MITRO207
Homework 3: Solutions

Problem 1: carrier maps

Prove that a map $\Phi : \mathcal{A} \to 2^{\mathcal{B}}$ is carrier if and only if $\forall \sigma, \tau \in \mathcal{A} : \Phi(\sigma \cap \tau) \subseteq \Phi(\sigma) \cap \Phi(\tau)$.

Suppose that $\forall \sigma, \tau \in \mathcal{A} :\ \Phi(\sigma \cap \tau) \subseteq \Phi(\sigma) \cap \Phi(\tau)$. In the special case when $\tau \subseteq \sigma$, we get $\Phi(\tau) = \Phi(\sigma \cap \tau) \subseteq \Phi(\sigma) \cap \Phi(\tau) \subseteq \Phi(\sigma)$, i.e., $\Phi(\tau) \subseteq \Phi(\sigma)$. Hence, Φ is carrier.

Now suppose that Φ is carrier and consider $\tau, \sigma \in \mathcal{A}$. Since $\sigma \cap \tau \subseteq \sigma$ and $\sigma \cap \tau \subseteq \tau$, we have $\Phi(\sigma \cap \tau) \subseteq \Phi(\sigma) \cap \Phi(\tau)$.

Problem 2: rigid carrier maps

Give an example of a rigid carrier map that is not strict. Give an example of a strict carrier map that is not rigid.

A rigid carrier map Φ ensures that for all σ, $\Phi(\sigma)$ is a pure complex of dimension $dim(\sigma)$. A carrier map Φ is strict if it guarantees that for all σ, τ, $\Phi(\sigma \cap \tau) = \Phi(\sigma) \cap \Phi(\tau)$.

Recall our famous example of consensus: the carrier map Δ of the consensus task sends vertex $\{P0\}$ to complex $\{\{P0\}\}$, vertex $\{P1\}$ to complex $\{\{P1\}\}$, vertex $\{Q0\}$ to complex $\{\{Q0\}\}$, simplex $\sigma = \{P0, Q0\}$ to complex $\{\{P0, Q0\}\}$ (plus all subsets), and simplex $\tau = \{P0, Q1\}$ to complex $\{\{P0, Q0\}, \{P1, Q1\}\}$ (plus all subsets). The map is obviously rigid: no simplex gets a complex of smaller dimension as an image.

We have $\Delta(\sigma \cap \tau) = \{\{P0\}\}$ and $\Delta(\sigma) \cap \Delta(\tau) = \{\{P0, Q0\}\}$ (plus all subsets), i.e., Δ is not strict.

Similarly, not any strict carrier map must be rigid: a trivial map that sends all simplices to a single vertex is strict, but, obviously, not rigid.

Problem 3: mesh-shrinking subdivision

The mesh *of a geometric simplicial complex $\mathcal{K}$ is the length of its longest edge. Let Δ be the simplicial complex consisting of a simplex together with all it faces,*

$Bary^N\Delta$ be the iterated barycentric subdivision of Δ, and c_N be the mesh *of $Bary^N\Delta$ (see Chapter 3.6.5 in the textbook).*

Prove that barycentric subdivision is mesh-shrinking*, i.e.,* $\lim_{N\to\infty} c_N = 0$.

Recall that any point in a *geometric* simplex $\Delta_n = [v_0, \ldots, v_n]$ has a unique representation $\sum_i t_i v_i$, where each $t_i \in [0,1]$ and $\sum_i t_i = 1$. For example, for a vertex v_i, we have $t_i = 1$, and for all $j \neq i$, $t_j = 0$.

The barycenter of a face $[w_0, \ldots, w_m]$ of Δ_n is then defined as $\frac{1}{m+1}\sum_i w_i$. In particular, the barycenter of $\Delta_n = [v_0, \ldots, v_n]$ is $\frac{1}{n+1}\sum_i v_i$.

Observe now that, for any point $v \in |\Delta_n|$, the most distant from v point of $|\Delta_n|$ is necessarily a vertex v_i. Indeed, for any other point $v' = \sum_i t_i v_i$, we have:

$$(*)\ |v - \sum_i t_i v_i| = |\sum_i t_i(v - v_i)| \leq \sum_i t_i |v - v_i| \leq \sum_i t_i \max_j |v - v_j| = \max_j |v - v_j|$$

We are going to show that the mesh of $Bary\Delta_n$ is at most $\frac{n}{n+1}d$, where d is the mesh of Δ_n. As a result, we get $c_N \leq (\frac{n}{n+1})^N mesh(\Delta_n) \to_{N\to\infty} 0$, i.e., *Bary* is a mesh-shrinking subdivision.

To proceed by induction on n, consider $n = 1$ as a base case. The mesh of $Bary\Delta_1$ is $\frac{1}{2}$ of the length of the longest edge of $|\Delta_1|$ and we are done.

Now suppose the claim is true for all $m < n$ and consider any edge u, w of $Bary\Delta_n$. Note that the only vertex of $Bary\Delta_n$ that lies in the interior of $|\Delta_n|$ is the barycenter $b = \frac{1}{n+1}\sum_i v_i$.

If neither u nor w is the barycenter, then both vertices belong to a proper face of $Bary\Delta_n$ of dimension $m < n$. By the induction hypothesis, $|u - w| \leq \frac{m}{m+1}d_m$, where d_m is the mesh of the m-dimensional face. Since $d_m \leq d$ and $\frac{m}{m+1} < \frac{n}{n+1}$, we have $|u - w| < \frac{n}{n+1}d$.

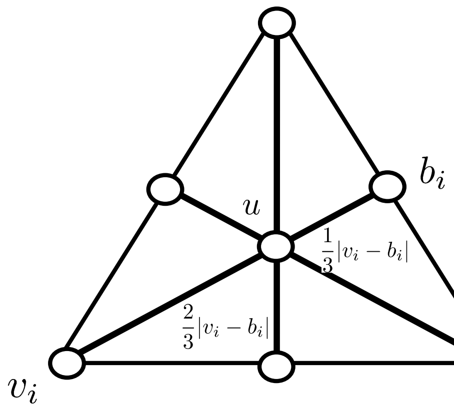

Figure 1: The distance between b_i and v_i in $Bary\Delta_2$.

Now suppose that, say, u is the barycenter of Δ_n. By inequality $(*)$ above, $|u - w| \leq \max_i |u - v_i|$. For each vertex v_i, u can be represented as $\frac{n}{n+1}b_i + \frac{1}{n+1}v_i$, where $b_i = \frac{1}{n}\sum_{j\neq i} v_j$ is the barycenter of the face $[v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n]$. Thus, $|v_i - u| = \frac{n}{n+1}|v_i - b_i|$ (see the example for Δ_2 in Figure 1).

Thus, using again inequality $(*)$, we obtain $|u - w| \leq \max_i |u - v_i| = \frac{n}{n+1} \max_i |v_i - b_i| \leq \frac{n}{n+1} \max_{i,j} |v_i - v_j| = \frac{n}{n+1} d$.