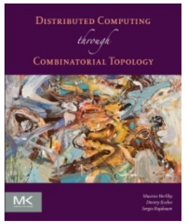
Elements of combinatorial topology

MITRO207, P4, 2019



Road Map

Simplicial Complexes

Standard Constructions

Carrier Maps

Connectivity

Subdivisions

Simplicial & Continuous Approximations

Road Map

Simplicial Complexes

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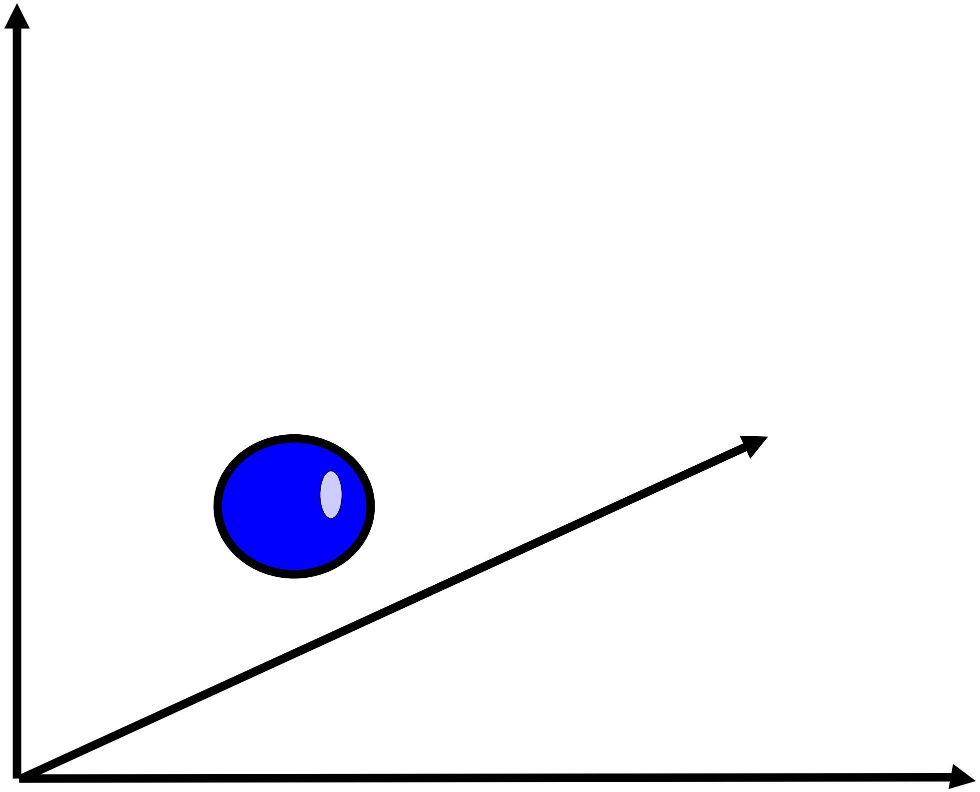
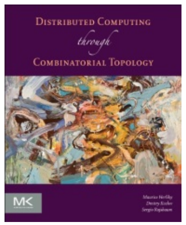
Simplicial & Continuous Approximations

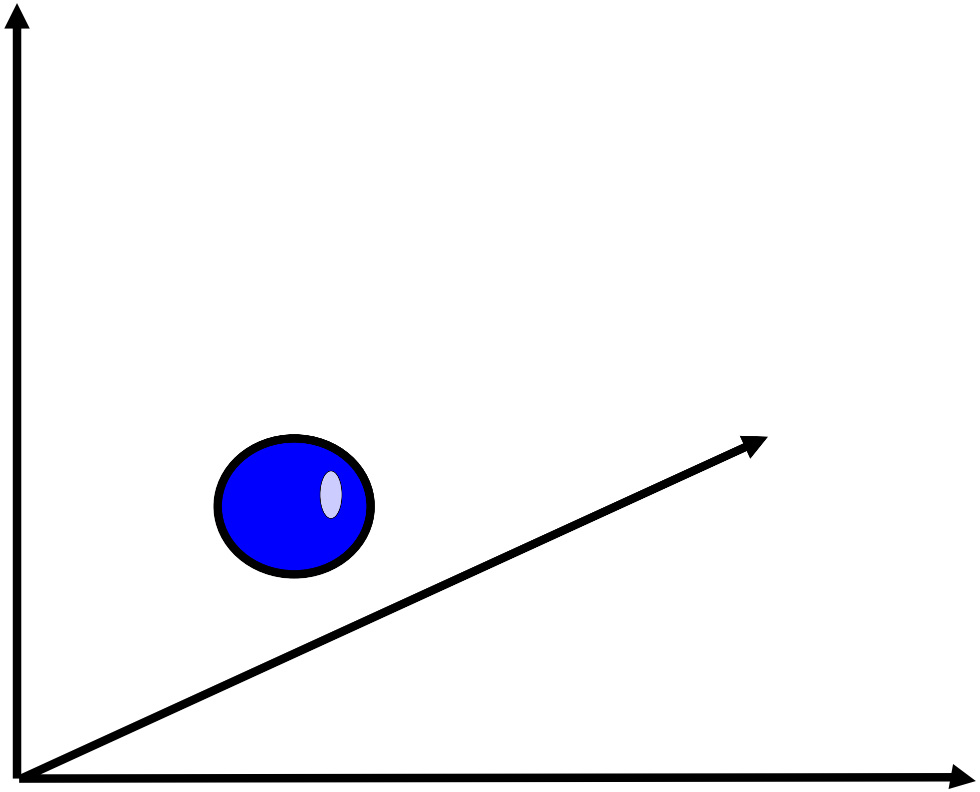
A Vertex

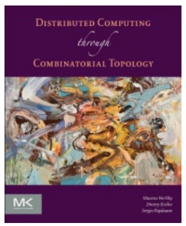




A Vertex

Combinatorial: an element of a set.

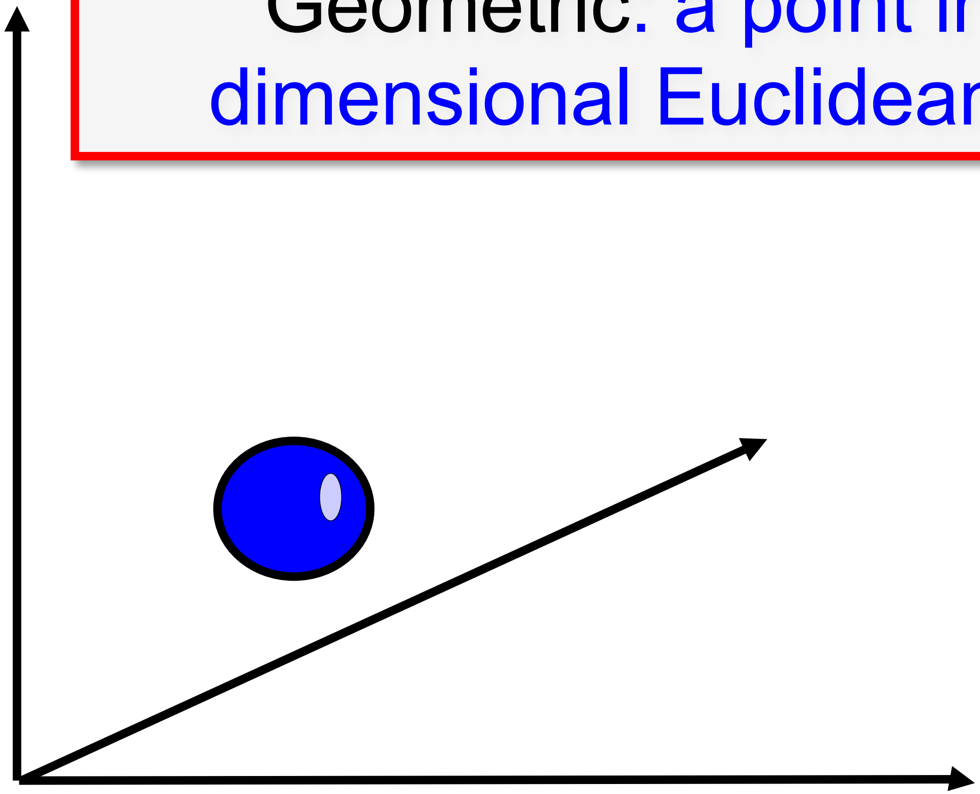


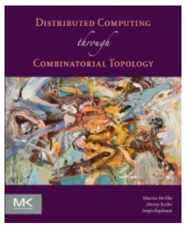


A Vertex

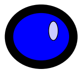
Combinatorial: an element of a set

Geometric: a point in high-dimensional Euclidean Space



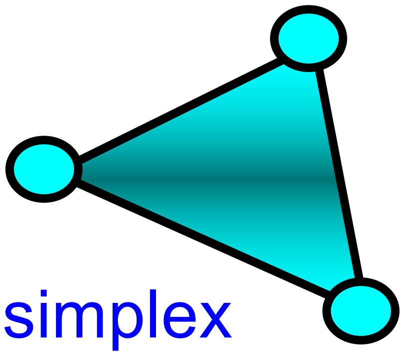


Simplexes

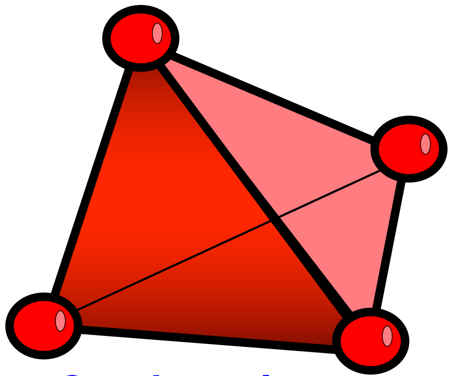


0-simplex

1-simplex



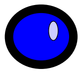
2-simplex



3-simplex

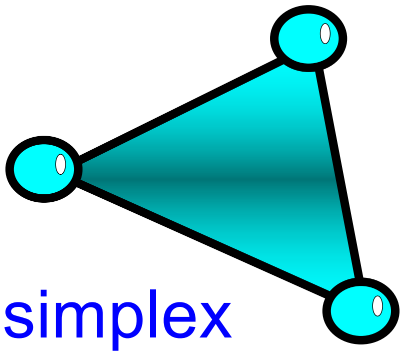
Simplexes

Combinatorial: a set of vertexes.

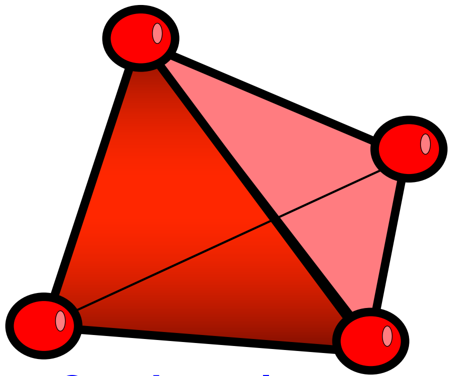


0-simplex

1-simplex



2-simplex

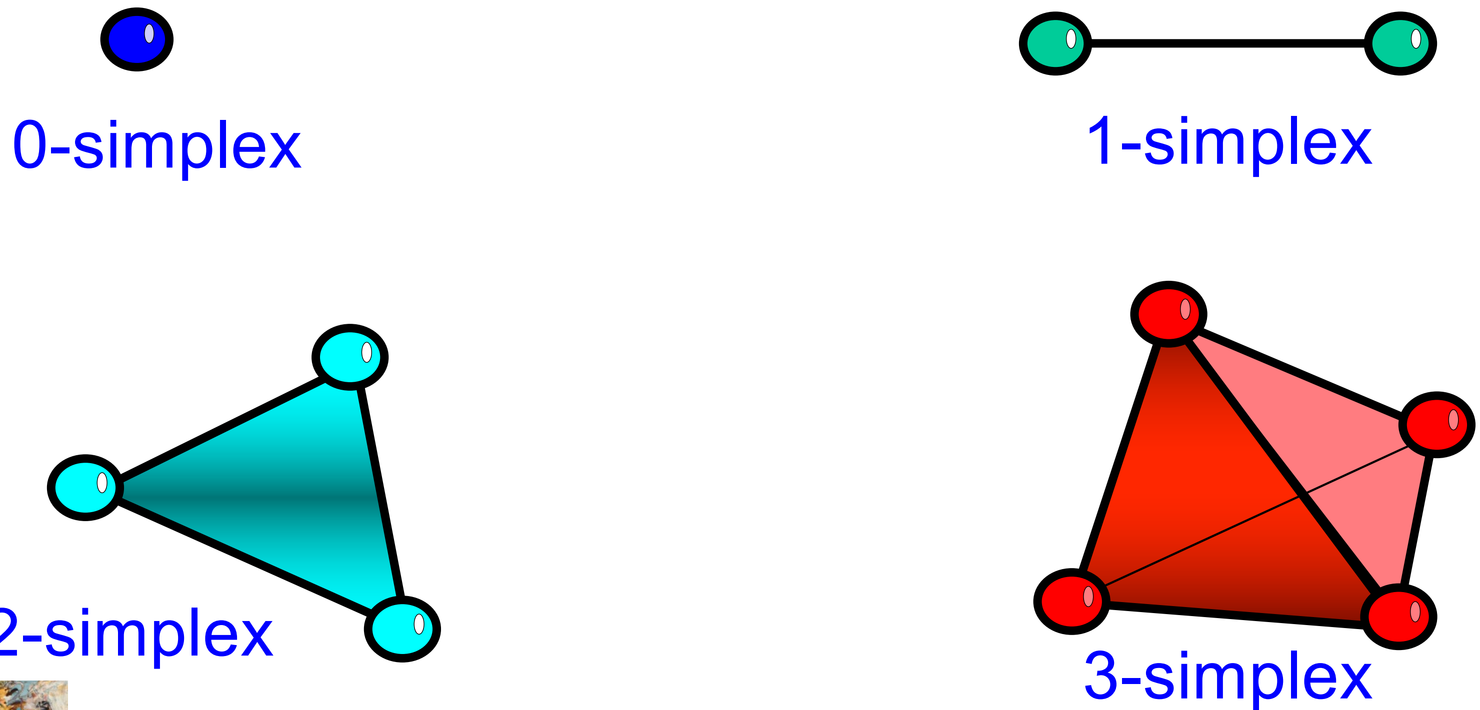


3-simplex

Simplexes

Combinatorial: a set of vertexes.

Geometric: convex hull of points in general position



Simplexes

Combinatorial: a set of vertexes.

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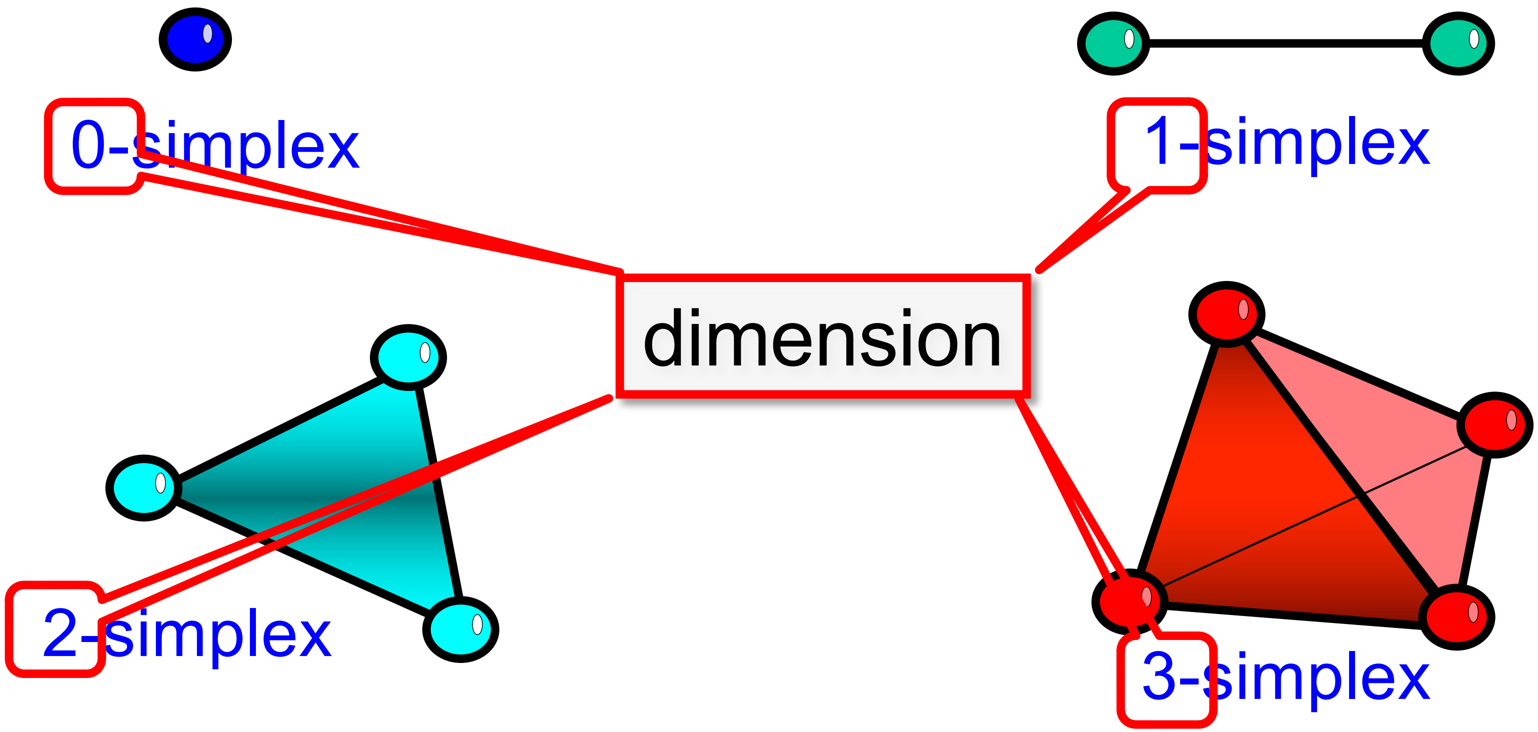
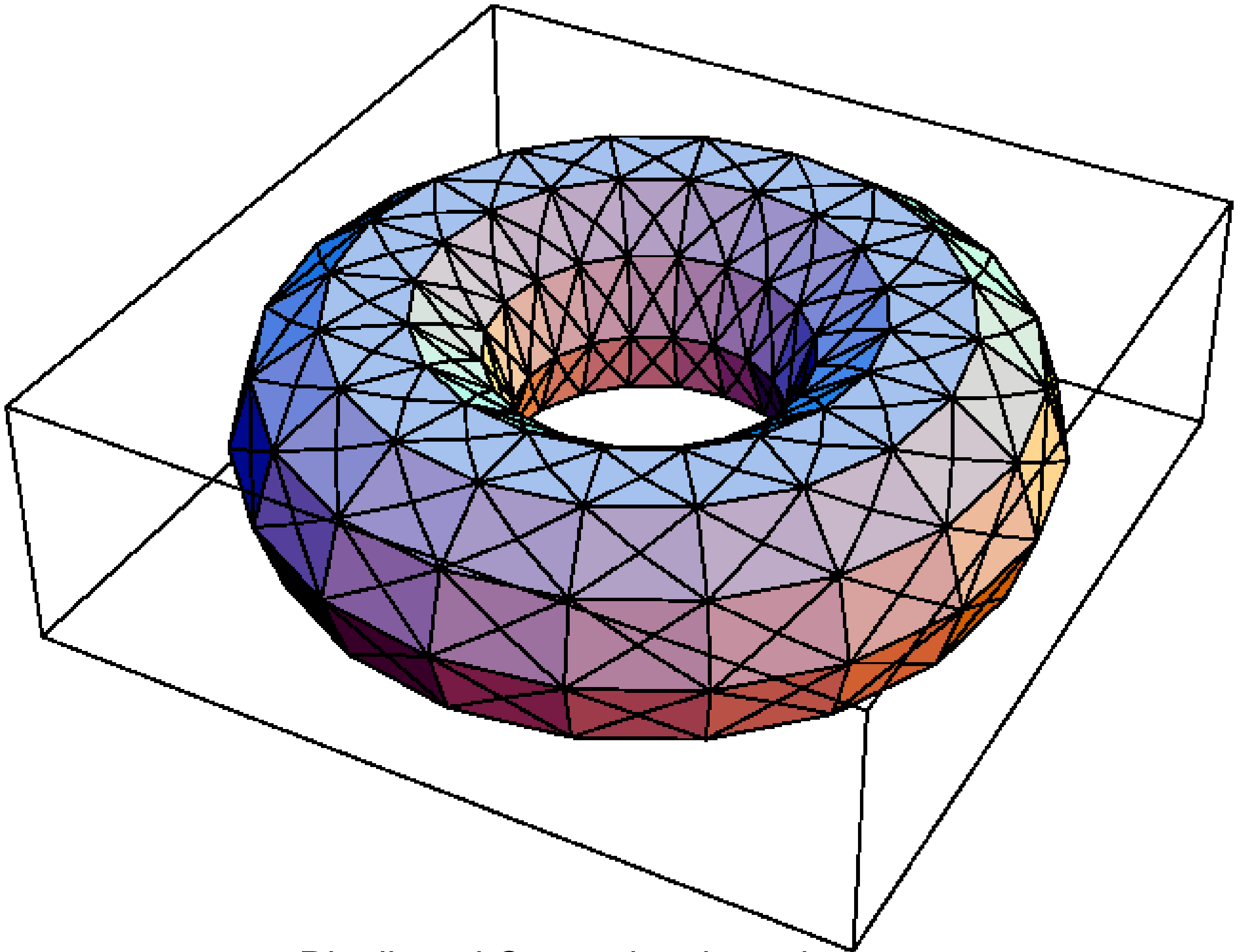
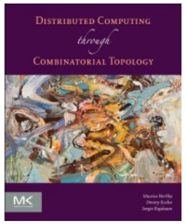


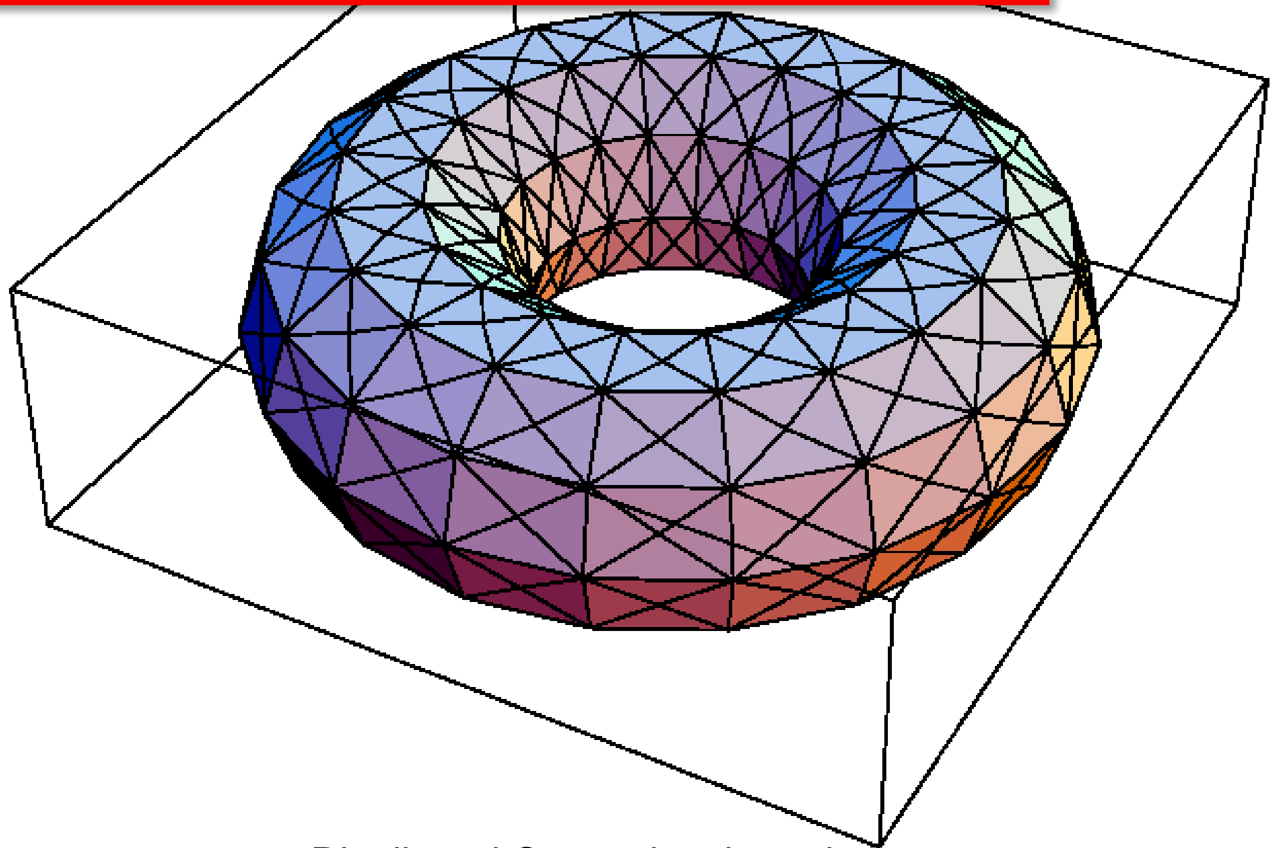
Simplicial Complex

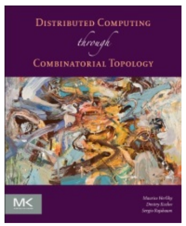




Simplicial Complex

Combinatorial: a set of simplexes close under inclusion.

