MITRO207
Homework 2: Solutions

Problem 1: connected graphs

Consider a simplicial map φ from a graph $\mathcal{G}$ to a graph $\mathcal{H}$. Prove that if $\mathcal{G}$ is connected, so is the image $\varphi(\mathcal{G})$.

Let Φ be a carrier map from $\mathcal{G}$ to $\mathcal{H}$. Show that it is not *always the case that if $\mathcal{G}$ is connected, then $\Phi(\mathcal{G})$ is connected. Show that if $\mathcal{G}$ is connected and, for every edge $\sigma \in \mathcal{G}$, $\Phi(\sigma)$ is connected, then $\Phi(\mathcal{G})$ is connected.*

An example of a carrier map $\Phi : \mathcal{G} \to 2^{\mathcal{H}}$ that does not generate a connected image of a connected graph $\mathcal{G}$ is the carrier map of the coordinated attack task (slides 60-64 in class02). Here a connected input graph $\mathcal{I}$ is mapped to three disconnected segments of the output graph $\mathcal{O}$: "both armies attack at noon", "both armies attack at dawn", "do not attack". This is because each of the two edges of $\mathcal{I}$ is mapped to a *disconnected* subcomplex of $\mathcal{O}$.

Suppose now that Φ maps each edge of $\mathcal{G}$ to a *connected* subgraph of $\mathcal{H}$. Consider two edges σ and τ that share a vertex: $\sigma \cap \tau \neq \emptyset$. Since $\Phi(\sigma)$ and $\Phi(\tau)$ are connected, $\Phi(\sigma \cap \tau) \in \Phi(\sigma)$ and $\Phi(\sigma \cap \tau) \in \Phi(\tau)$ imply that $\Phi(\sigma \cup \tau)$ is connected. Applying this step to all paths in $\mathcal{G}$, we derive that $\Phi(\mathcal{G})$ is connected.

Since a simplicial map is a special case of a carrier map (check if the statement is true), we derive that a simplicial map φ from a connected graph $\mathcal{G}$ generates a connected image $\varphi(\mathcal{G})$.

Problem 2: map compositions

Prove that a composition of two simplicial maps is a simplicial map. Prove that if both maps are rigid, so is their composition.

Consider two simplicial maps $\varphi : V(\mathcal{A}) \to V(\mathcal{B})$ and $\psi : V(\mathcal{B}) \to V(\mathcal{C})$, for some simplicial complexes $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$. Consider a simplex $\sigma \in \mathcal{A}$. Since φ and ψ are simplicial, we have that $\varphi(V(\sigma))$ (the set of vertices in the image of σ under ϕ) is a simplex in $\mathcal{B}$ and $\psi(\varphi(V(\sigma)))$ is a simplex in $\mathcal{C}$. Thus, $\psi \circ \varphi : V(\mathcal{A}) \to V(\mathcal{C})$ is a simplicial map.

Moreover, if the two maps are rigid, we have $dim(\sigma) = dim(\varphi(\sigma)) = dim(\psi(\varphi(\sigma)))$, and, thus, $\psi \circ \varphi : \mathcal{A} \to \mathcal{C}$ is rigid.

Problem 3: carrier compositions

Define the composition of a carrier map followed by a simplicial map. Prove that the composition is a carrier map. Moreover, if both maps are chromatic, then their composition is chromatic.

Let $\Phi : \mathcal{A} \to 2^{\mathcal{B}}$ be a carrier map, and $\psi : V(\mathcal{B}) \to V(\mathcal{C})$ be a simplicial map. The composition $\psi \circ \Phi : \mathcal{A} \to 2^{\mathcal{C}}$ is then defined as $\psi \circ \Phi : \sigma \mapsto \{\psi(V(\mu)) | \mu \in \Phi(\sigma)\}$. Note that, since ψ is simplicial and Φ is carrier, for each $\sigma \in \mathcal{A}$, $\psi \circ \Phi(\sigma) \subseteq \mathcal{C}$.

Let σ and τ be simplices in $\mathcal{A}$ such that $\tau \subseteq \sigma$. Since Φ is carrier, we have $\Phi(\tau) \subseteq \Phi(\sigma)$.

Consider any simplex $\mu \in \Phi(\tau)$ and its image $\psi(V(\mu)) \in \psi \circ \Phi(\tau)$. Since $\mu \in \Phi(\sigma)$, we also have $\psi(V(\mu)) \in \psi \circ \Phi(\sigma)$. Thus, $\psi \circ \Phi(\tau) \subseteq \psi \circ \Phi(\sigma)$ and $\psi \circ \Phi$ is carrier.

If, additionally, Φ and ψ are chromatic with a coloring χ, then for each $\sigma \in \mathcal{G}$, we have $\chi(\Phi(\sigma)) = \chi(\sigma)$, and for each $\mu \in \Phi(\sigma)$, we have $\chi(\psi(\mu)) = \chi(\mu)$. Thus, $\chi(\psi \circ \Phi(\sigma)) = \chi(\sigma)$.