Neural Computation*, 13(7), 1443–1471.

Literature (mentioned in the tutorial) (4)

- Serfling, R. (2002). A depth function and a scale curve based on spatial quantiles. In: *Statistical Data Analysis Based on the L_1-Norm and Related Methods*m Birkhäser, Basel, 25-38.
- Stahel W. (1981). *Robust Estimation: Infinitesimal Optimality and Covariance Matrix Estimators (In German).* Ph.D. thesis, Swiss Federal Institute of Technology in Zurich.
- Tukey J.W. (1975). Mathematics and the picturing of data. In: *Proceedings of the International Congress of Mathematicians*, volume 2, Canadian Mathematical Congress, 523–531.
- Vapnik, V. and Chervonenkis, A. (1974). *Theory of Pattern Recognition* (in Russian). Nauka, Moscow.
- Vapnik, V. and Lerner, A. (1963). Pattern recognition using generalized portraits. *Avtomatika i Telemekhanika*, 24, 774–780.
- Vardi Y., Zhang C. (2000). The multivariate L_1-median and associated data depth. *Proceedings of the National Academy of Sciences of the United States of America*, 97, 1423–1426.
- Zuo Y., Serfling R. (2000). General notions of statistical depth function. *The Annals of Statistics*, 28, 461–482.