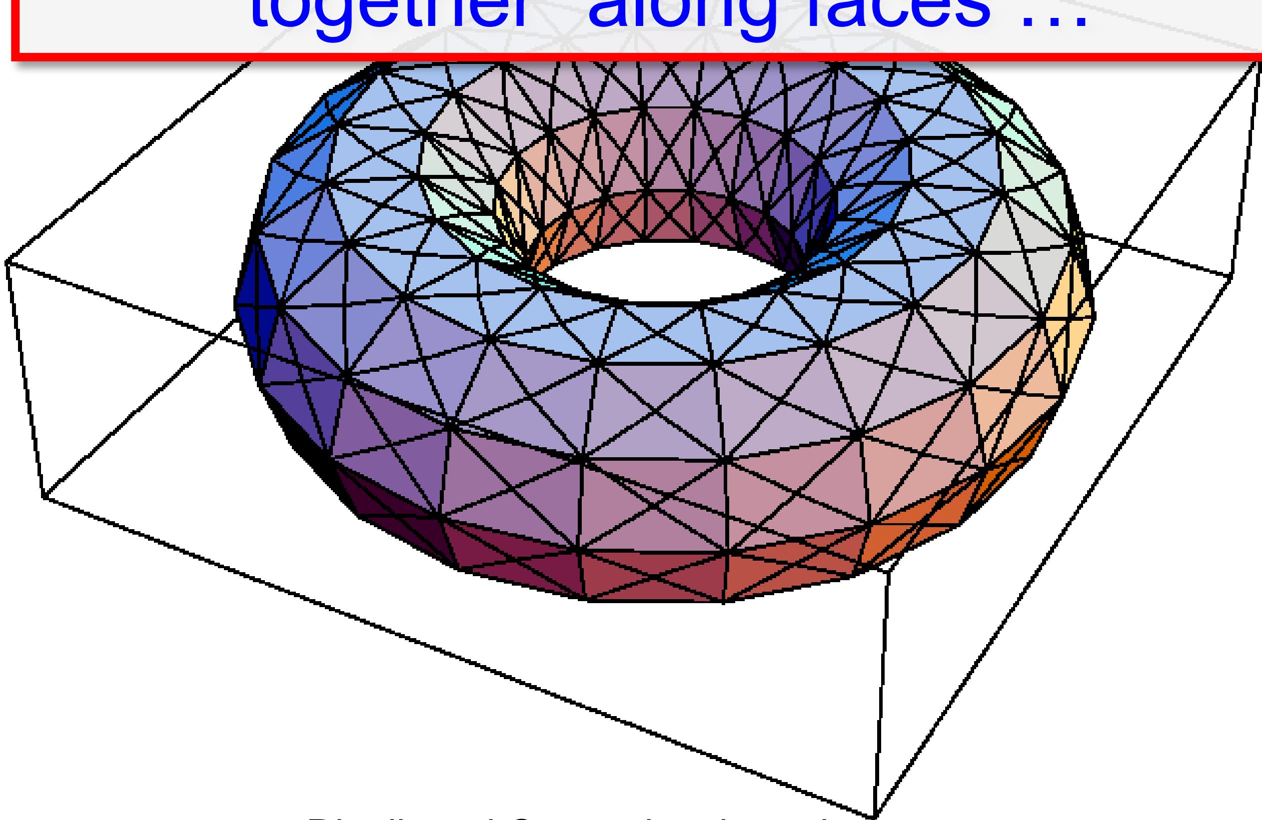


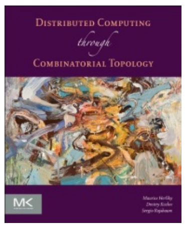


Simplicial Complex

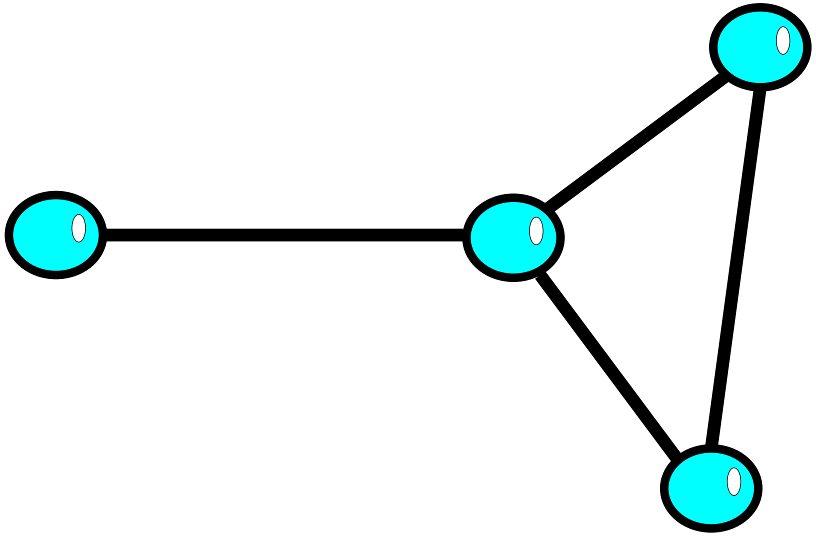
Combinatorial: a set of simplexes closed under inclusion

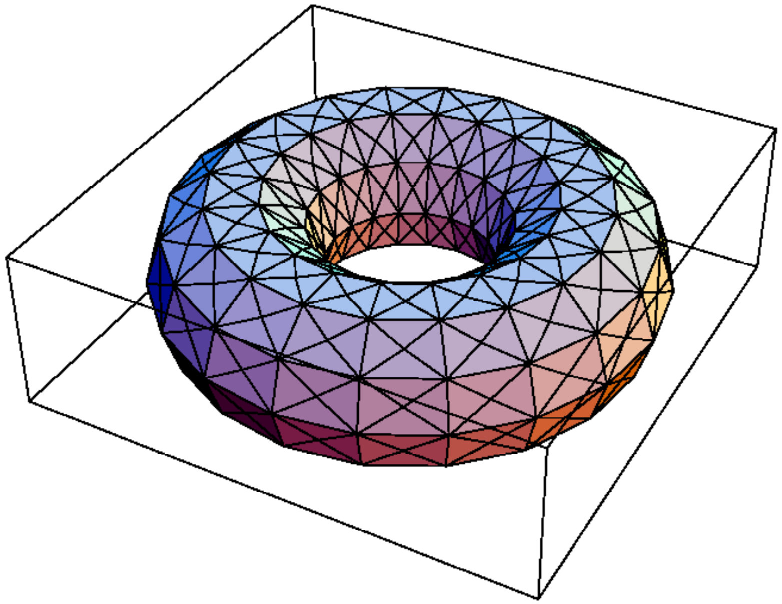
Geometric: simplexes "glued together" along faces …

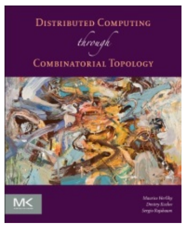




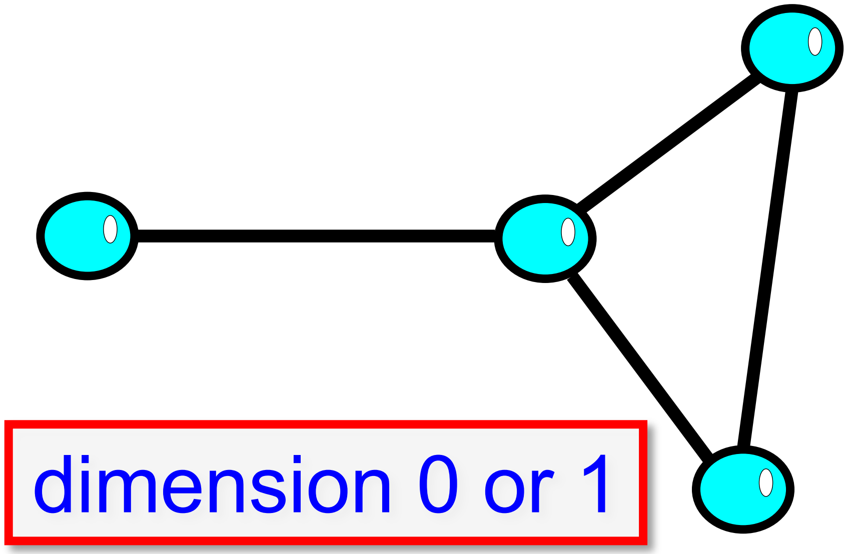
Graphs vs Complexes

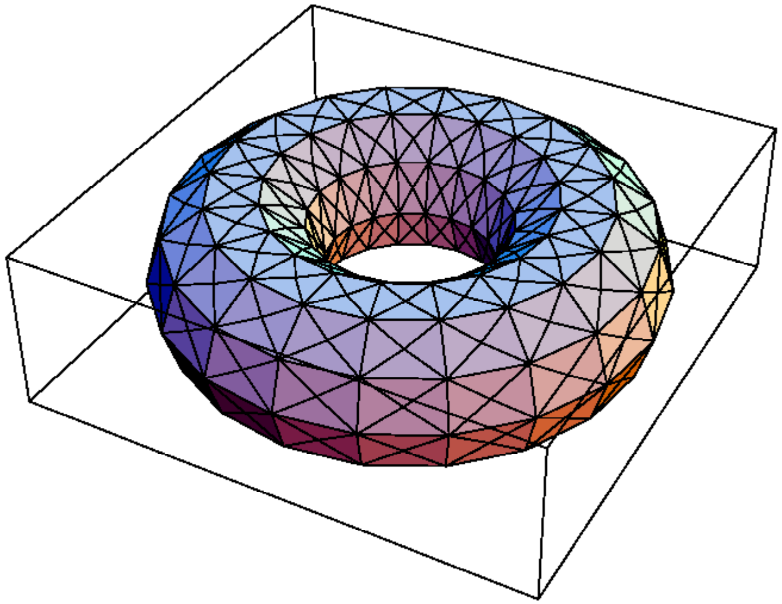


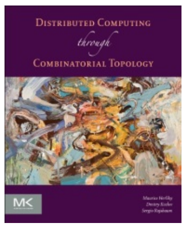




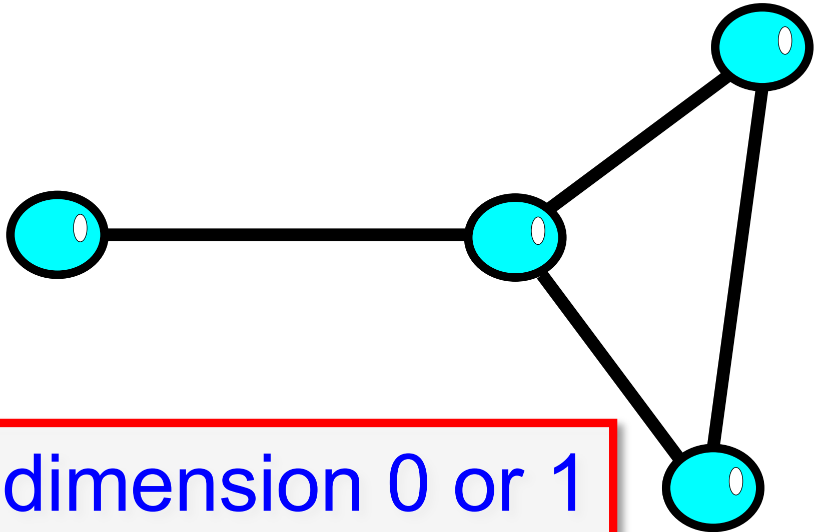
Graphs vs Complexes



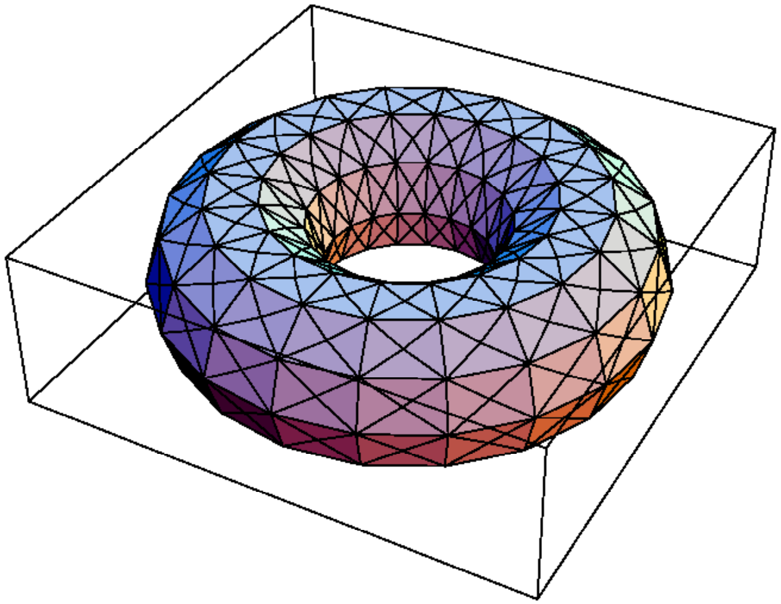




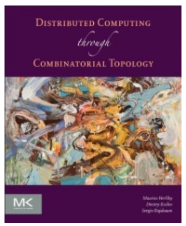
Graphs vs Complexes



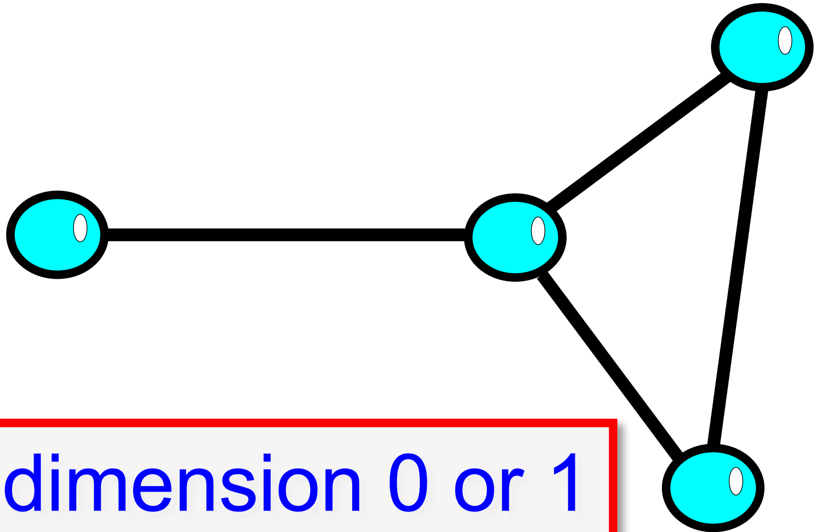
dimension 0 or 1



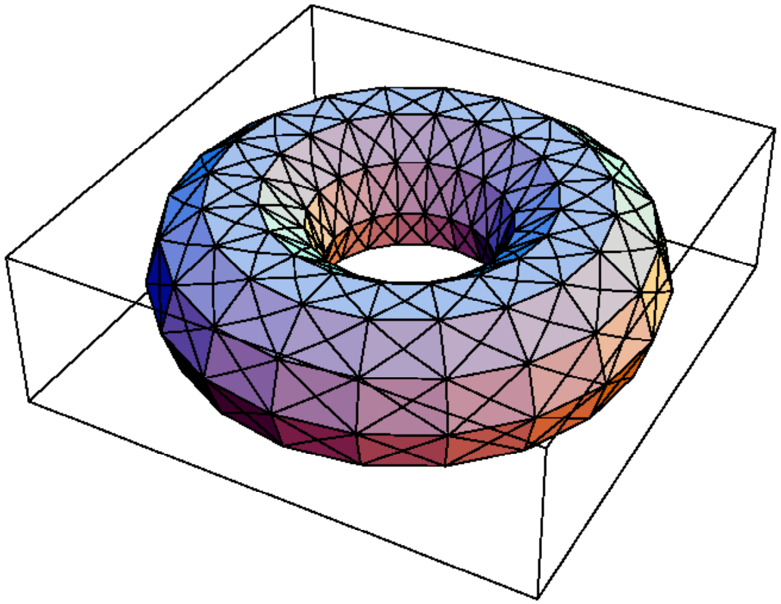
arbitrary dimension



Graphs vs Complexes



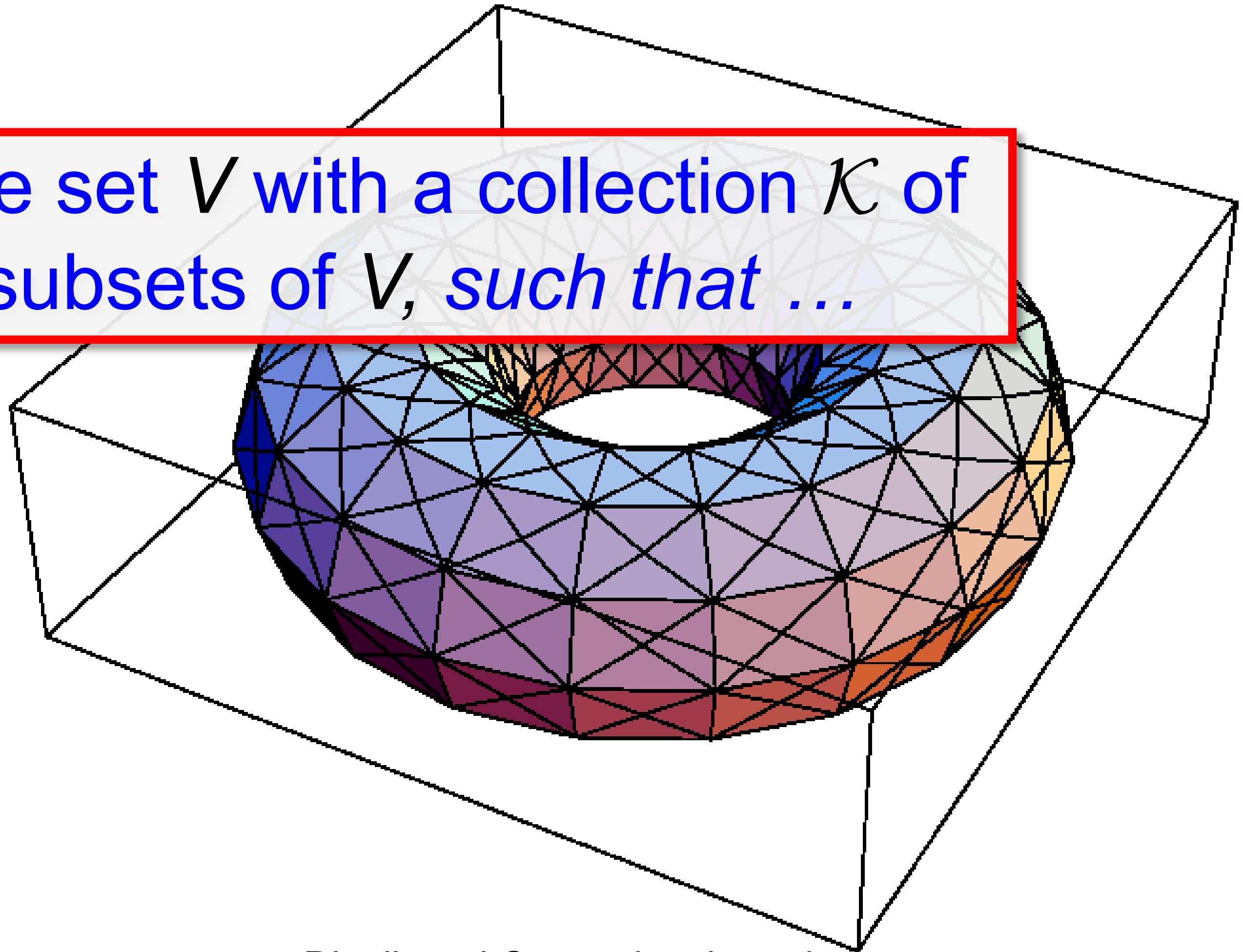
dimension 0 or 1



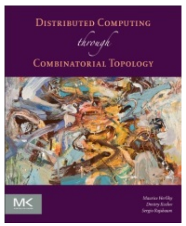
arbitrary dimension

complexes are a natural *generalization* of graphs

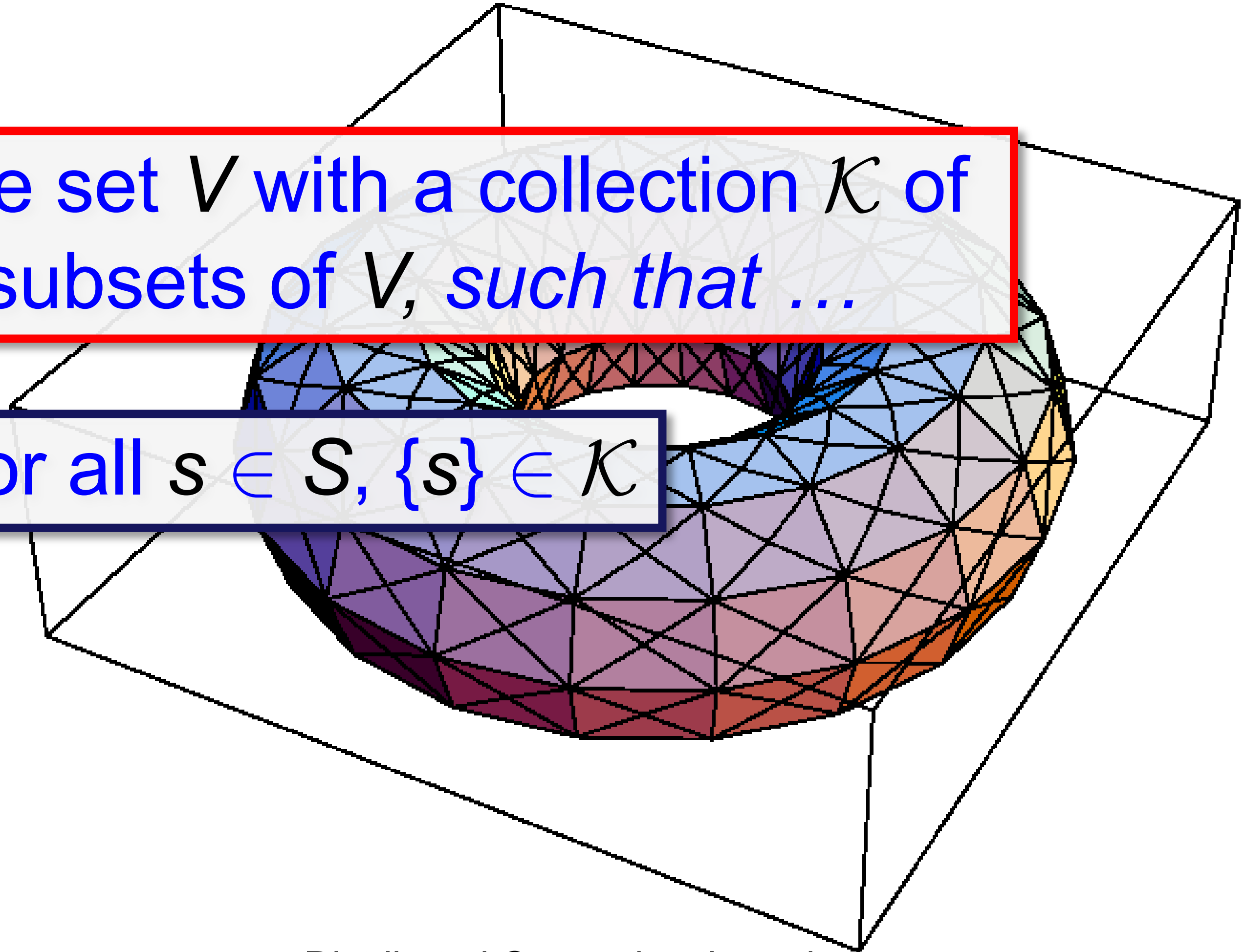
Abstract Simplicial Complex



finite set V with a collection $\mathcal{K}$ of subsets of V, *such that …*

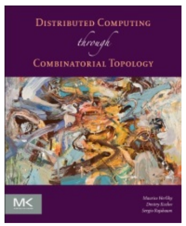


Abstract Simplicial Complex

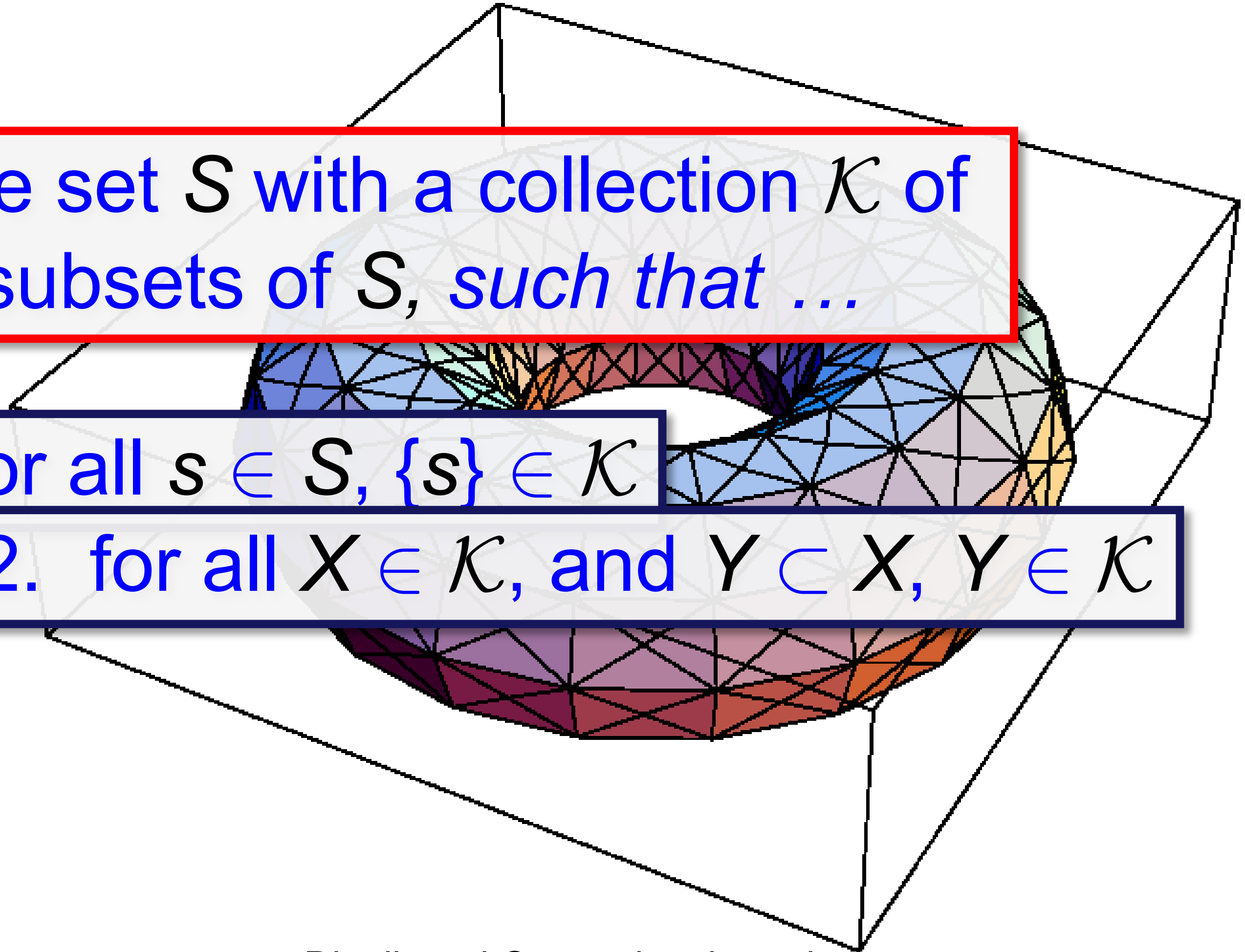


finite set V with a collection $\mathcal{K}$ of subsets of V, *such that …*

1. for all $s \in S$, $\{s\} \in \mathcal{K}$

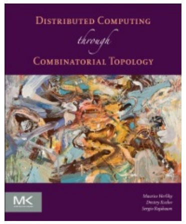


Abstract Simplicial Complex



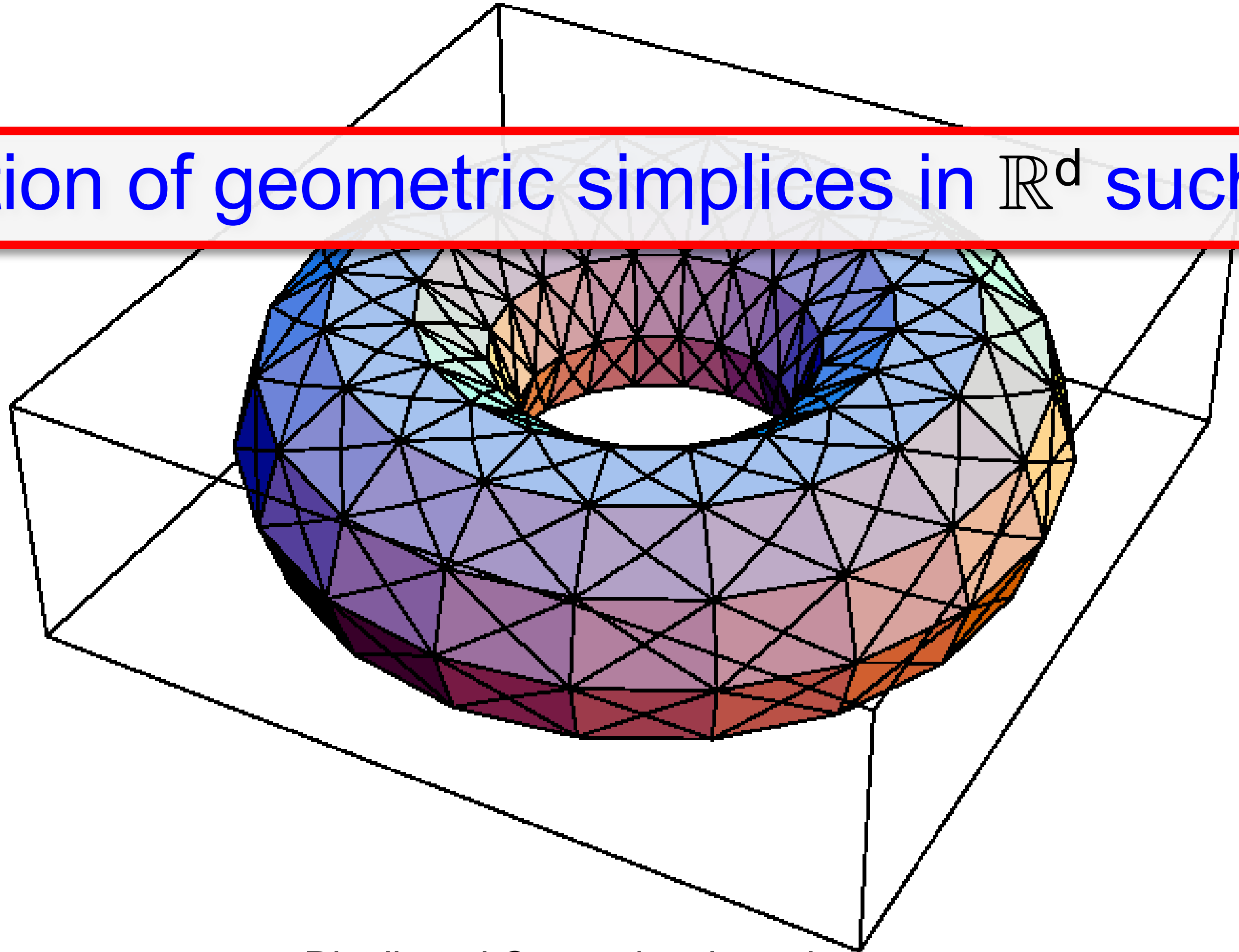
finite set S with a collection $\mathcal{K}$ of subsets of S, *such that …*

