MITRO207
Homework 4: Solutions

Problem 1: test-and-set

Consider the model in which $n+1$ processes $p_0, \ldots, p_n$ communicate via a test-and-set *(T&S) object. The object is initialized to 0 and exports one atomic operation TAS() which returns the value of the object and then sets the value to 1.*

1. *Draw the protocol complexes of the model for $n = 1$ (two processes) and $n = 2$ (three processes).*
2. *The task of* input-less consensus *is defined as a tuple $(\mathcal{I}, \mathcal{O}, \Delta)$, where the input complex $\mathcal{I}$ is the simplex $\{(0,0), \ldots, (n,n)\}$ plus all its faces, the output complex $\mathcal{O}$ is a set of simplices $\{(0,0), \ldots, (n,0)\}$, $\{(0,1), \ldots, (n,1)\}$, $\ldots$, $\{(0,n), \ldots, (n,n)\}$ plus all their faces, and Δ maps each $\sigma = \{(i_1,i_1),\ldots,(i_k,i_k)\}$ to the set of simplices of $\mathcal{O}$ in which vertices have the form (i,j) such that $i,j \in \{i_1, \ldots, i_k\}$.*

   Intuitively, every process has its identifier as an input, and the goal of the task is to agree on the identifier of one of the participating processes.

   Is the task of simplified consensus task solvable in the model above for $n = 1$? For $n = 2$? Explain why.

1. The protocol complexes for $n = 1$ and $n = 2$ are depicted in Figure 1.
2. For the case $n = 1$, the decision map for consensus δ is defined as $\delta([p_i, 0]) = (i,i)$ and $\delta([p_i, 1]) = (i, 1-i)$. One can easily see that both processes decide on the identifier of the "winner" of the T&S object.

   For the case $n = 3$, no simplicial decision map carried by Δ exists: the "corners" of the protocol complex must be mapped to distinct disconnected components of $\mathcal{O}$.

Problem 2: skeleton maps (Exercise 4.6)

Prove that $skel^k$ is a strict carrier map.

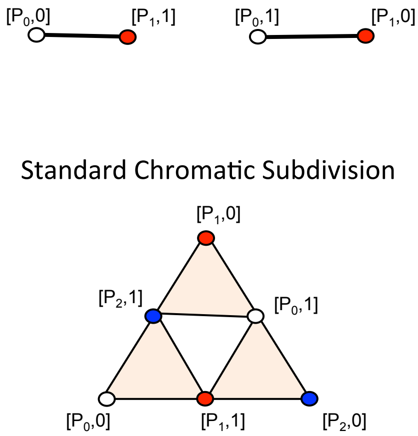

Figure 1: T&S complexes for $n = 1$ and $n = 2$.

Recall that, for a complex $\mathcal{C}$, $skel^k : \mathcal{C} \to 2^{\mathcal{C}}$, sends each simplex $\sigma \in \mathcal{C}$ to the set of all faces of σ of dimensions up to k.

Consider simplices τ and σ in $\mathcal{I}$, and consider $skel^k(\tau)$ and $skel^k(\sigma)$.

Each face of $\tau \cap \sigma$ of dimension k or less is also a face of both σ and τ of dimension k or less. Thus, $\mu \in skel^k(\tau \cap \sigma)$ if and only if $\mu \in skel^k(\tau)$ and $\mu \in skel^k(\sigma)$, i.e., $\mu \in skel^k(\tau) \cap skel^k(\sigma)$. Thus, $skel^k$ is strict.

Problem 3: commit-adopt task (Exercise 4.10)

The commit-adopt *task is a variant of consensus, where each process is assigned an input value in some set V and it chooses, as an output, a pair $[decision, v]$, where decision is either commit (C) or adopt (A) and v is* one of the input values *in the execution. Moreover, (i) if a process decides $[C, v]$, then every other decision is $[., v]$, and (ii) if all inputs are the same, then every decision value must be $[C, .]$.*

Define the task formally as a colorless task and show that it can be solved by a 2-layer colorless protocol, but not by a 1-layer colorless protocol.

Recall that a colorless task is defined on the set of allowed input values, set of allowed output values and the carrier map that relates inputs with outputs. The input complex $\mathcal{I}$ of the commit-adopt task is just a simplex $\{0, 1\}$ plus its two vertices. The output complex $\mathcal{O}$ is depicted in Figure 2. Note that only the only vertices that carry different values are $[A, 0]$ and $[A, 1]$.

The carrier map $\Delta' : \mathcal{I} \to \mathcal{O}$ is defined as:

- $\Delta(\{0\}) = \{\{[C, 0]\}\}$;
- $\Delta(\{1\}) = \{\{[C, 1]\}\}$;

- $\Delta(\{0,1\}) = \mathcal{O}$.

Figure 2: The output complex of the *commit-adopt* task.

As we learned, a colorless protocol is equivalent to the barycentric subdivision of the input complex. Therefore, the 1-layer protocol complex (Figure 3) cannot be mapped simplicially to $\mathcal{O}$ so that it is carried by Δ: the endpoints must go to the endpoints of $\mathcal{O}$, and then we have a problem with mapping the barycenter to the any vertex of $\mathcal{O}$ so that the map is simplicial.

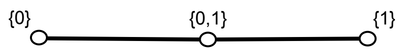

Figure 3: The 1-layer colorless protocol complex..

In the 2-layer protocol, we have four 1-dimensional edges that can be mapped simplicially to $\mathcal{O}$ satisfying Δ, e.g. as shown in Figure 4.

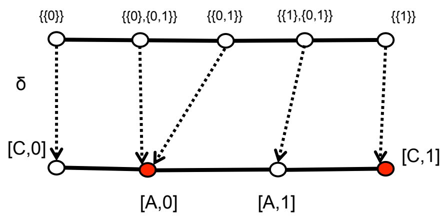

Figure 4: The 2-layer colorless protocol complex and the corresponding simplicial map to $\mathcal{O}$.