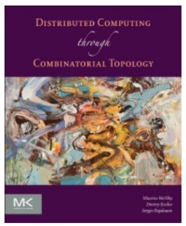
1. for all $s \in S$, $\{s\} \in \mathcal{K}$
2. for all $X \in \mathcal{K}$, and $Y \subset X$, $Y \in \mathcal{K}$



Geometric Simplicial Complex

A collection of geometric simplices in $\mathbb{R}^d$ such that

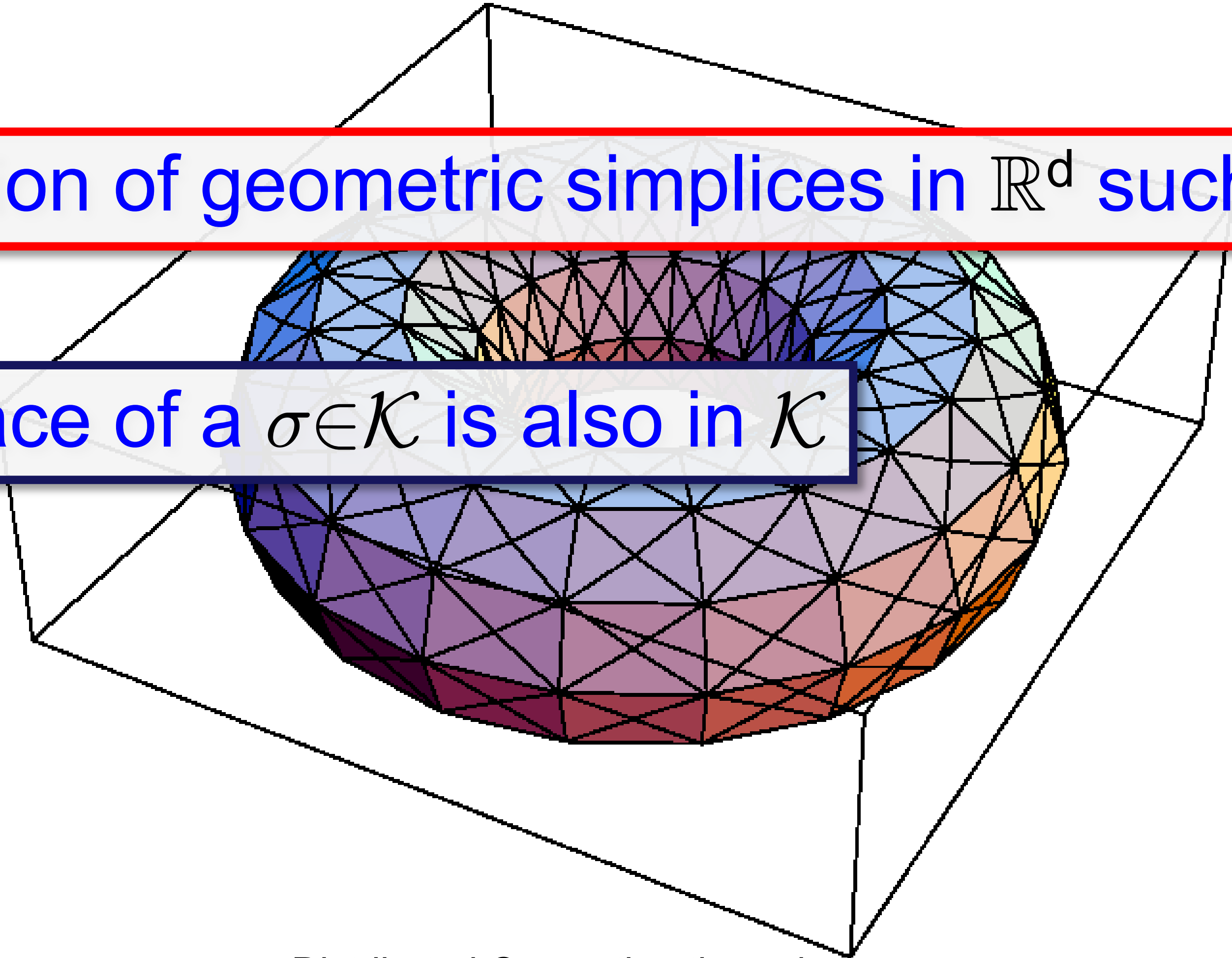


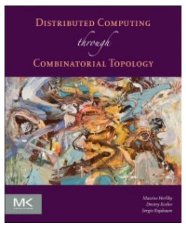


Geometric Simplicial Complex

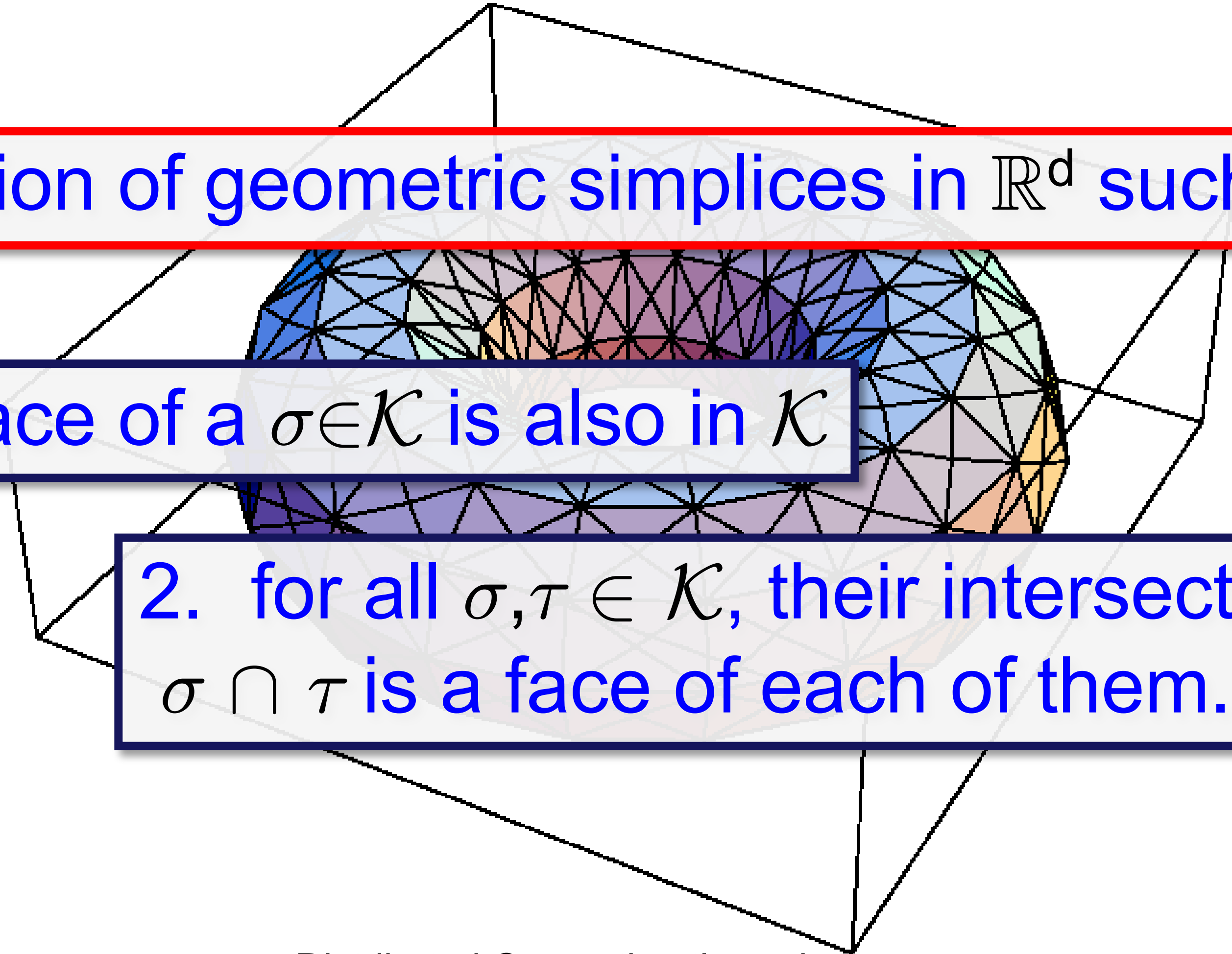
A collection of geometric simplices in $\mathbb{R}^d$ such that

1. any face of a $\sigma \in \mathcal{K}$ is also in $\mathcal{K}$



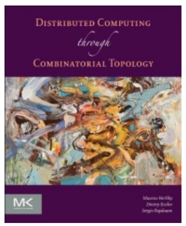


Geometric Simplicial Complex

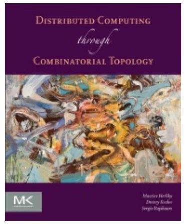


A collection of geometric simplices in $\mathbb{R}^d$ such that

1. any face of a $\sigma \in \mathcal{K}$ is also in $\mathcal{K}$
2. for all $\sigma, \tau \in \mathcal{K}$, their intersection $\sigma \cap \tau$ is a face of each of them.

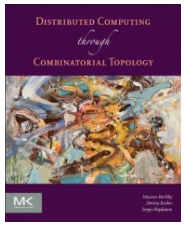


Abstract vs Geometric Complexes



Abstract vs Geometric Complexes

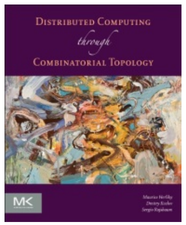
$\boxed{\text{Abstract: } \mathcal{A}}$



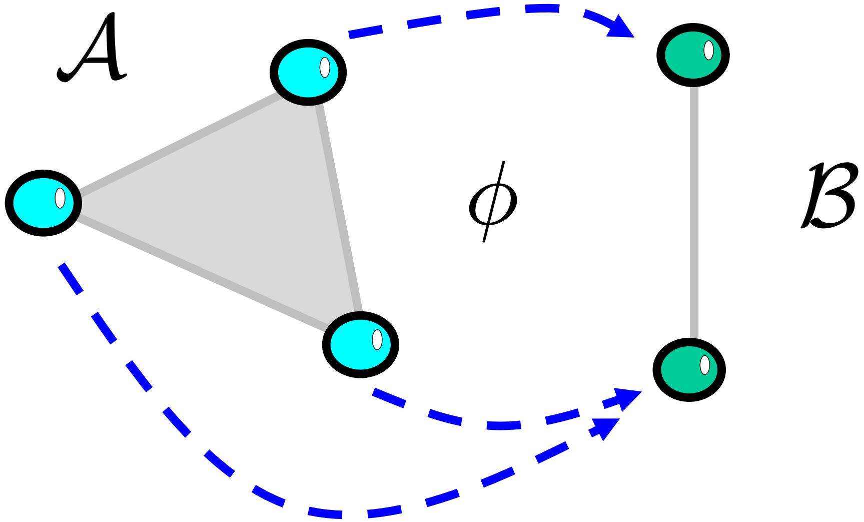
Abstract vs Geometric Complexes

$\boxed{\text{Abstract: } \mathcal{A}}$

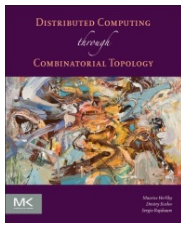
$\boxed{\text{Geometric: } |\mathcal{A}|}$



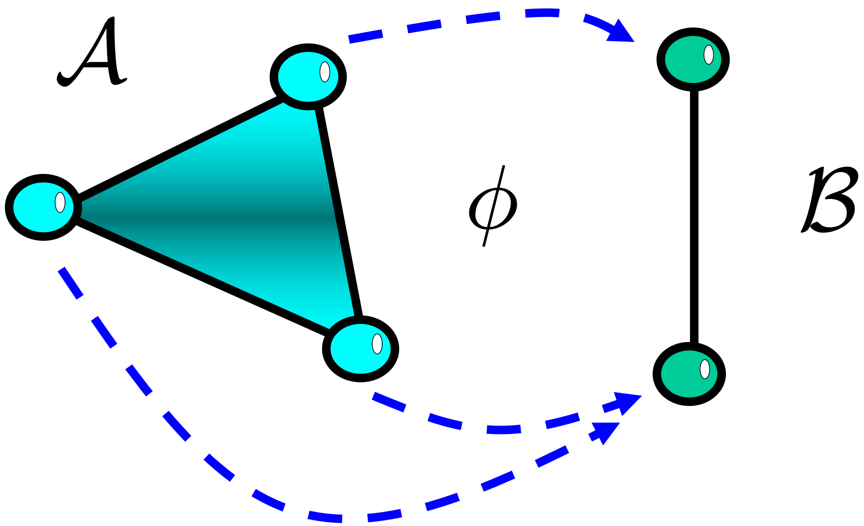
Simplicial Maps



Vertex-to-vertex map …



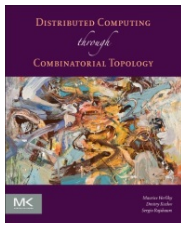
Simplicial Map



Vertex-to-vertex map …

that sends simplexes to simplexes

$$\boxed{\phi \colon \mathcal{A} \to \mathcal{B}}$$



Road Map

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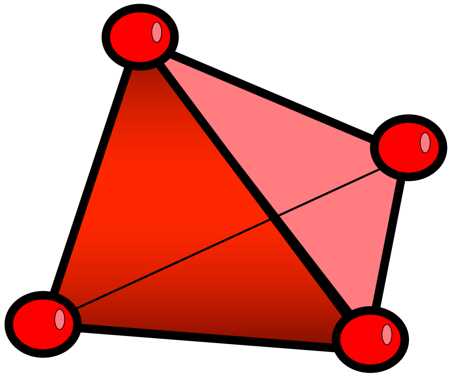
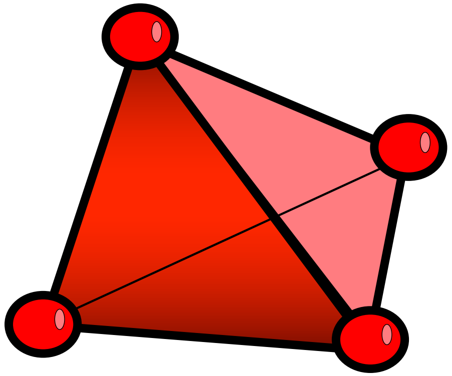
Skeleton



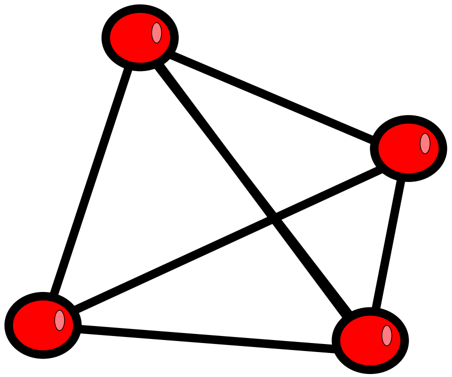
$\mathcal{C}$

(solid tetrahedron)

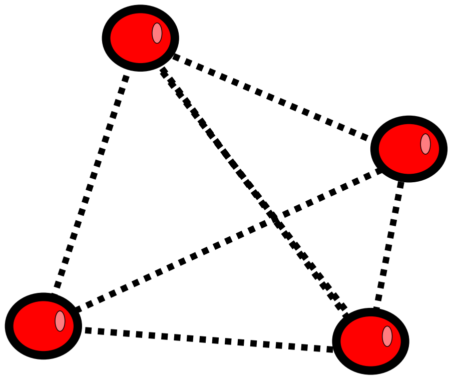


skel^2 $\mathcal{C}$

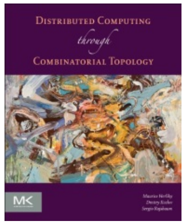
(hollow tetrahedron)



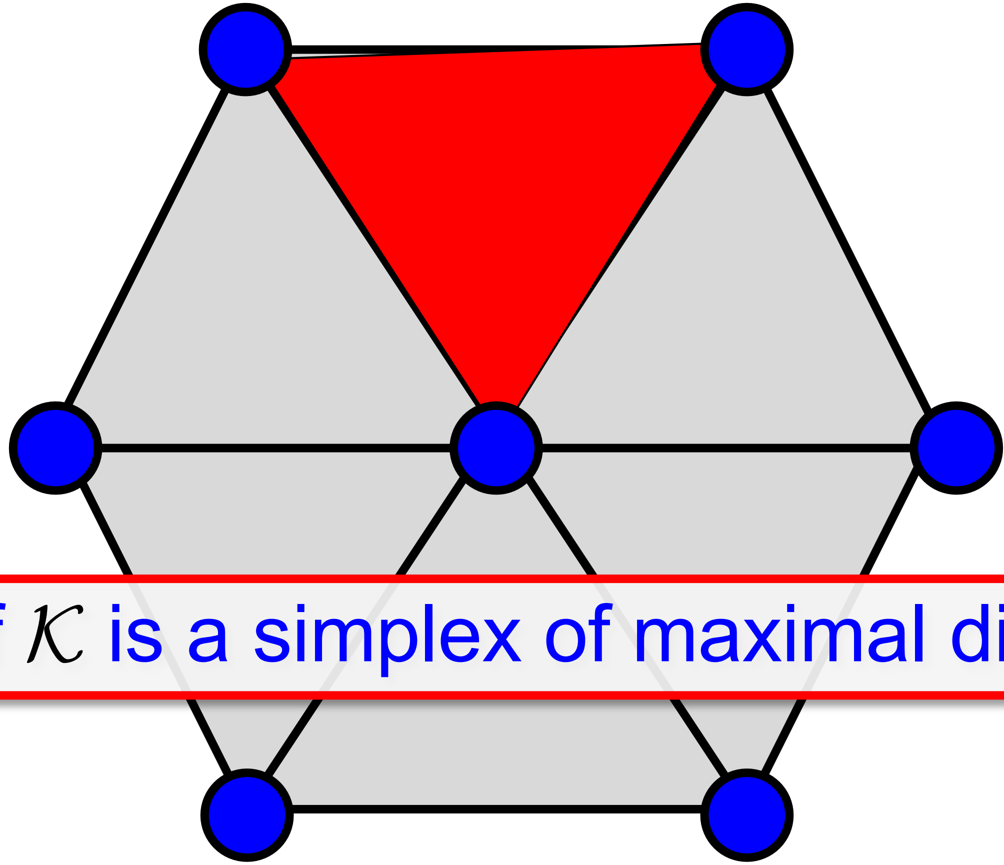
skel^1 $\mathcal{C}$



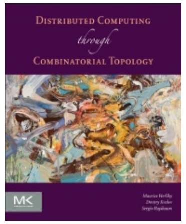
skel^0 $\mathcal{C}$



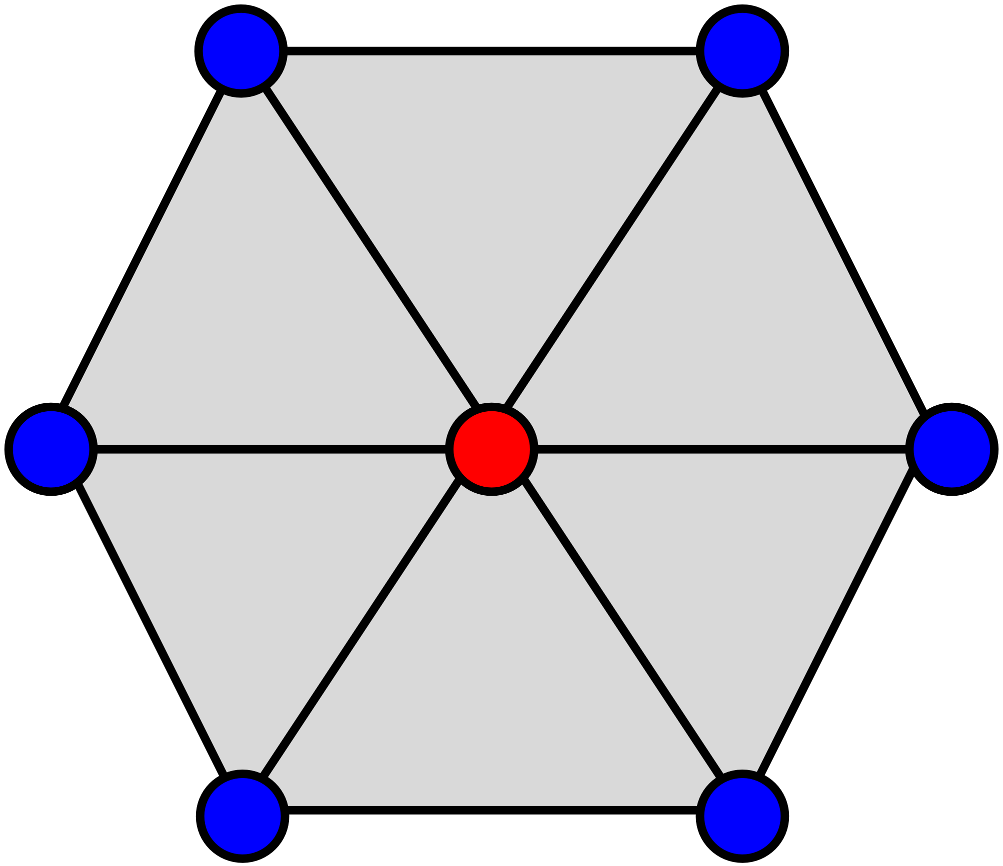
Facet



A facet of $\mathcal{K}$ is a simplex of maximal dimension



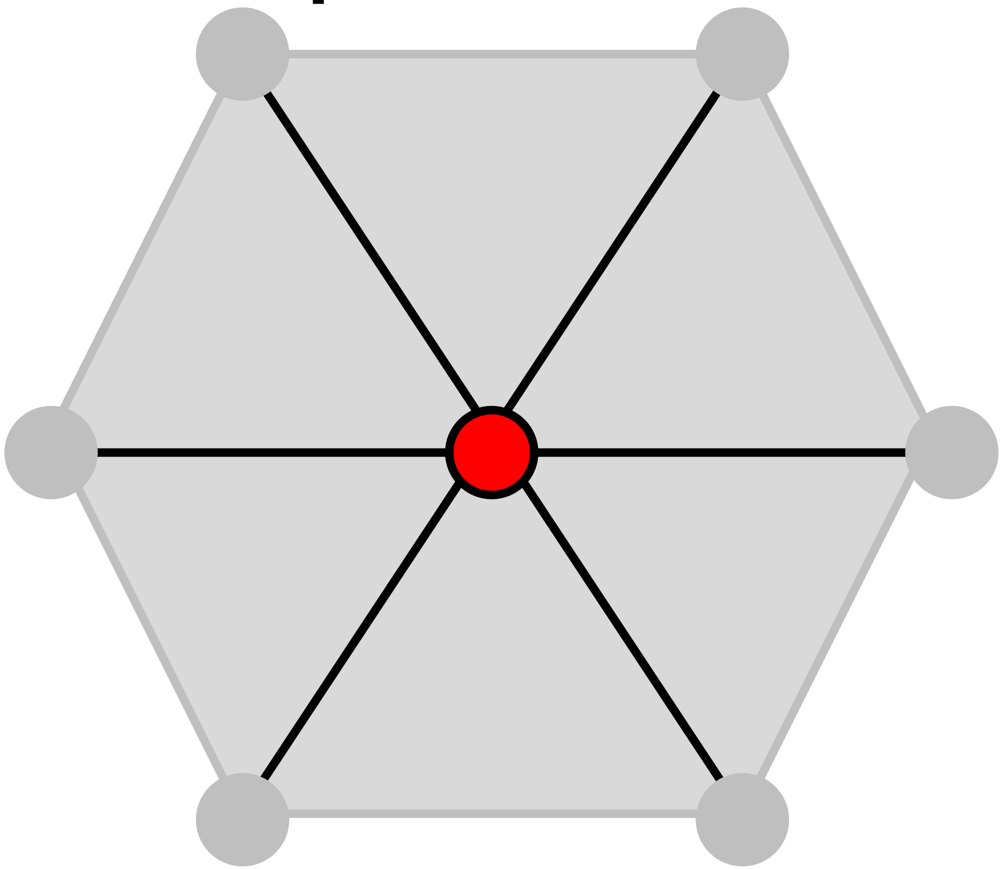
Star



Star(σ,$\mathcal{K}$) is the complex of facets of $\mathcal{K}$ containing σ

Complex

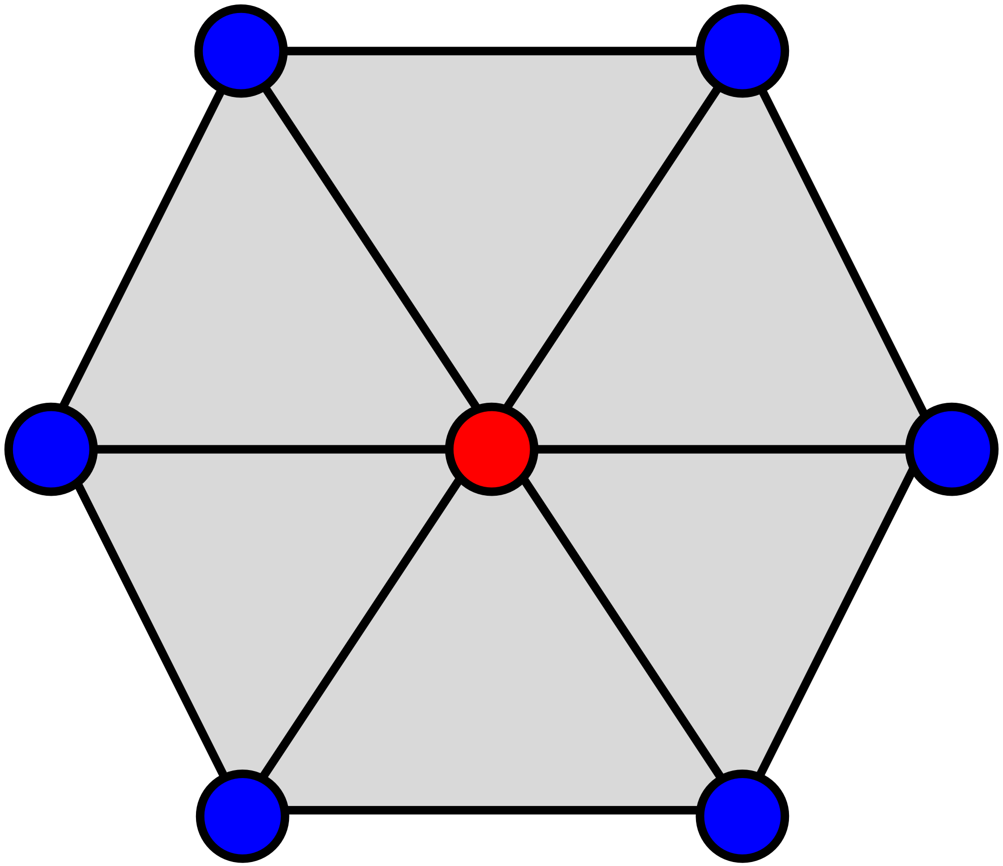
Open Star



$\text{Star}^{o}(\sigma,\mathcal{K})$ union of interiors of simplexes containing σ

Point Set

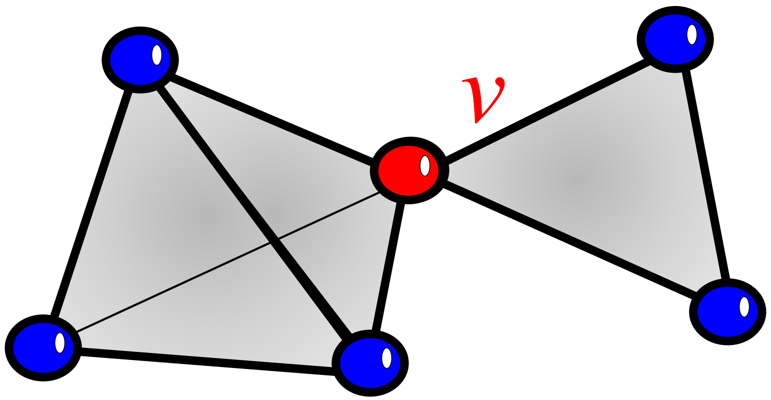
Link



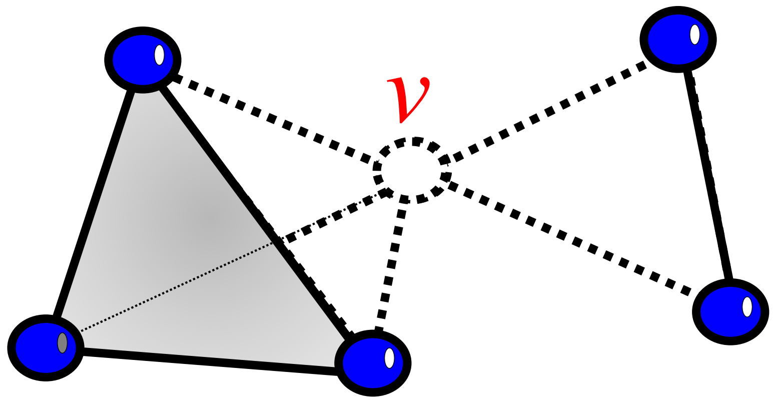
Link(σ,$\mathcal{K}$) is the complex of simplices of Star(σ,$\mathcal{K}$) not containing σ

Complex

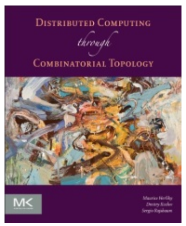
More Links



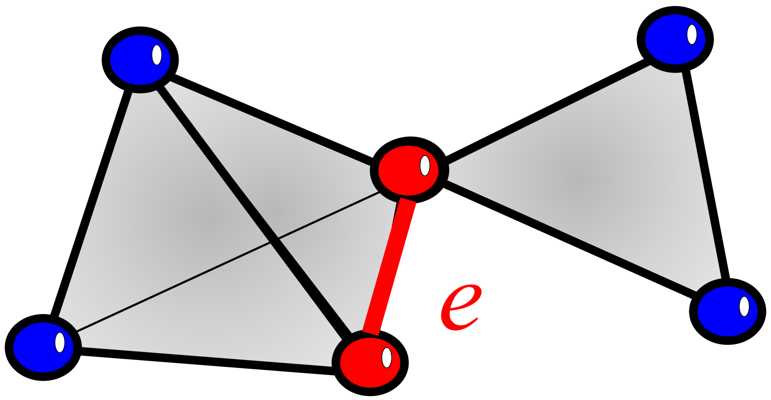
$\mathcal{C}$



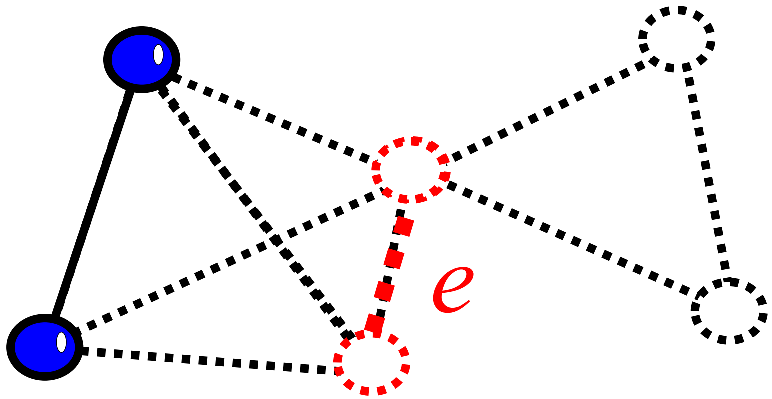
Link(v,$\mathcal{C}$)



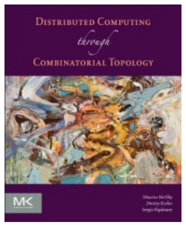
More Links



$\mathcal{C}$



Link(e,$\mathcal{C}$)



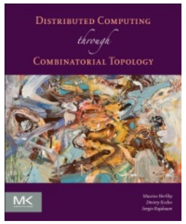
Join

Let $\mathcal{A}$ and $\mathcal{B}$ be complexes with disjoint sets of vertices

their *join* $\mathcal{A}*\mathcal{B}$ is the complex

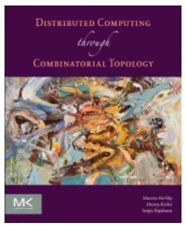
with vertices $V(\mathcal{A}) \cup V(\mathcal{B})$

and simplices $\alpha \cup \beta$, where $\alpha \in \mathcal{A}$, and $\beta \in \mathcal{B}$.



Join

$\mathcal{A}$ $\qquad$ $\mathcal{B}$ $\qquad\qquad$ $\mathcal{A}*\mathcal{B}$



Road Map

Simplicial Complexes

Standard Constructions

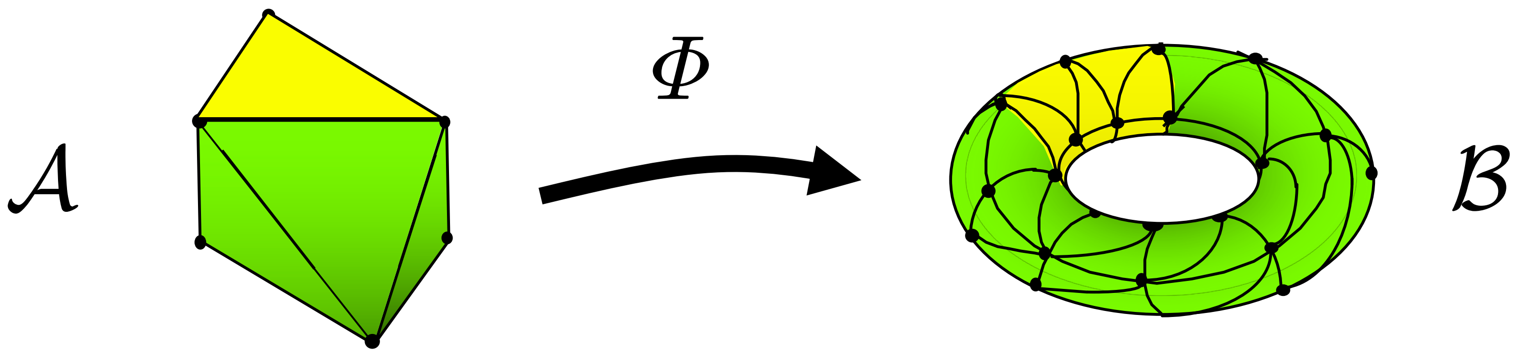
Carrier Maps

Connectivity

Subdivisions

Simplicial & Continuous Approximations

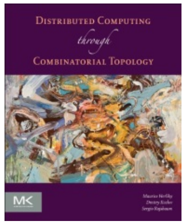
Carrier Map



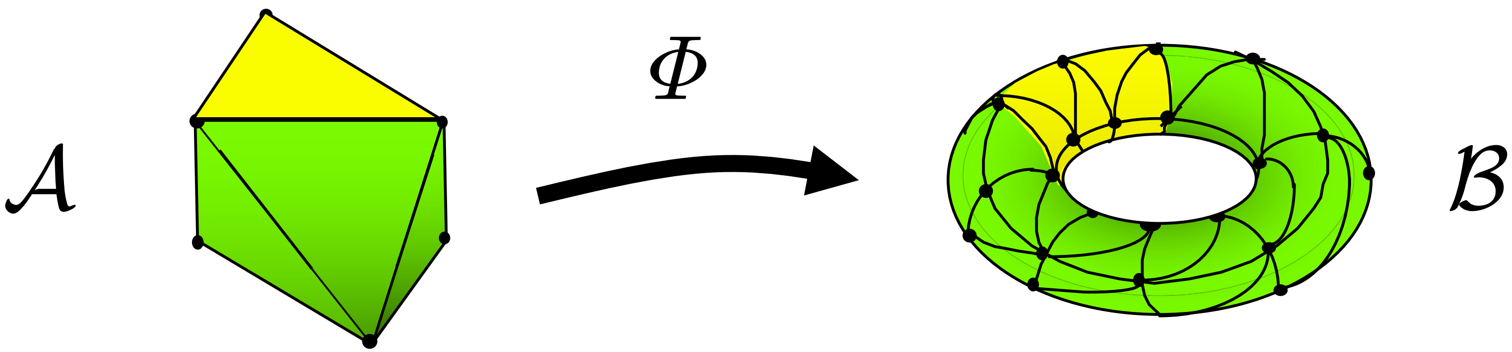
Maps simplex of $\mathcal{A}$

to subcomplex of $\mathcal{B}$

$$\boxed{\Phi: \mathcal{A} \to 2^{\mathcal{B}}}$$



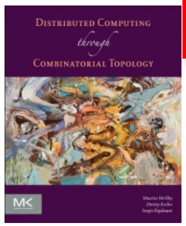
Carrier Maps are Monotonic



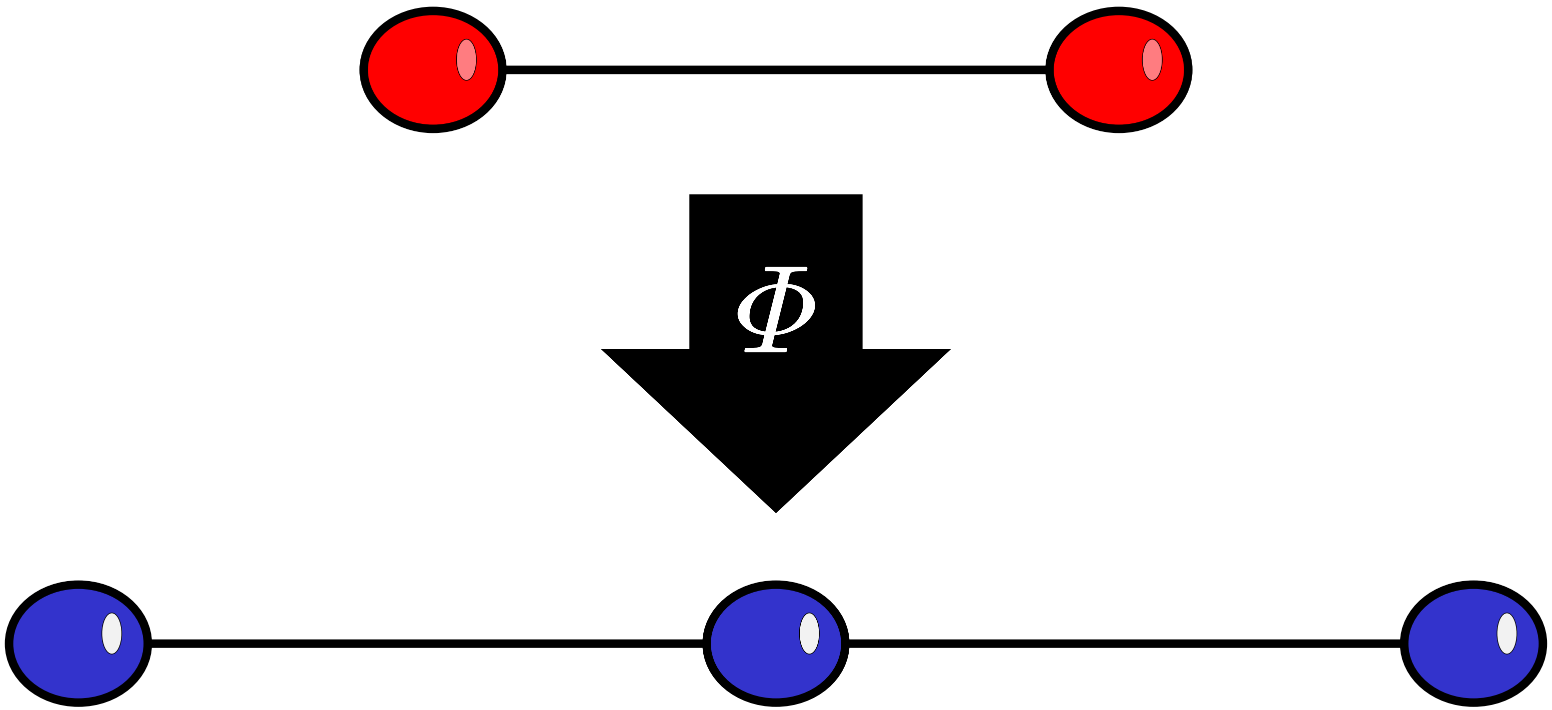
$$\boxed{\text{If } \tau \subseteq \sigma \text{ then } \Phi(\tau) \subseteq \Phi(\sigma)}$$

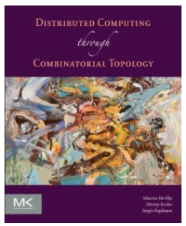
$\boxed{\textit{or}}$

$$\boxed{\text{for } \sigma, \tau \in \mathcal{A},\ \Phi(\sigma \cap \tau) \subseteq \Phi(\sigma) \cap \Phi(\tau)}$$

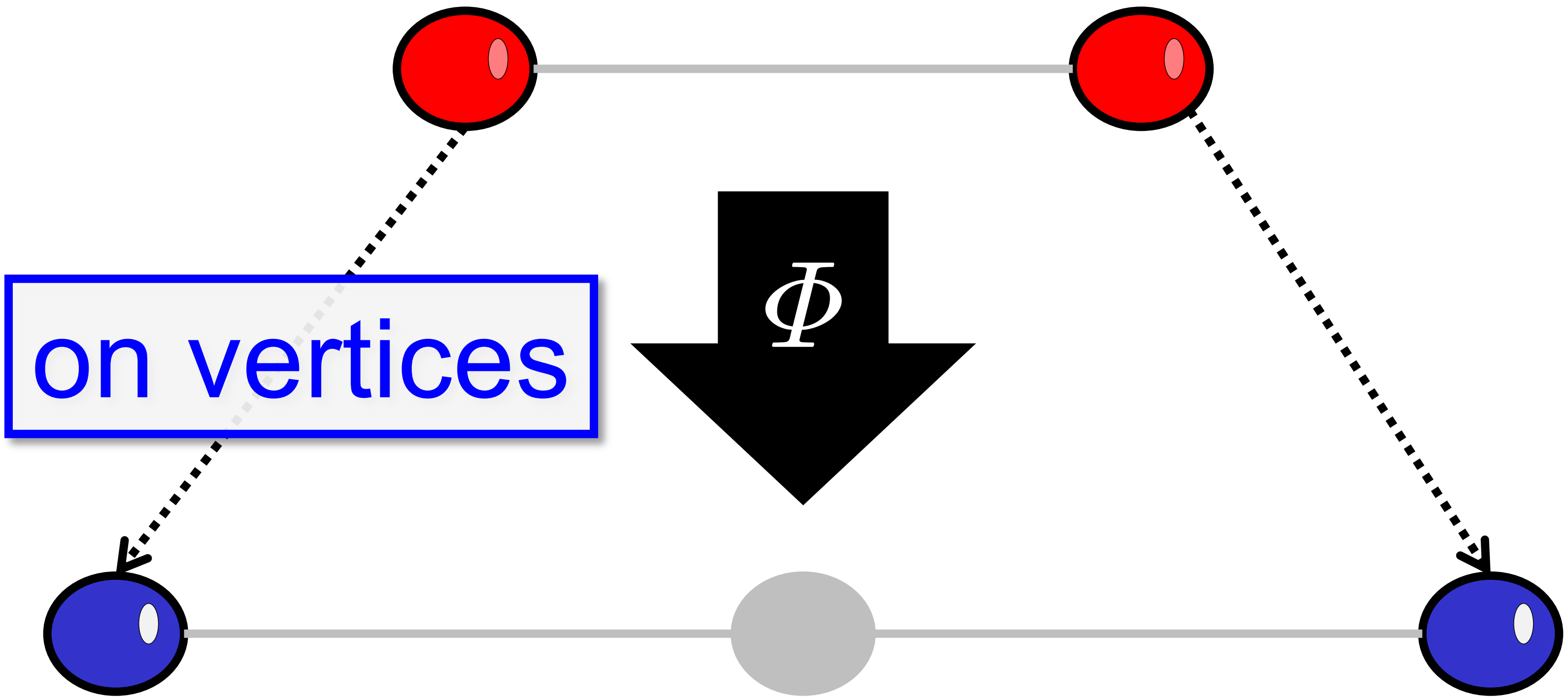


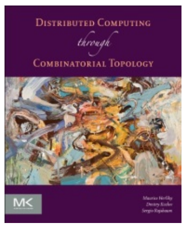
Example


Φ

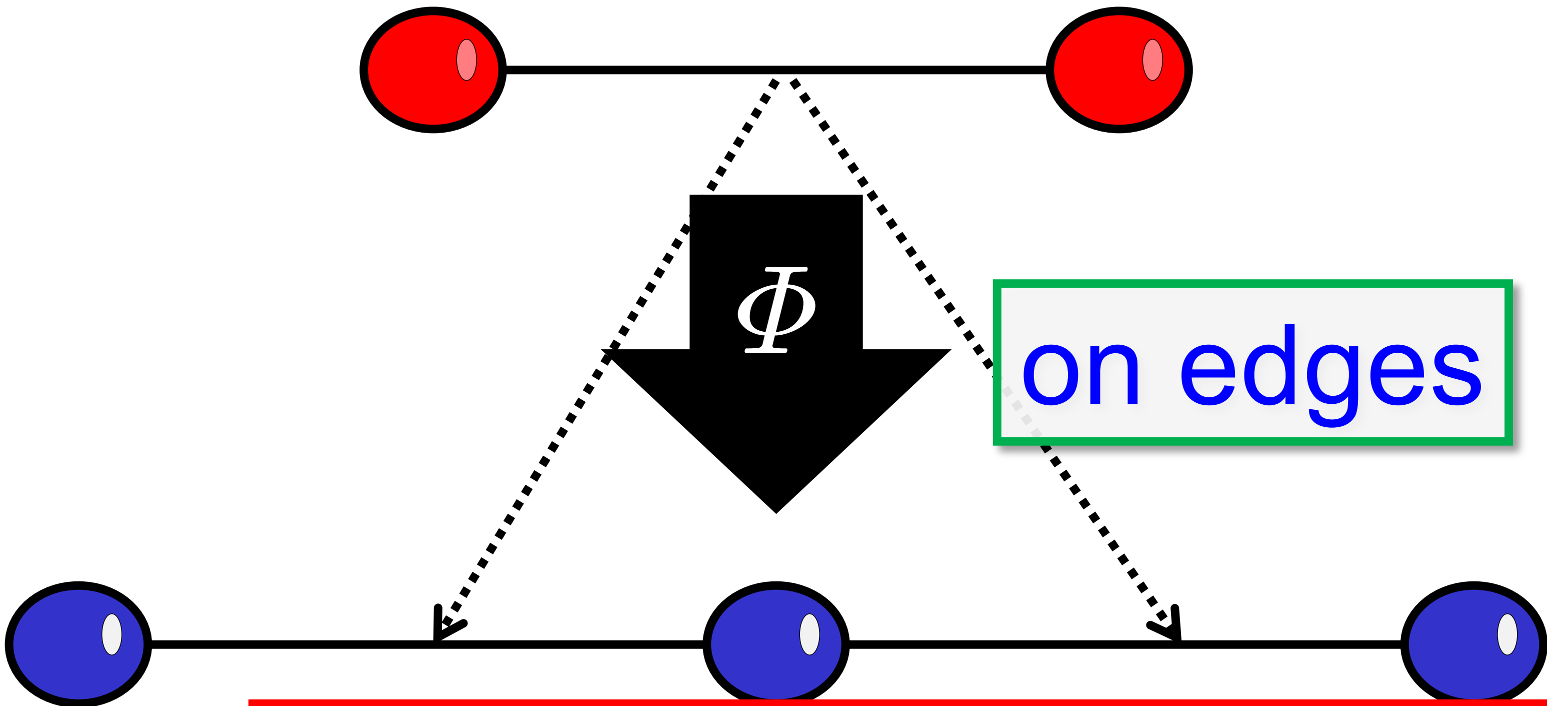

Distributed Computing
through
Combinatorial Topology

Example

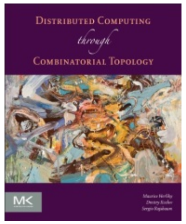

on vertices
Φ


Distributed Computing
through
Combinatorial Topology

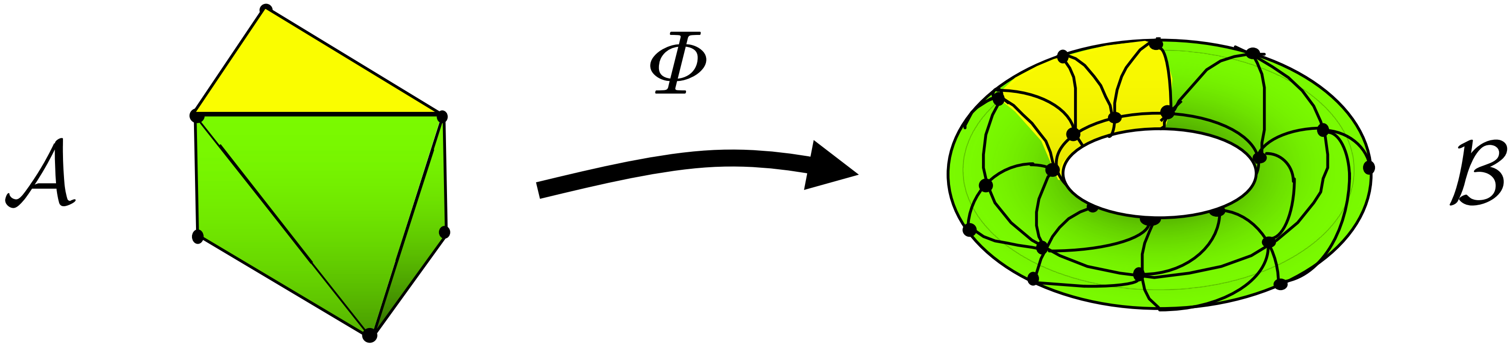
Example



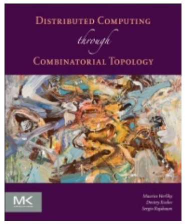
There is no simplicial map carried by Φ:
endpoints must be sent to endpoints!



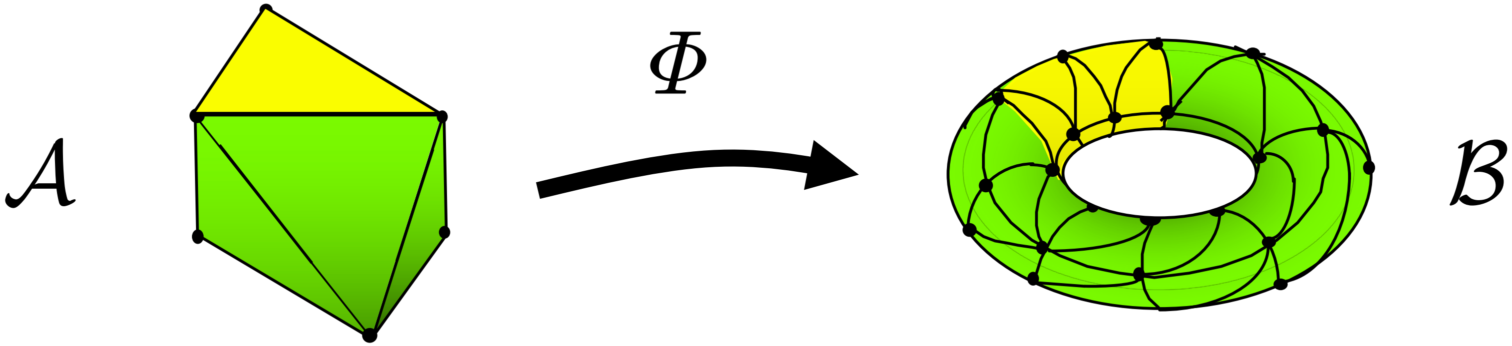
Strict Carrier Maps



for all $\sigma,\tau \in \mathcal{A}$, $\Phi(\sigma \cap \tau) = \Phi(\sigma) \cap \Phi(\tau)$

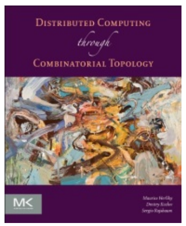


Strict Carrier Maps

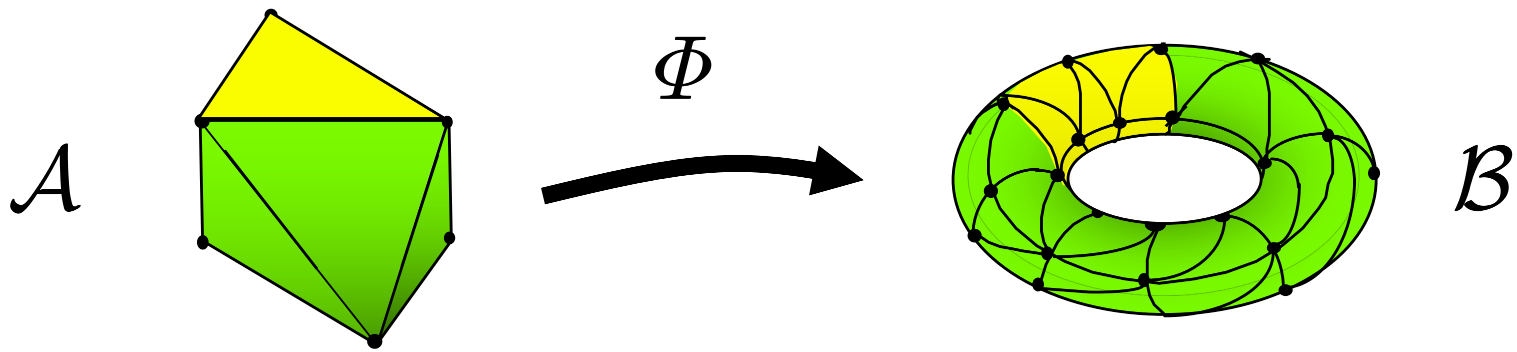


for all $\sigma,\tau \in \mathcal{A}$, $\Phi(\sigma\cap\tau) = \Phi(\sigma)\cap\Phi(\tau)$

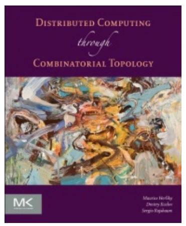
replace $\subseteq$ with $=$



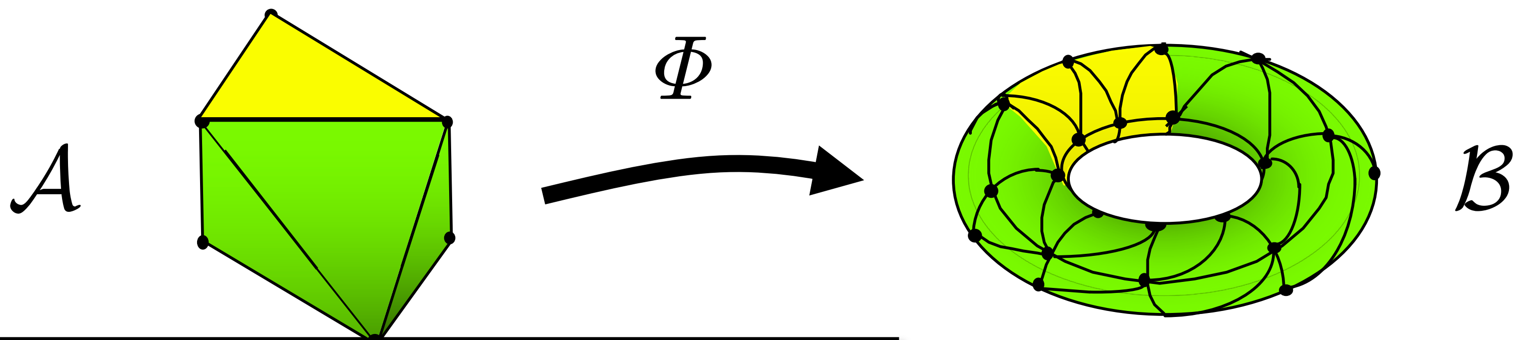
Rigid Carrier Maps



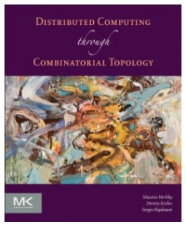
for $\sigma \in \mathcal{A}$, $\Phi(\sigma)$ is pure of dimension $\dim \sigma$



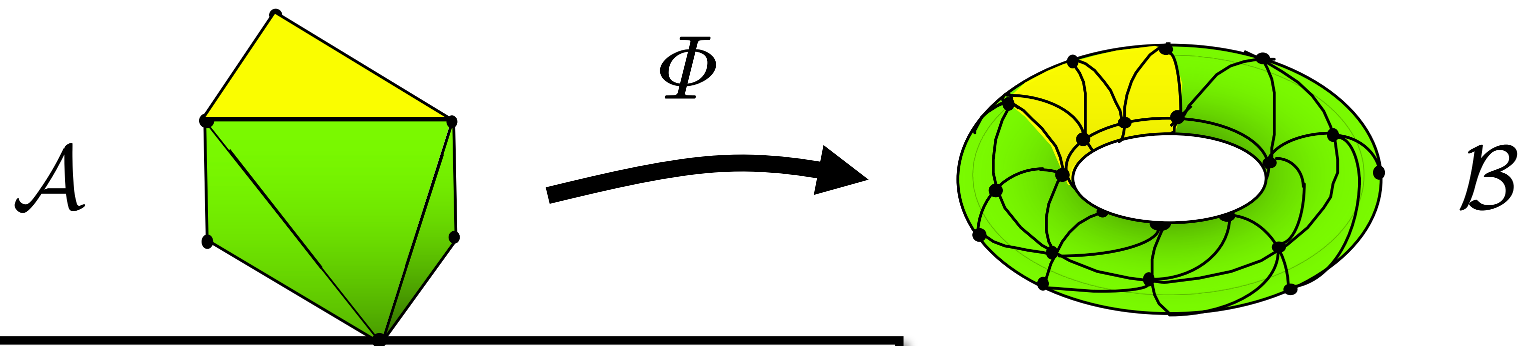
Carrier of a Simplex



given *strict* $\Phi: \mathcal{A} \to 2^{\mathcal{B}}$



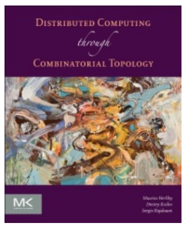
Carrier of a Simplex



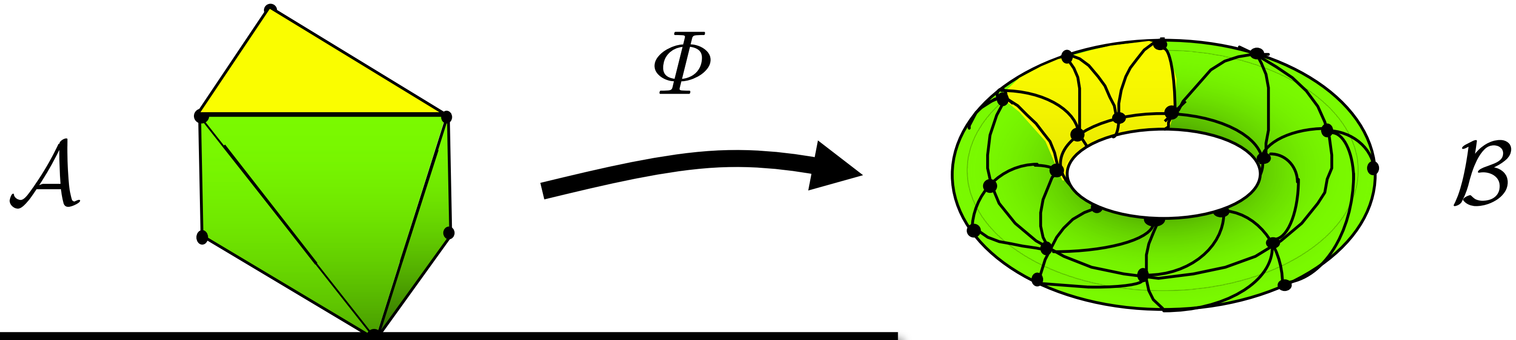
given *strict* Φ: $\mathcal{A} \rightarrow 2^{\mathcal{B}}$

for each $\tau \in \mathcal{B}$,

$\exists$ unique smallest $\sigma \in \mathcal{A}$ such that $\tau \in \Phi(\sigma)$.



Carrier of a Simplex

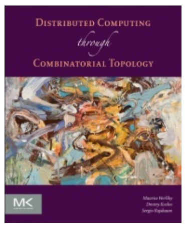


given *strict* $\Phi: \mathcal{A} \to 2^{\mathcal{B}}$

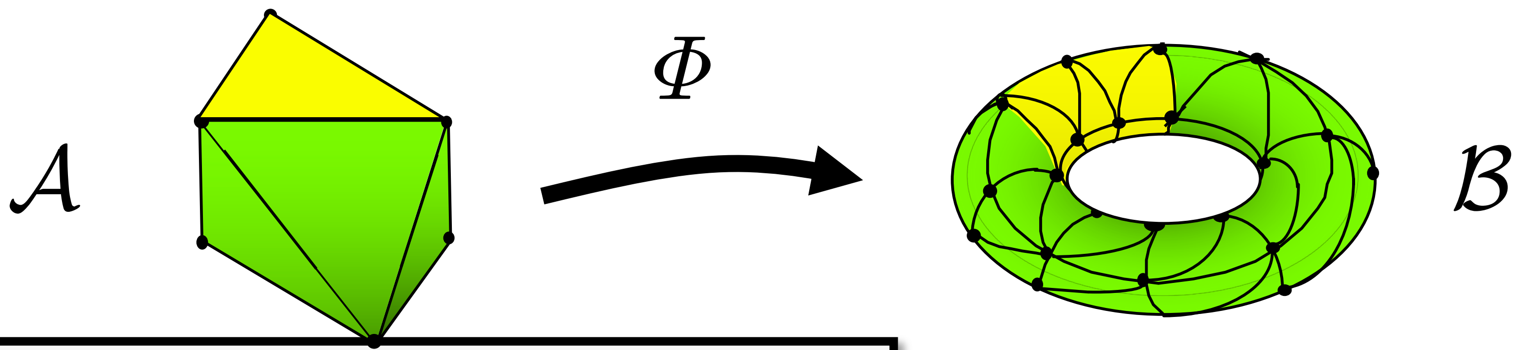
for each $\tau \in \mathcal{B}$,

$\exists$ unique smallest $\sigma \in \mathcal{A}$ such that $\tau \in \Phi(\sigma)$.

$\sigma = \mathrm{Car}(\tau, \Phi)$



Carrier of a Simplex



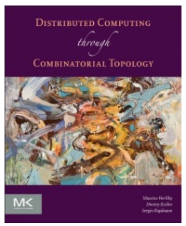
given *strict* Φ: $\mathcal{A} \rightarrow 2^{\mathcal{B}}$

for each $\tau \in \mathcal{B}$,

$\exists$ unique smallest $\sigma \in \mathcal{A}$ such that $\tau \in \Phi(\sigma)$.

σ = Car(τ, Φ)

sometimes omitted

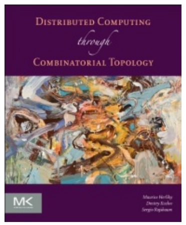


Carrier Map Carried By Carrier Map

Given carrier maps

$$\Phi\colon \mathcal{A} \to 2^{\mathcal{B}}$$

$$\Psi\colon \mathcal{A} \to 2^{\mathcal{B}}$$



Carrier Map Carried By Carrier Map

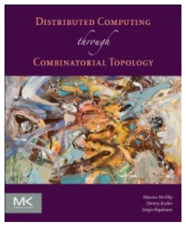
Given carrier maps

$\Phi: \mathcal{A} \to 2^{\mathcal{B}}$

$\Psi: \mathcal{A} \to 2^{\mathcal{B}}$

Φ is *carried by* Ψ if

for all $\sigma \in \mathcal{A}$, $\Phi(\sigma) \subseteq \Psi(\sigma)$



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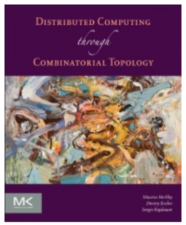
Φ is *carried by* Ψ if

for all $\sigma \in \mathcal{A}$, $\Phi(\sigma) \subseteq \Psi(\sigma)$

written: $\Phi \subseteq \Psi$

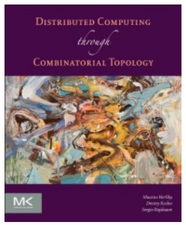


Simplicial Map Carried By Carrier Map

Given carrier and simplicial maps

$\mathit{\Phi}\colon \mathcal{A} \to 2^{\mathcal{B}}$

$\varphi\colon \mathcal{A} \to \mathcal{B}$



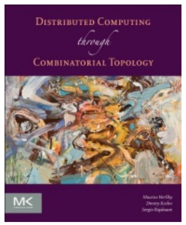
Simplicial Map Carried By Carrier Map

Given carrier and simplicial maps

$\Phi: \mathcal{A} \to 2^{\mathcal{B}}$

$\varphi: \mathcal{A} \to \mathcal{B}$

φ is *carried by* Φ if



Simplicial Map Carried By Carrier Map

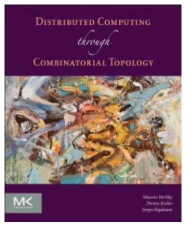
Given carrier and simplicial maps

$\Phi\colon \mathcal{A} \to 2^{\mathcal{B}}$

$\varphi\colon \mathcal{A} \to \mathcal{B}$

φ is *carried by* Φ if

for all $\sigma \in \mathcal{A}$, $\varphi(\sigma) \subseteq \Phi(\sigma)$



Simplicial Map Carried By Carrier Map

Given carrier and simplicial maps

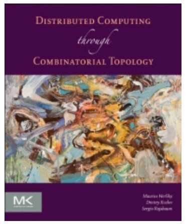
$\Phi\colon \mathcal{A} \to 2^{\mathcal{B}}$

$\varphi\colon \mathcal{A} \to \mathcal{B}$

φ is *carried by* Φ if

for all $\sigma \in \mathcal{A}$, $\varphi(\sigma) \subseteq \Phi(\sigma)$

written: $\varphi \subseteq \Phi$

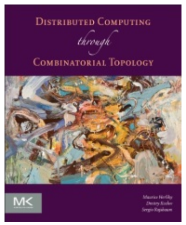


Continuous Map Carried By Carrier Map

Given carrier and continuous maps

$\Phi\colon \mathcal{A} \to 2^{\mathcal{B}}$

$f\colon |\mathcal{A}| \to |\mathcal{B}|$



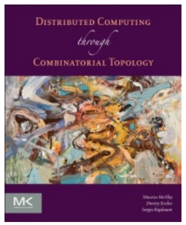
Continuous Map Carried By Carrier Map

Given carrier and continuous maps

$\Phi\colon \mathcal{A} \to 2^{\mathcal{B}}$

f: $|\mathcal{A}| \to |\mathcal{B}|$

f is *carried by* Φ if



Continuous Map Carried By Carrier Map

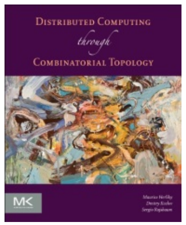
Given carrier and continuous maps

Φ: $\mathcal{A} \to 2^{\mathcal{B}}$

f: $|\mathcal{A}| \to |\mathcal{B}|$

f is *carried by* Φ if

for all $\sigma \in \mathcal{A}$, $f(|\sigma|) \subseteq |\Phi(\sigma)|$



Compositions

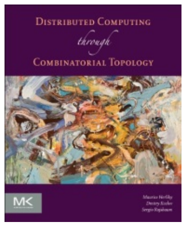
Given carrier maps

$\Phi\colon \mathcal{A} \to 2^{\mathcal{B}}$

$\Psi\colon \mathcal{B} \to 2^{\mathcal{C}}$

their *composition* is

$$(\Psi \circ \Phi)(\sigma) := \cup_{\tau \in \Phi(\sigma)} \Psi(\tau)$$



Theorem

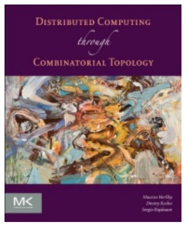
If Φ, Ψ are both

strict

so is $\Phi \circ \Psi$

rigid

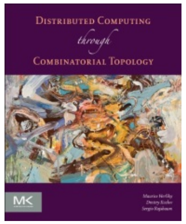
so is $\Phi \circ \Psi$



Compositions

Given carrier and simplicial maps

$\Phi: \mathcal{A} \to 2^{\mathcal{B}}$ $\qquad$ $\varphi: \mathcal{C} \to \mathcal{A}$



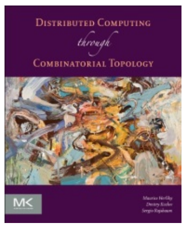
Compositions

Given carrier and simplicial maps

$\Phi: \mathcal{A} \to 2^{\mathcal{B}}$ $\quad$ $\varphi: \mathcal{C} \to \mathcal{A}$

their *composition* is the *carrier map*

$(\Phi \circ \varphi): \mathcal{C} \to 2^{\mathcal{B}}$



Compositions

Given carrier and simplicial maps

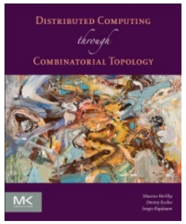
$\Phi: \mathcal{A} \to 2^{\mathcal{B}}$ $\quad$ $\varphi: \mathcal{C} \to \mathcal{A}$

their *composition* is the *carrier map*

$(\Phi \circ \varphi): \mathcal{C} \to 2^{\mathcal{B}}$

defined by

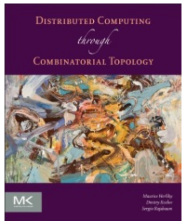
$$(\Phi \circ \varphi)(\sigma) := \Phi(\varphi(\sigma))$$



Compositions

Given carrier and simplicial maps

$\boxed{\Phi: \mathcal{A} \to 2^{\mathcal{B}}}$ $\boxed{\varphi: \mathcal{B} \to \mathcal{C}}$



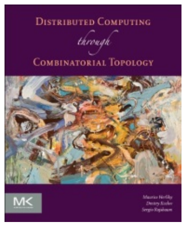
Compositions

Given carrier and simplicial maps

$\Phi: \mathcal{A} \to 2^{\mathcal{B}}$ $\varphi: \mathcal{B} \to \mathcal{C}$

their *composition* is the *carrier map*

$(\varphi \circ \Phi): \mathcal{A} \to 2^{\mathcal{C}}$



Compositions

Given carrier and simplicial maps

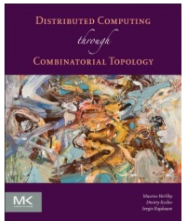
$$\Phi: \mathcal{A} \to 2^{\mathcal{B}} \qquad \varphi: \mathcal{B} \to \mathcal{C}$$

their *composition* is the *carrier map*

$$(\varphi \circ \Phi): \mathcal{A} \to 2^{\mathcal{C}}$$

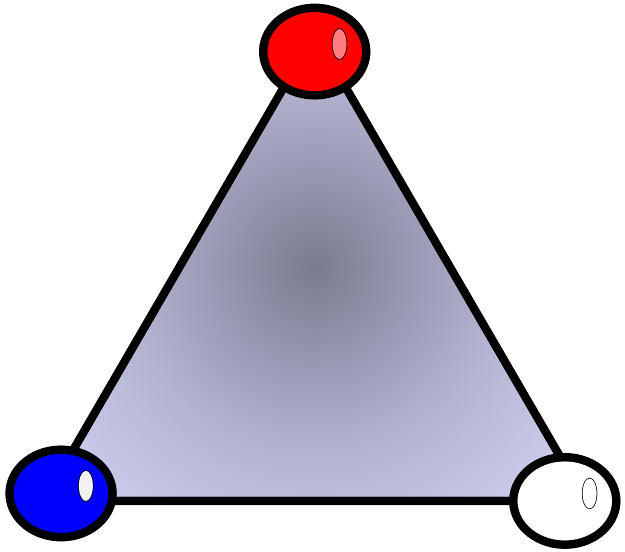
defined by

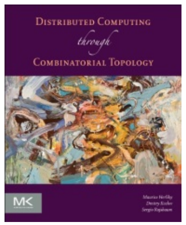
$$(\Phi \circ \varphi)(\sigma) := \bigcup_{\tau \in \Phi(\sigma)} \varphi(\tau)$$



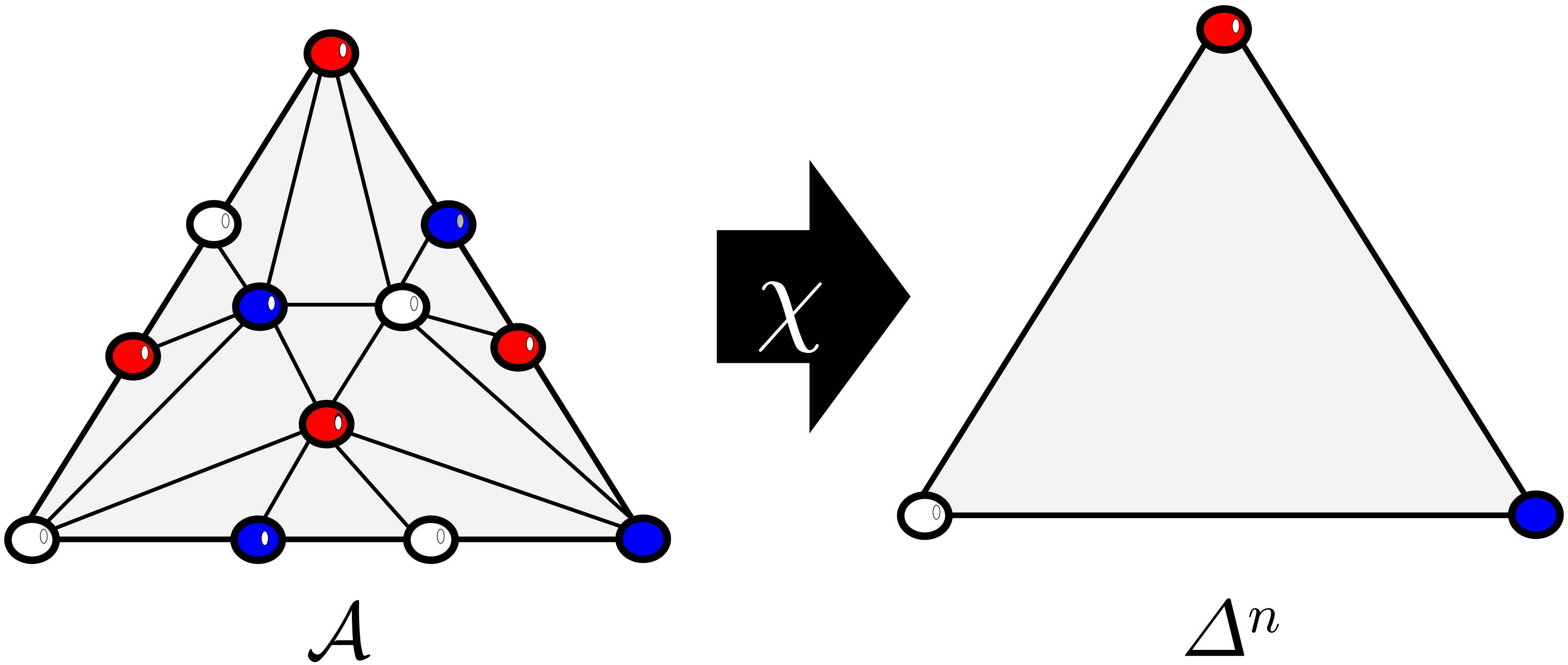
Colorings

$$\Delta^n :=$$

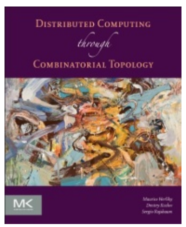




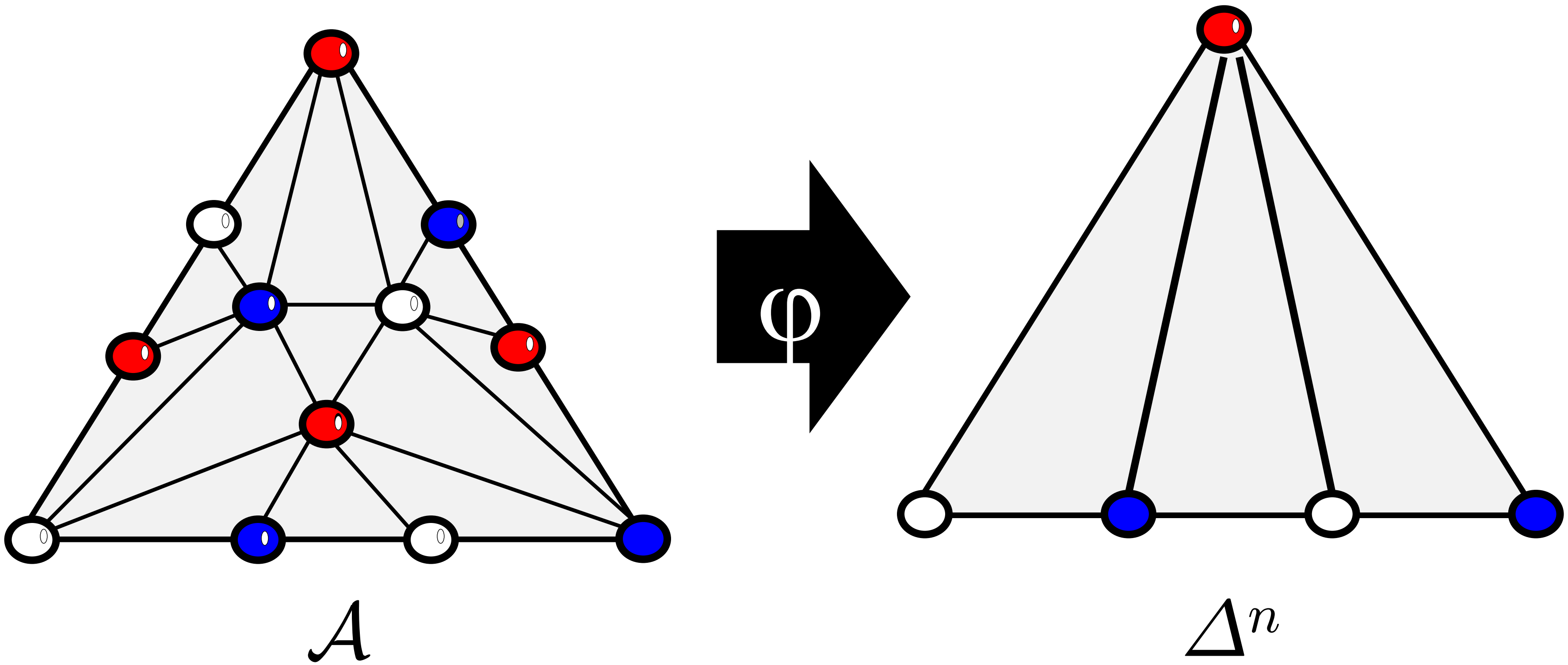
Chromatic Complex



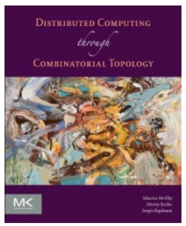
rigid simplicial map



Color-Preserving Simplicial Map



color of v = color of $\varphi(v)$



Road Map

Simplicial Complexes

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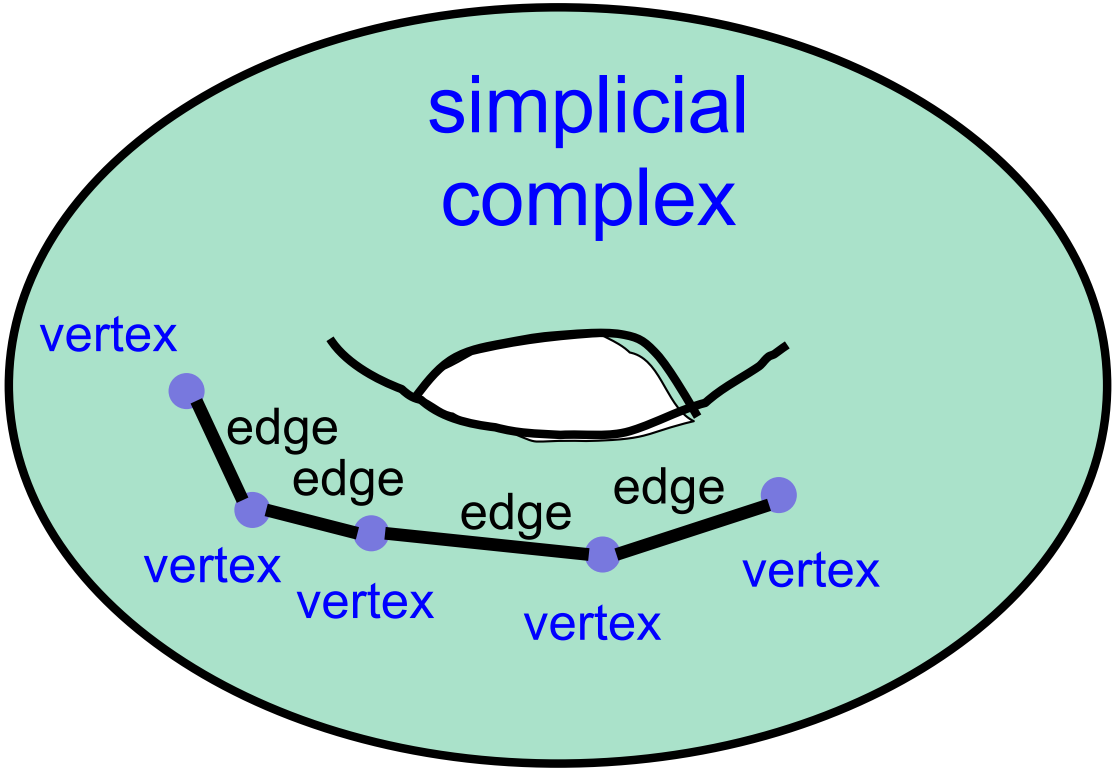
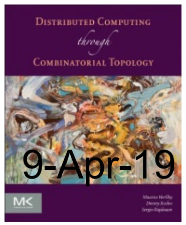
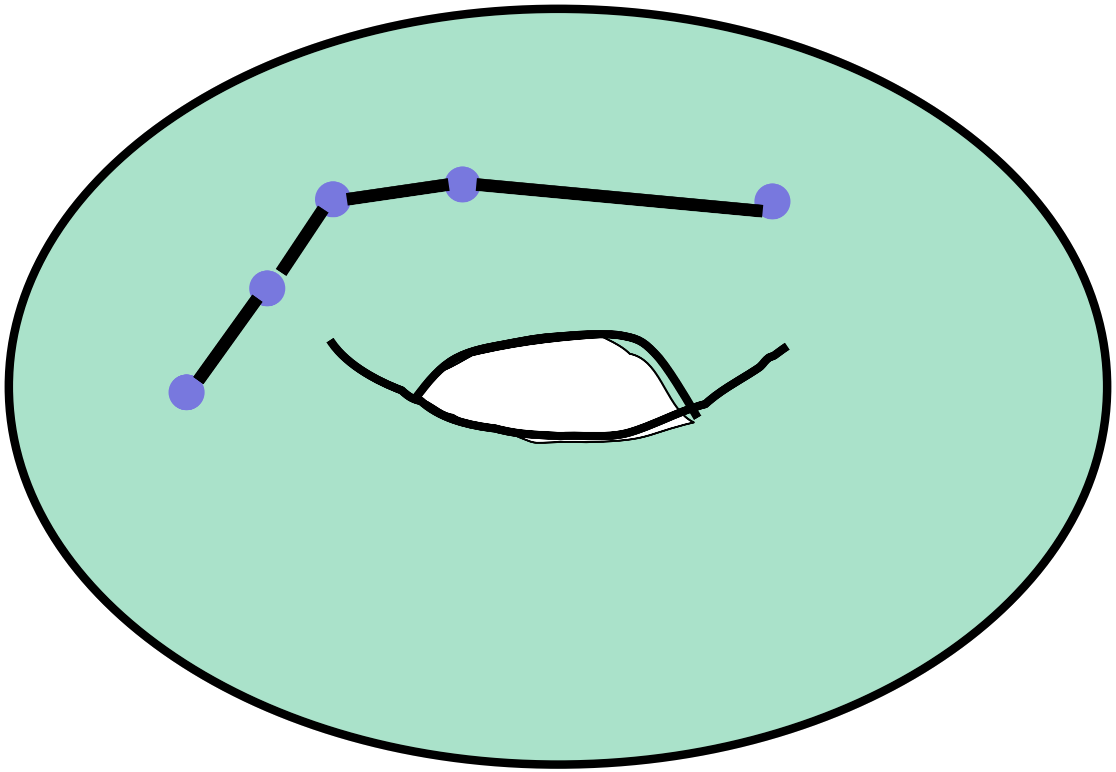
A Path


simplicial
complex
vertex
edge
edge
edge
edge
vertex
vertex
vertex
vertex


Distributed Computing
through
Combinatorial Topology
MK

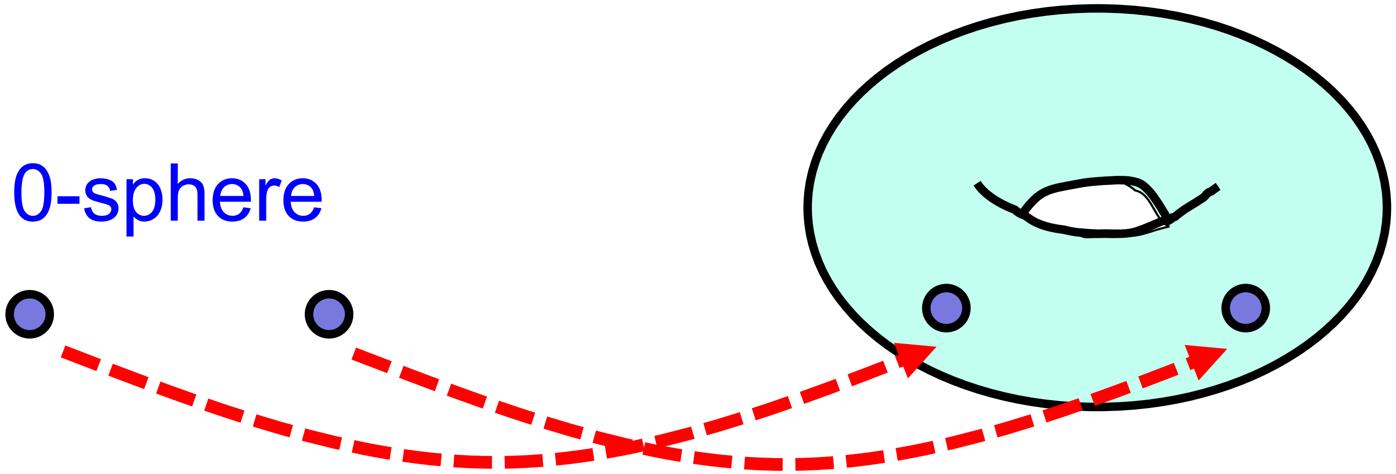
Path Connected



Any two vertexes can be linked by a path

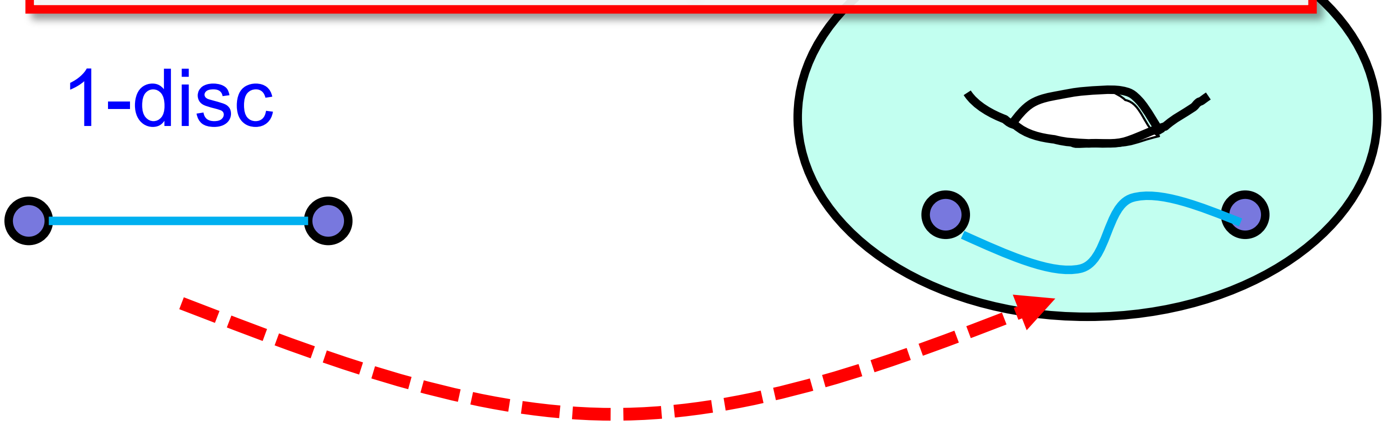
Rethinking Path Connectivity

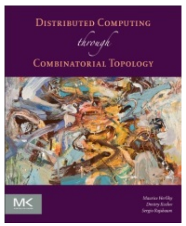
0-sphere



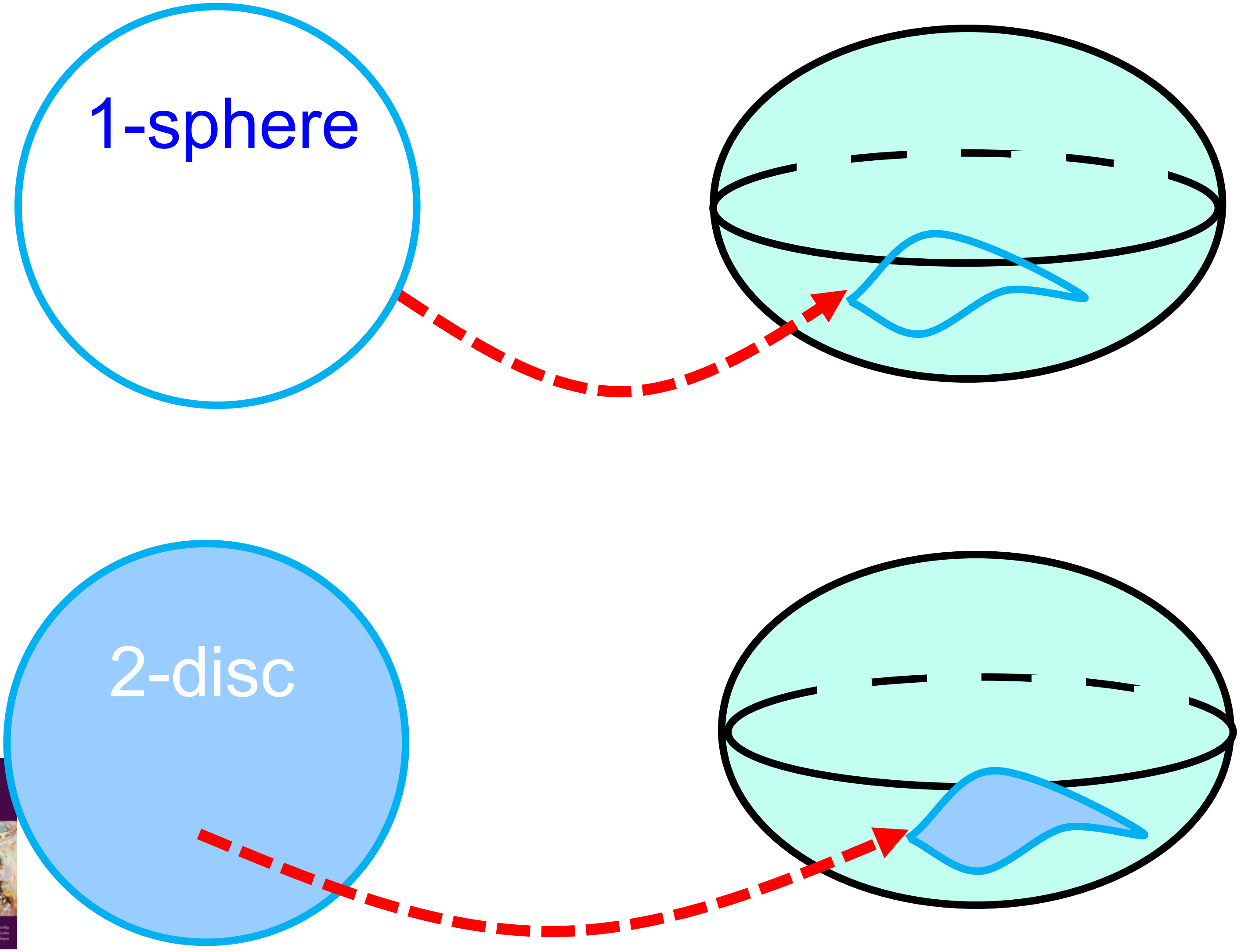
Let's call this complex *0-connected*

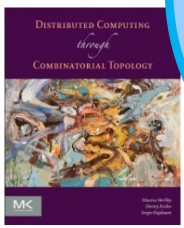
1-disc



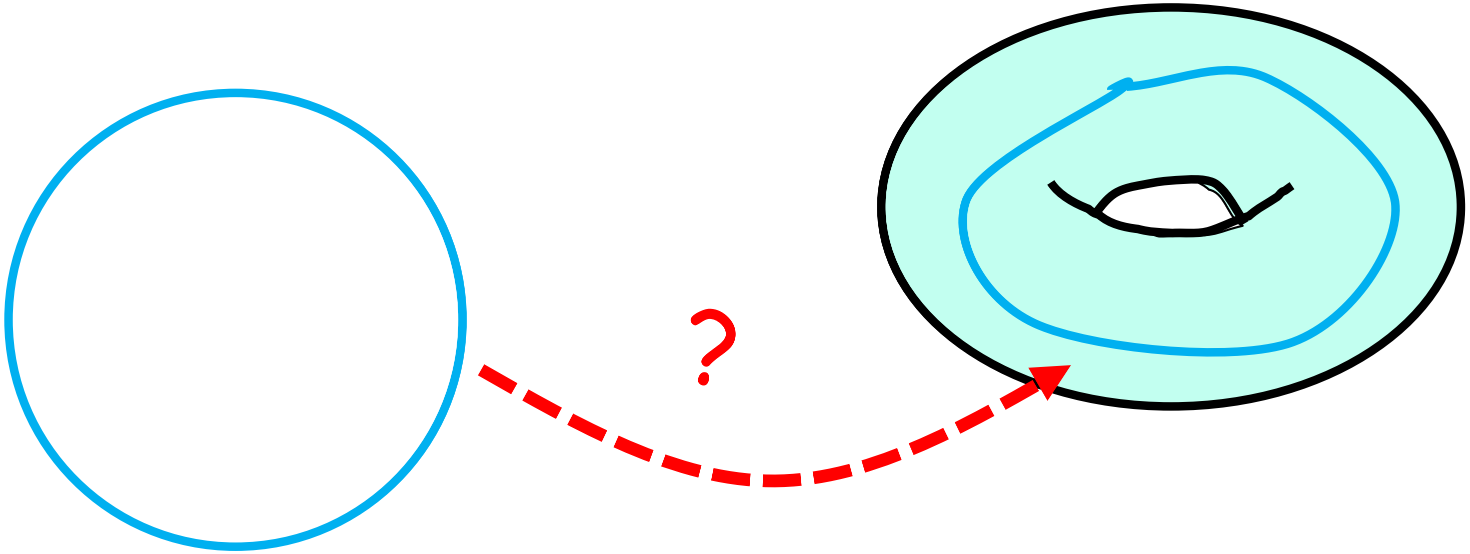


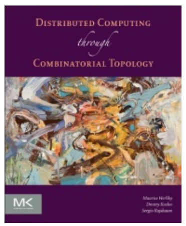
1-Connectivity





This Complex is not 1-Connected

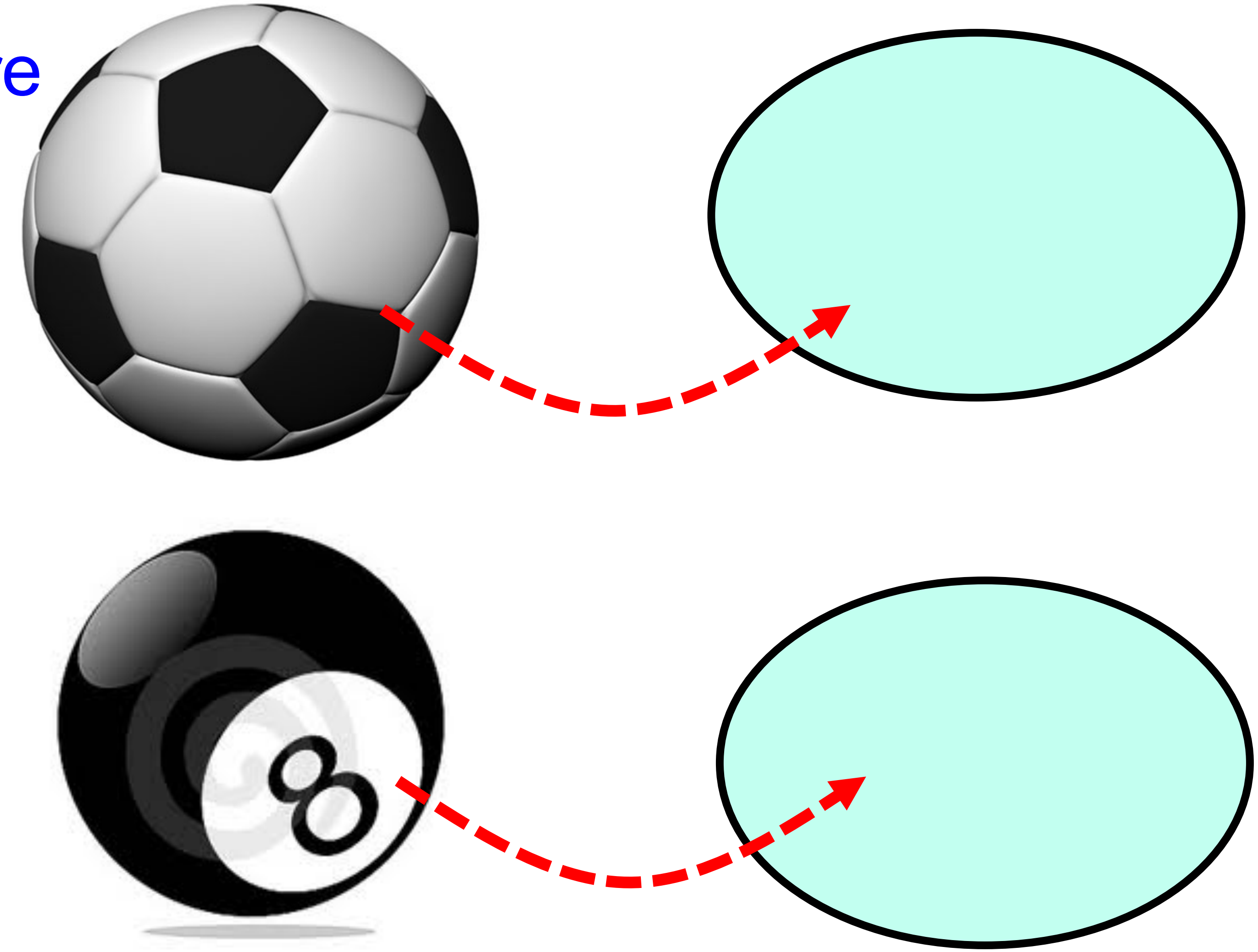


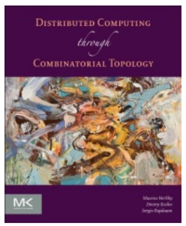


2-Connectivity

2-sphere

3-disk





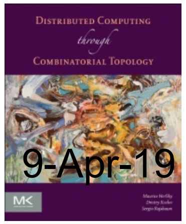
n-connectivity

$\mathcal{C}$ is *n*-connected, if, for $m \leq n$, every continuous map of the *m*-sphere

$$f : S^m \to |\mathcal{C}|$$

can be extended to a continuous map of the (*m+1*)-disk

$$f : D^{m+1} \to |\mathcal{C}|$$



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$\mathcal{C}$ is n-connected, if, for $m \leq n$, every continuous map of the m-sphere

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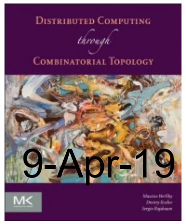
$$f : D^{m+1} \to |\mathcal{C}|$$

$\boxed{\text{(-1)-connected is non-empty}}$



Road Map

Simplicial Complexes

Standard Constructions

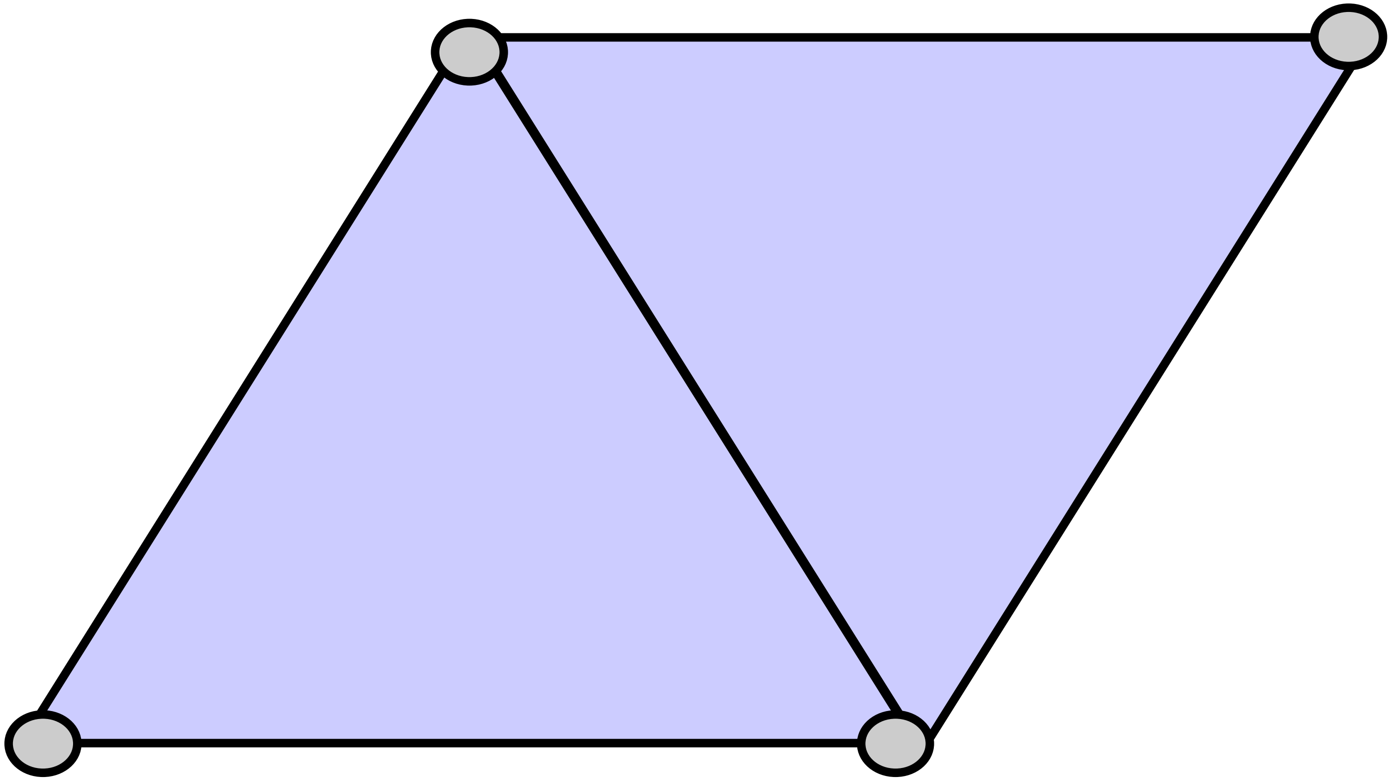
Carrier Maps

Connectivity

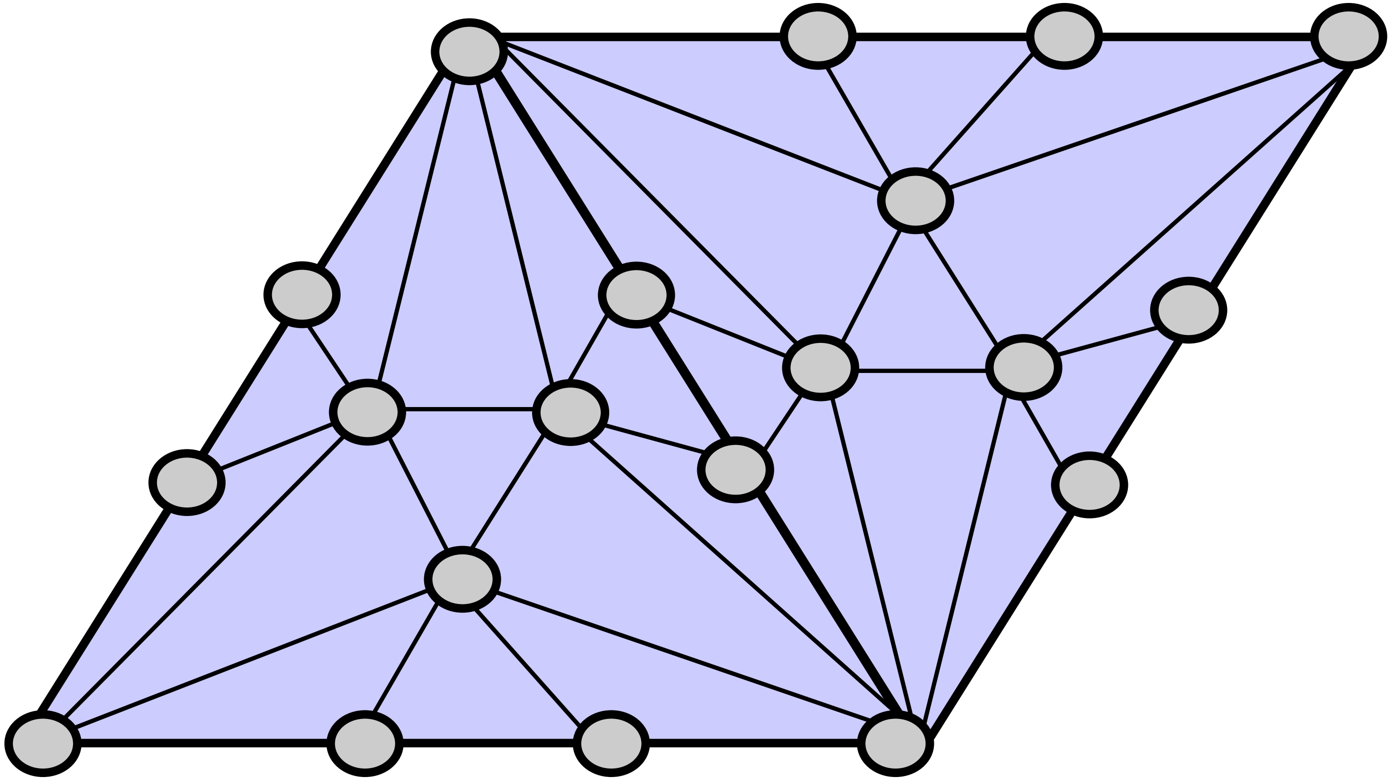
Subdivisions

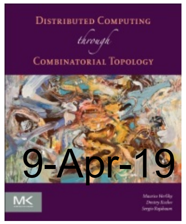
Simplicial & Continuous Approximations

Subdivisions

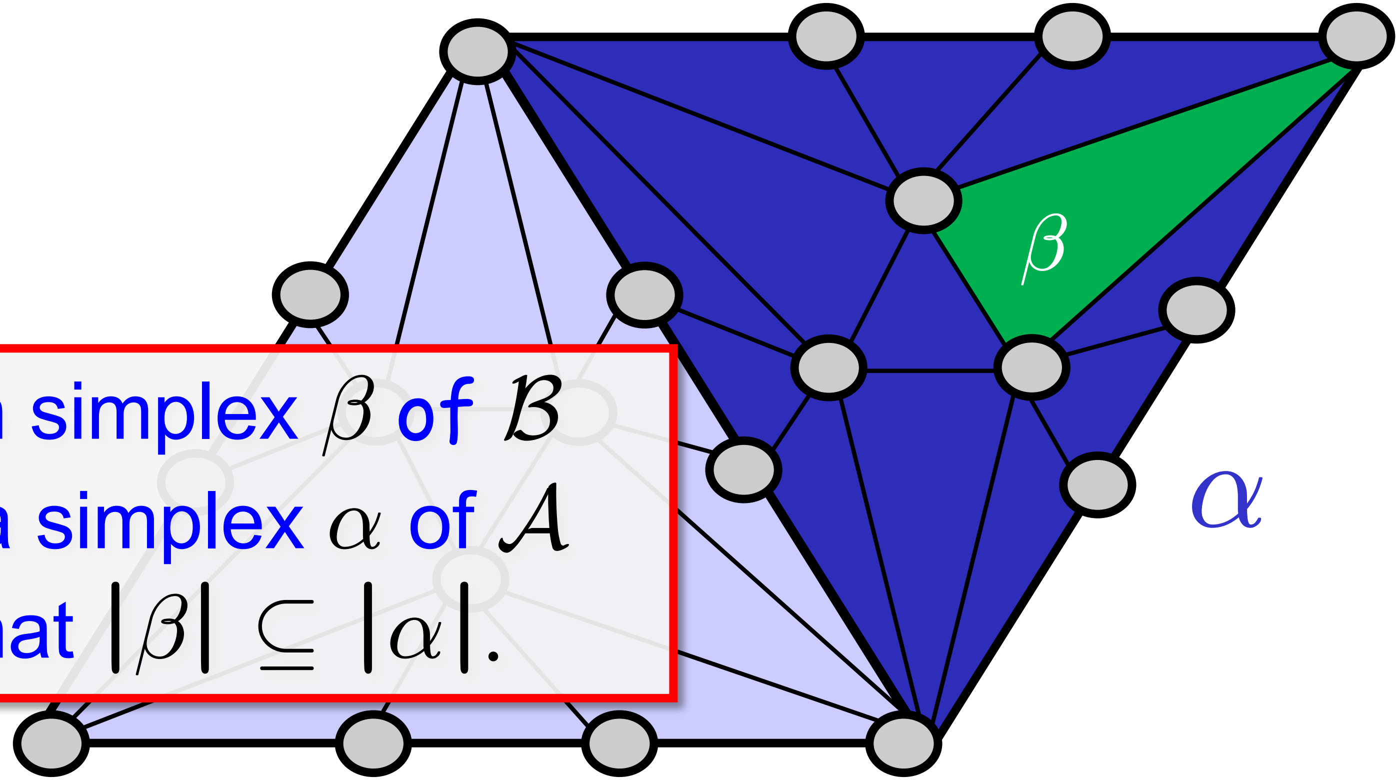


Subdivisions

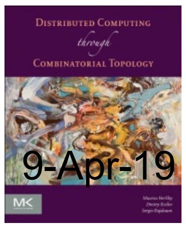




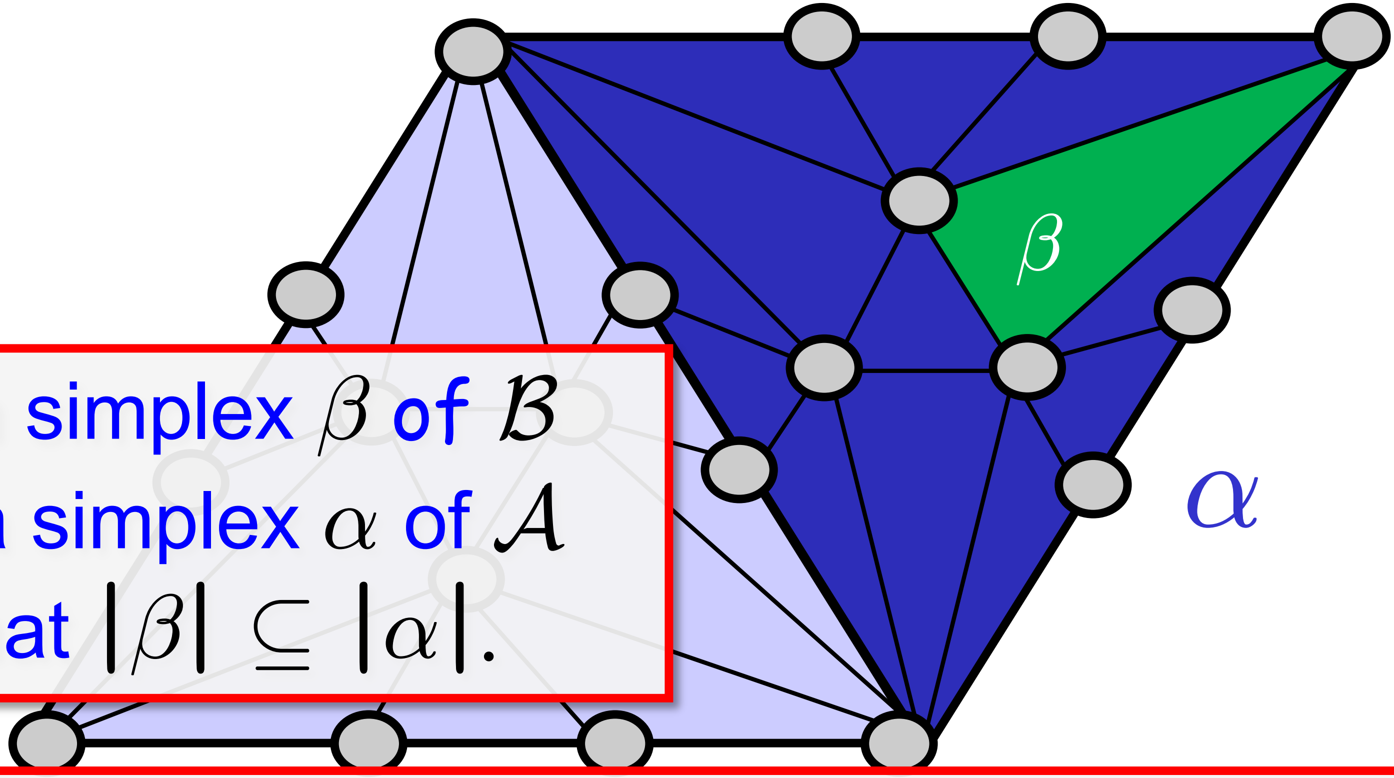
$\mathcal{B}$ is a subdivision of $\mathcal{A}$ if …



For each simplex β of $\mathcal{B}$ there is a simplex α of $\mathcal{A}$ such that $|\beta| \subseteq |\alpha|$.



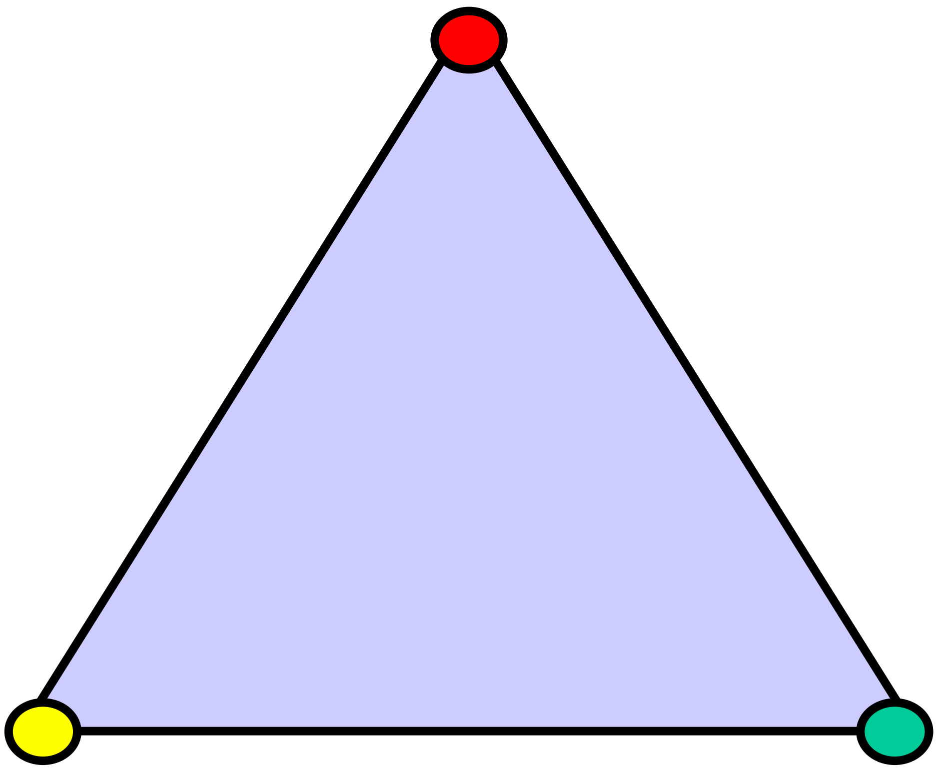
$\mathcal{B}$ is a subdivision of $\mathcal{A}$ if …



For each simplex β of $\mathcal{B}$ there is a simplex α of $\mathcal{A}$ such that $|\beta| \subseteq |\alpha|$.

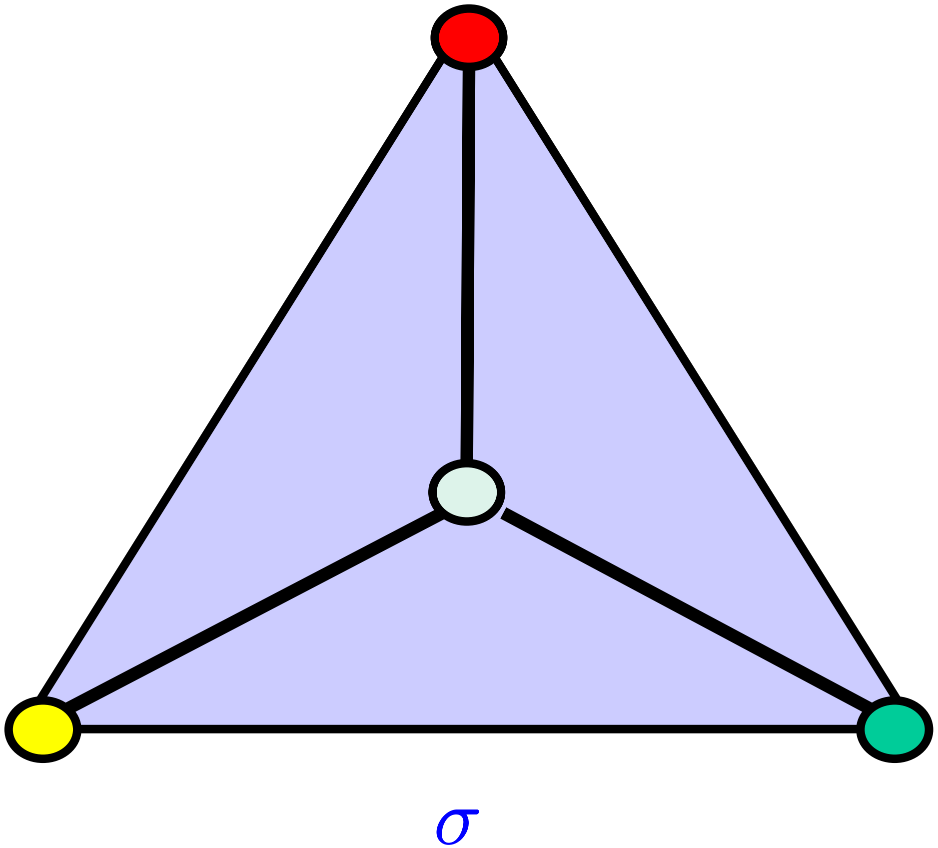
For each simplex α of $\mathcal{A}$, $|\alpha|$ is the union of a finite set of geometric simplexes of $\mathcal{B}$.

Stellar Subdivision

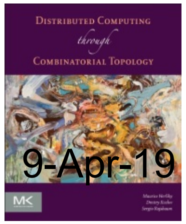


σ

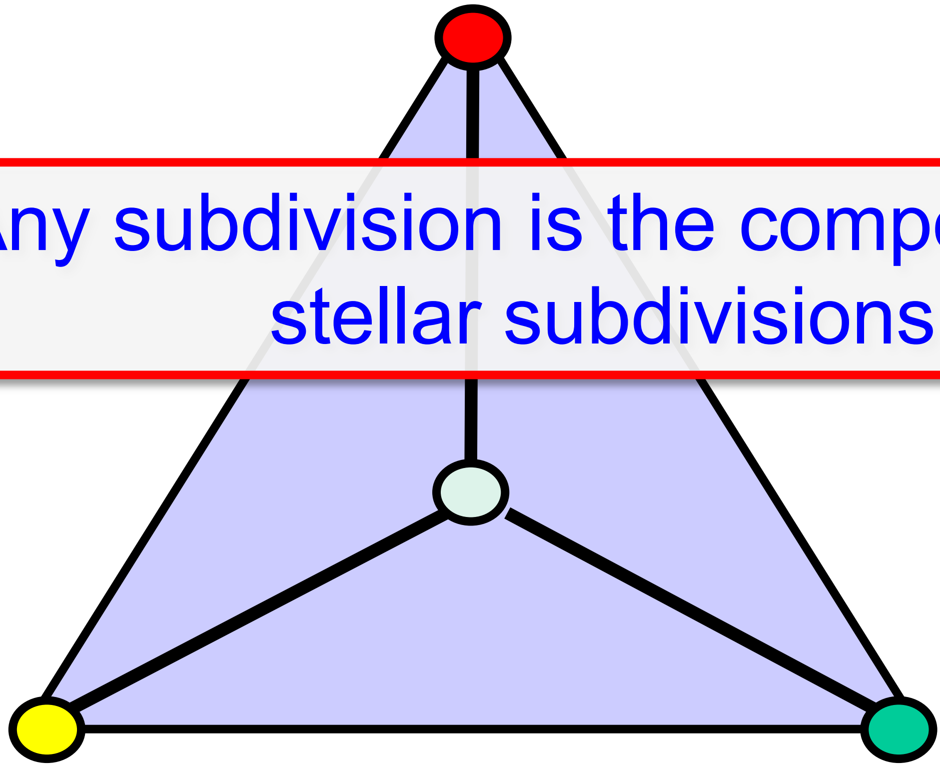
Stellar Subdivision



Stel σ



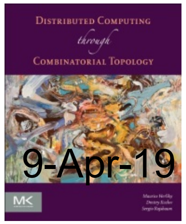
Stellar Subdivision



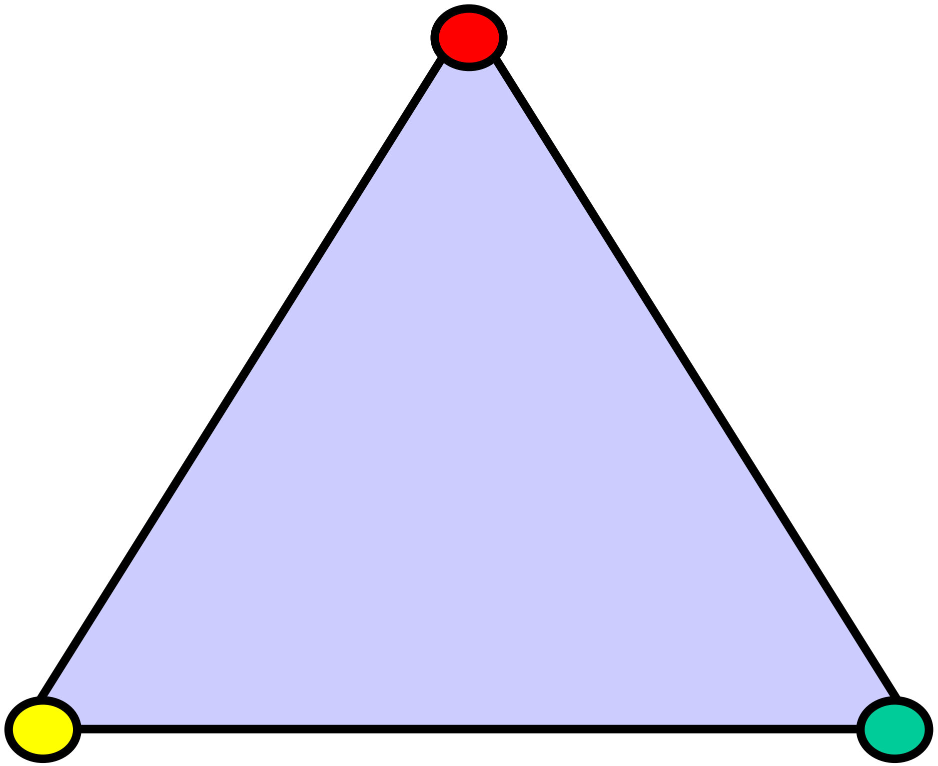
Any subdivision is the composition of stellar subdivisions

σ

Stel σ

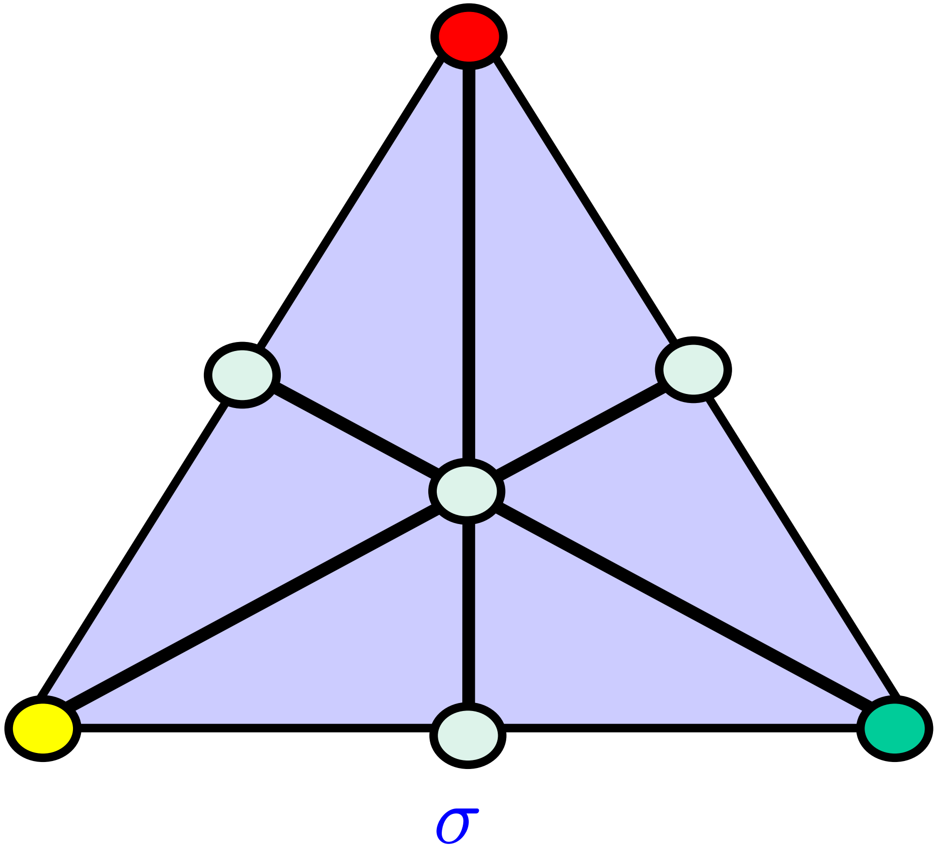


Barycentric Subdivision

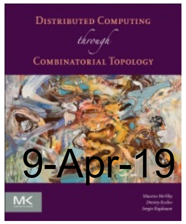


σ

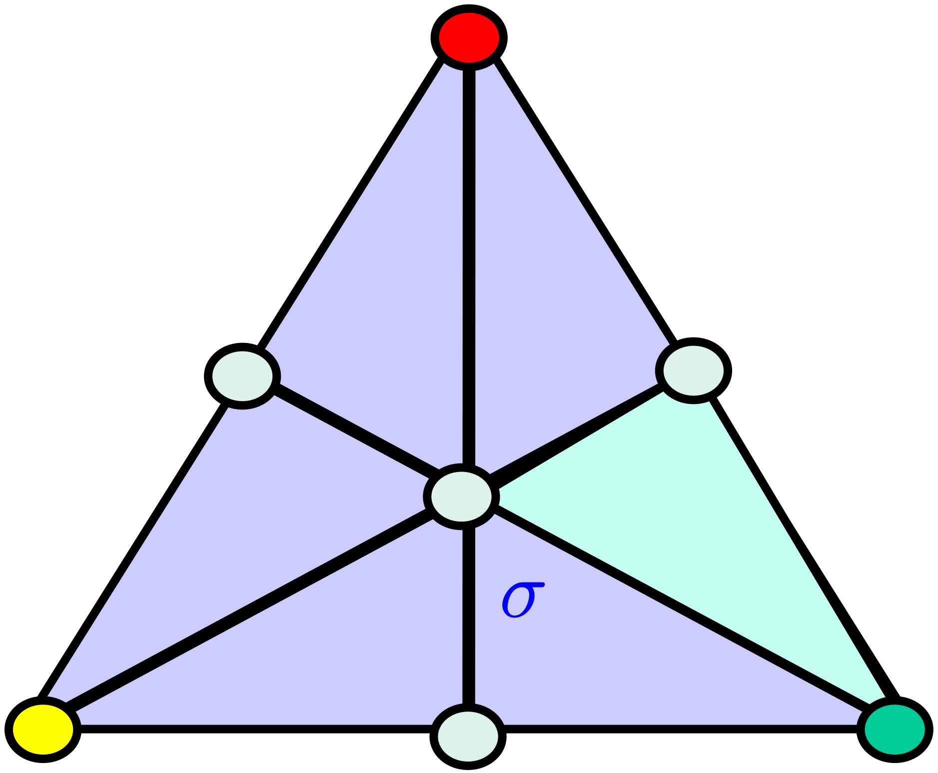
Barycentric Subdivision

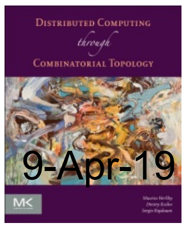


Bary σ



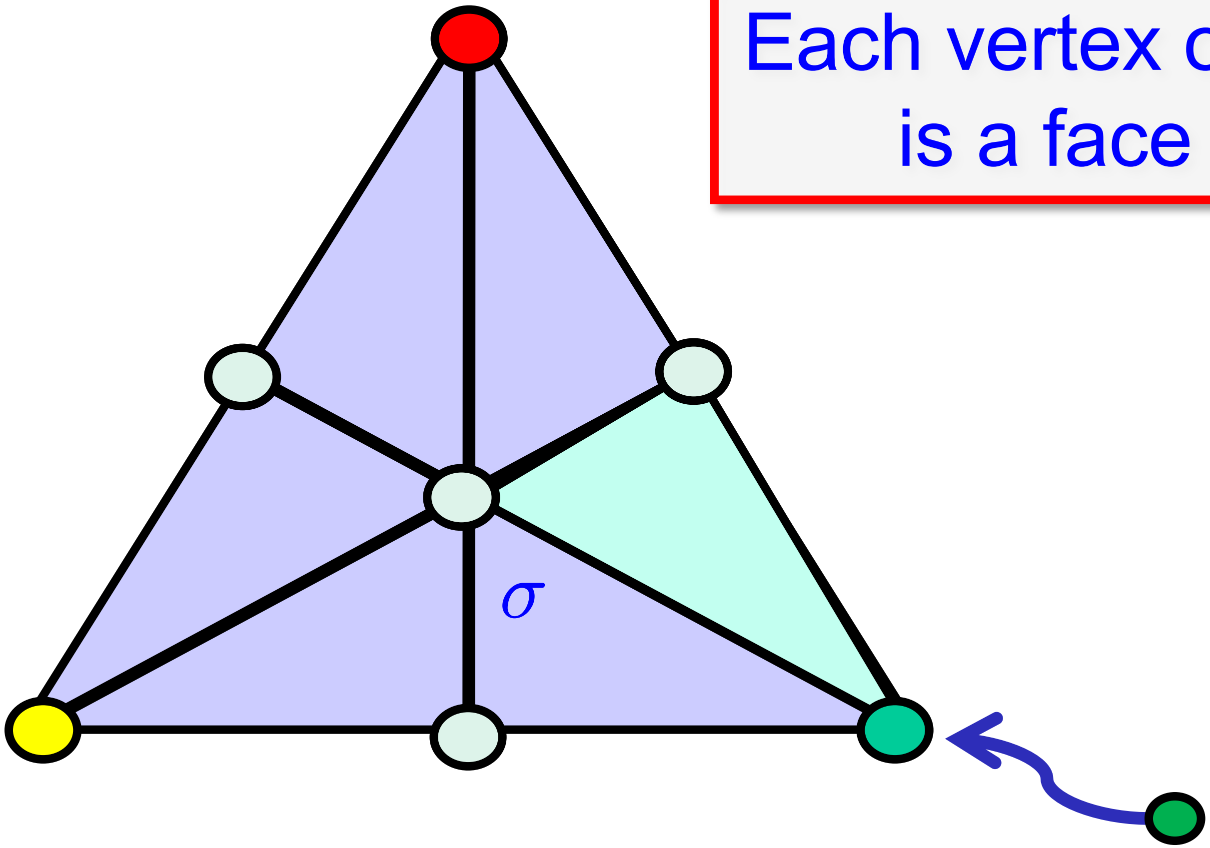
Barycentric Subdivision

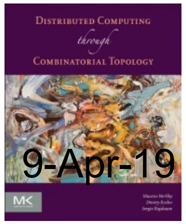




Barycentric Subdivision

Each vertex of Bary σ
is a face of σ





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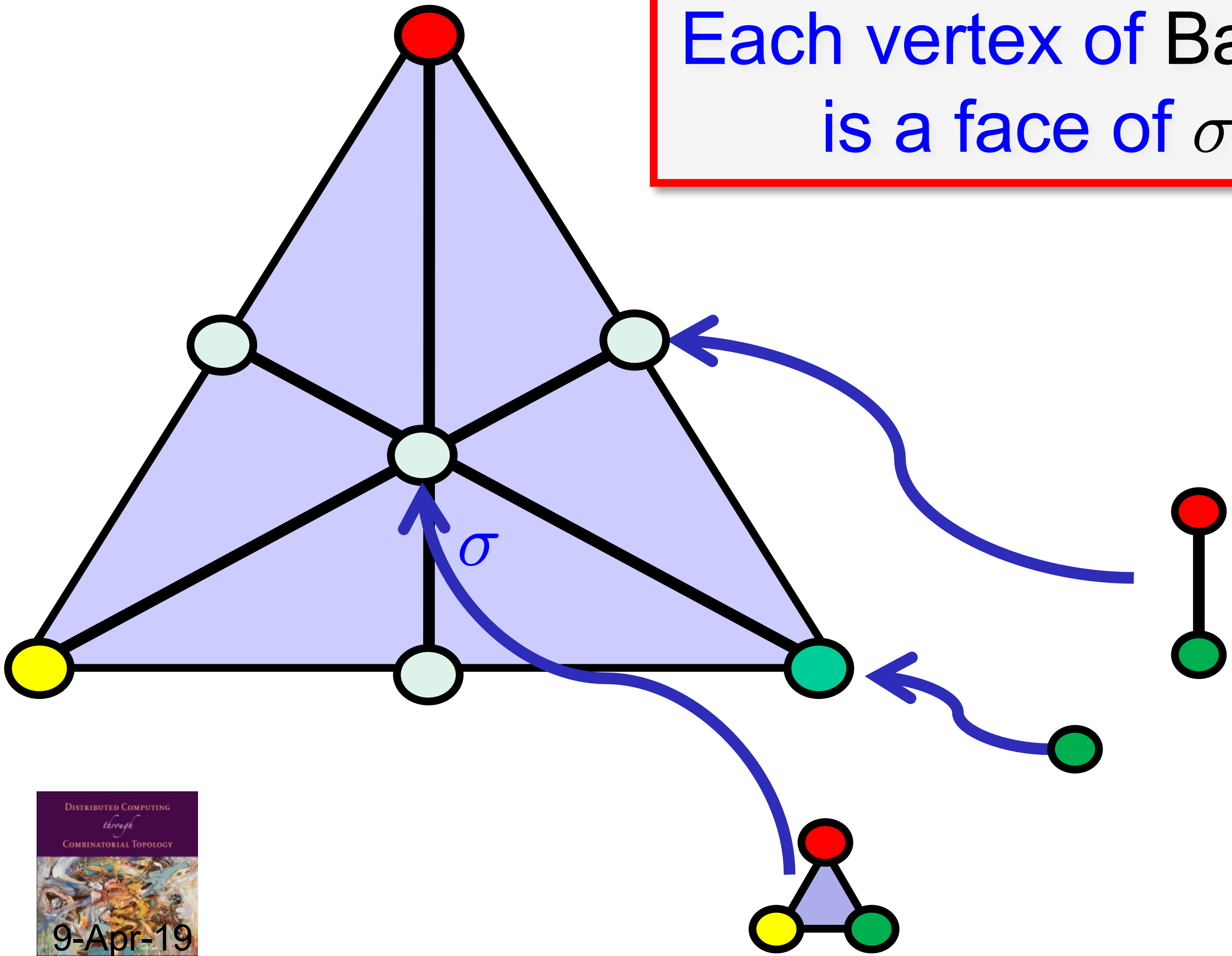
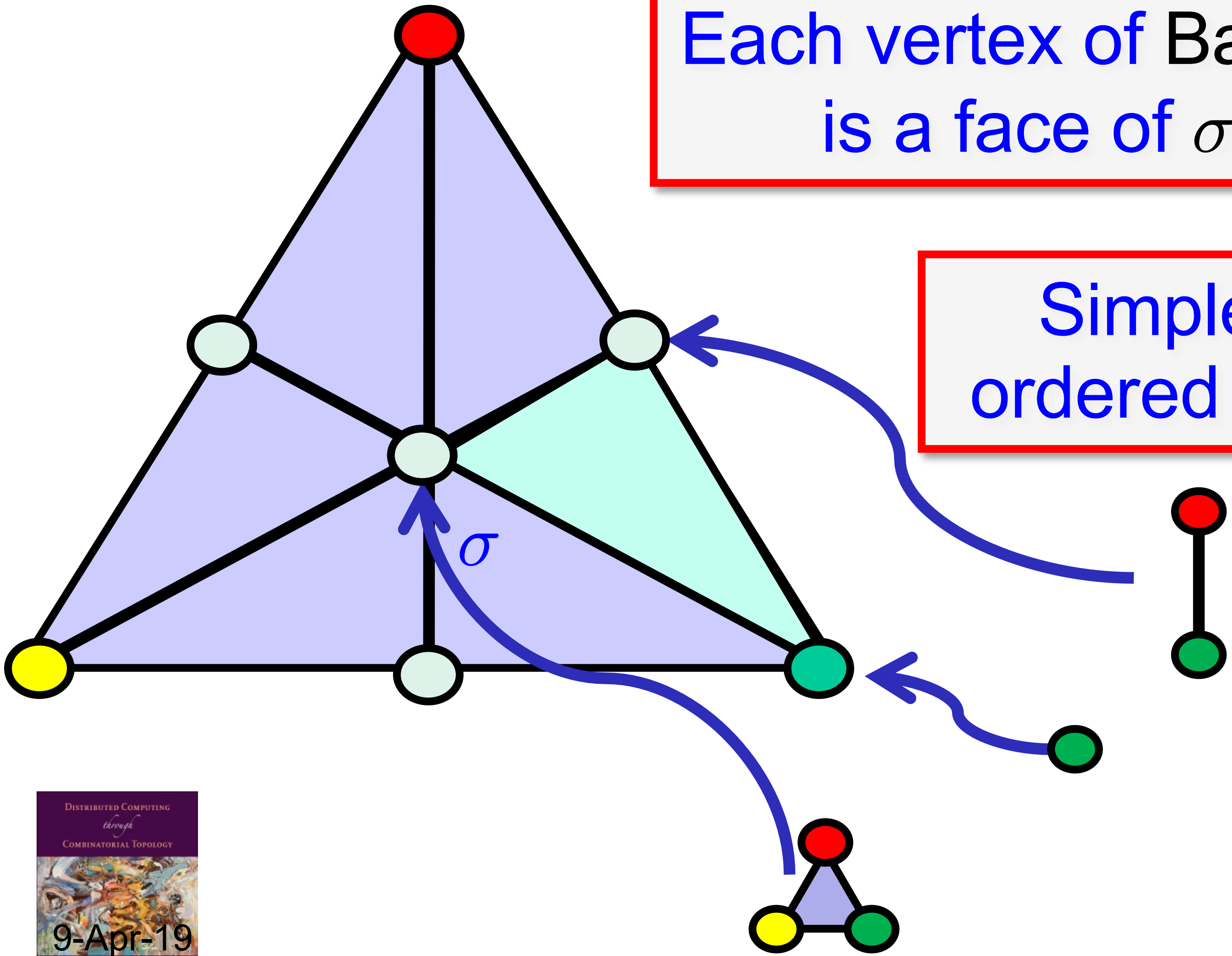


Barycentric Subdivision

Each vertex of Bary σ is a face of σ

Simplex = faces ordered by inclusion

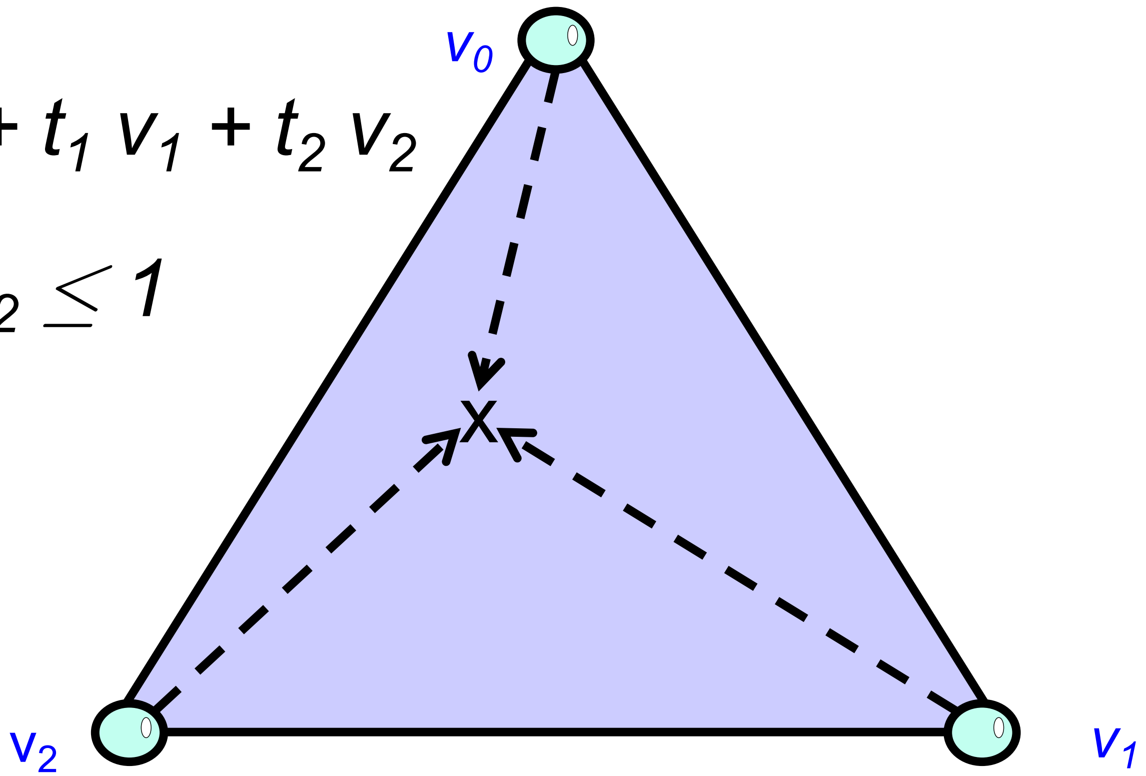


Barycentric Coordinates

$x = t_0\, v_0 + t_1\, v_1 + t_2\, v_2$

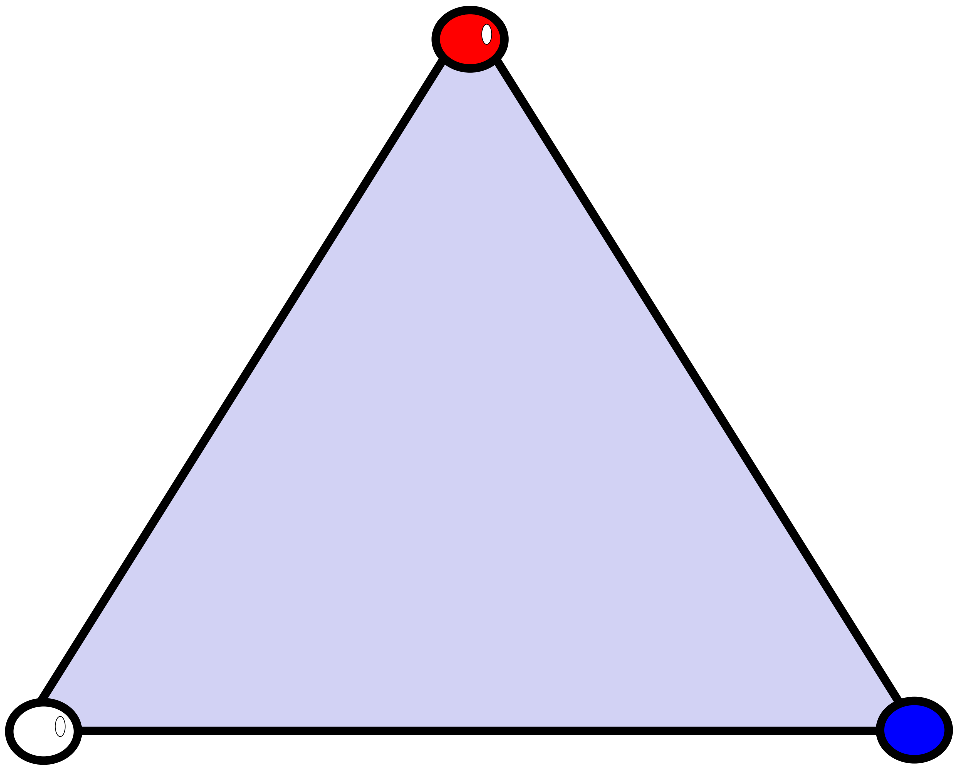
$0 \leq t_0, t_1, t_2 \leq 1$

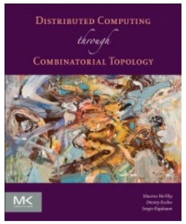
$\sum_i t_i = 1$



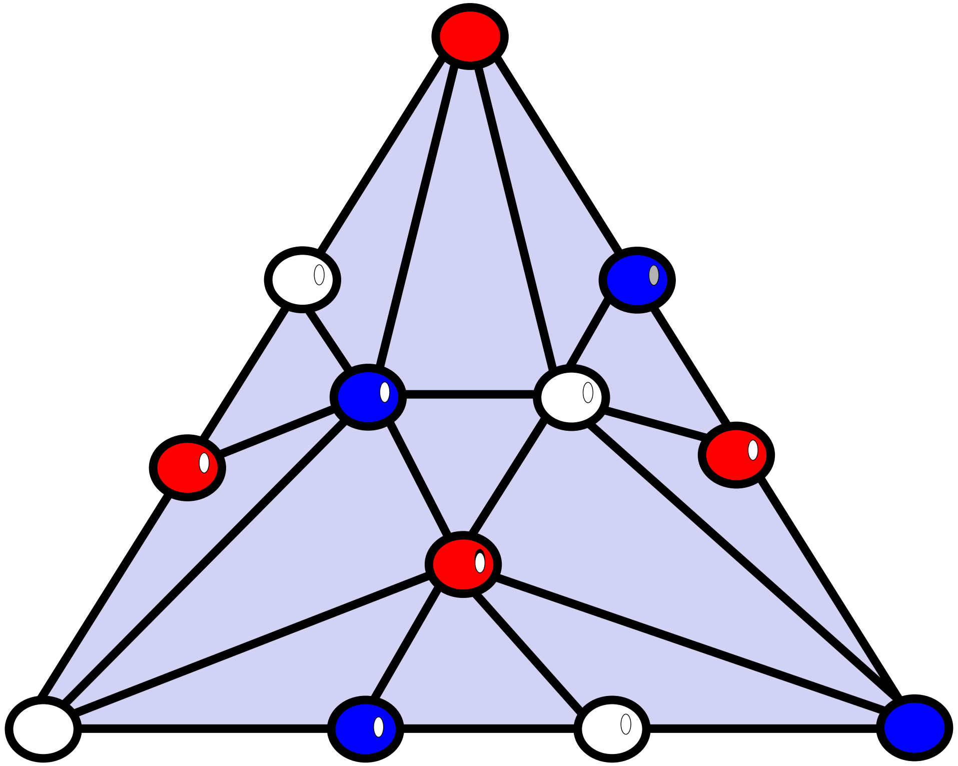
Every point of $|\mathcal{C}|$ has a unique representation using barycentric coordinates

Standard Chromatic Subdivision

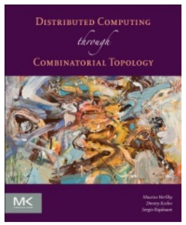




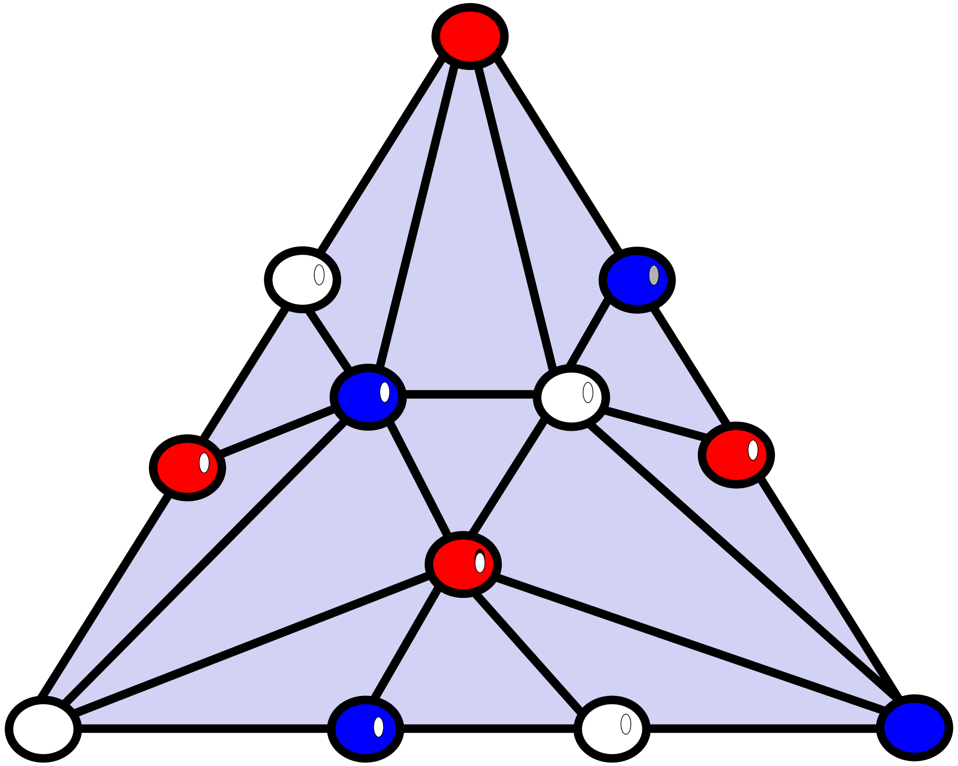
Standard Chromatic Subdivision



Ch σ

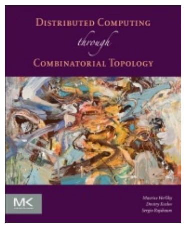


Standard Chromatic Subdivision



$\mathrm{Ch}\,\sigma$

Chromatic form of Barycentric



Road Map

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Simplicial & Continuous Approximations

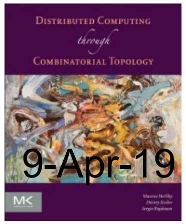
From Simplicial to Continuous

simplicial

$$\phi : \; A \to B$$

continuous

$$f : \; |A| \to |B|$$



From Simplicial to Continuous

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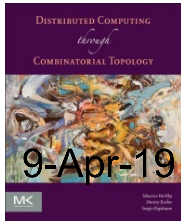


From Simplicial to Continuous

simplicial

$$\phi : \ A \to B$$

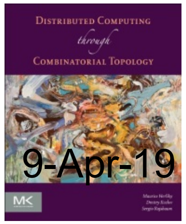
$$f(x) = \sum_i t_i \cdot |\phi(s_i)|$$

continuous

$$f : \ |A| \to |B|$$

extend over barycentric coordinates

(piece-wise linear map)



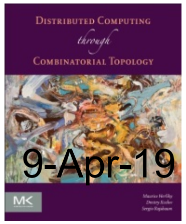
Maps

simplicial

$$\phi: \; A \to B$$

continuous

$$f: \; |A| \to |B|$$



Maps

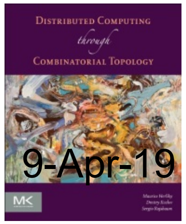
simplicial

$$\phi : \; A \to B$$

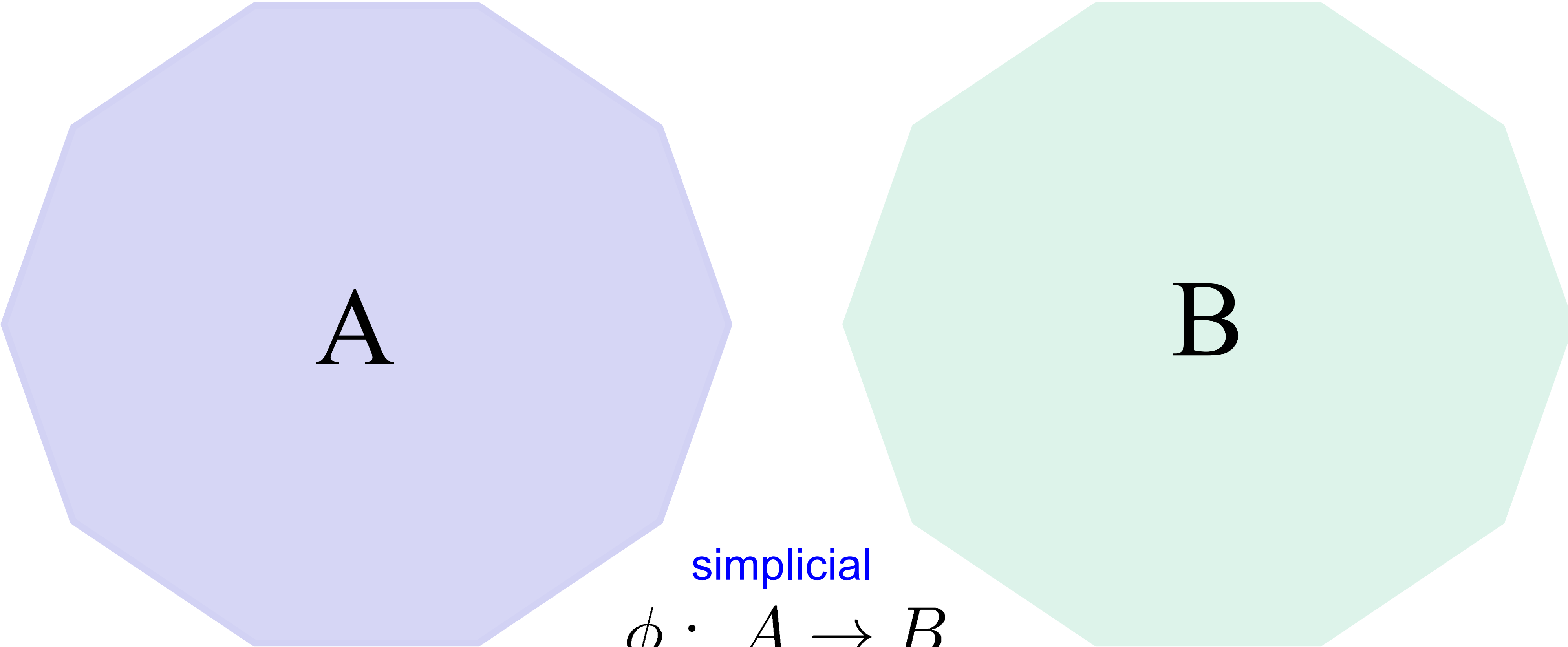
Simplicial Approximation Theorem

Continuous

$$f : \; |A| \to |B|$$



Simplicial Approximation

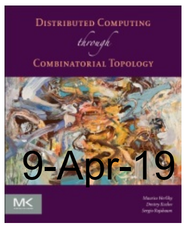


simplicial

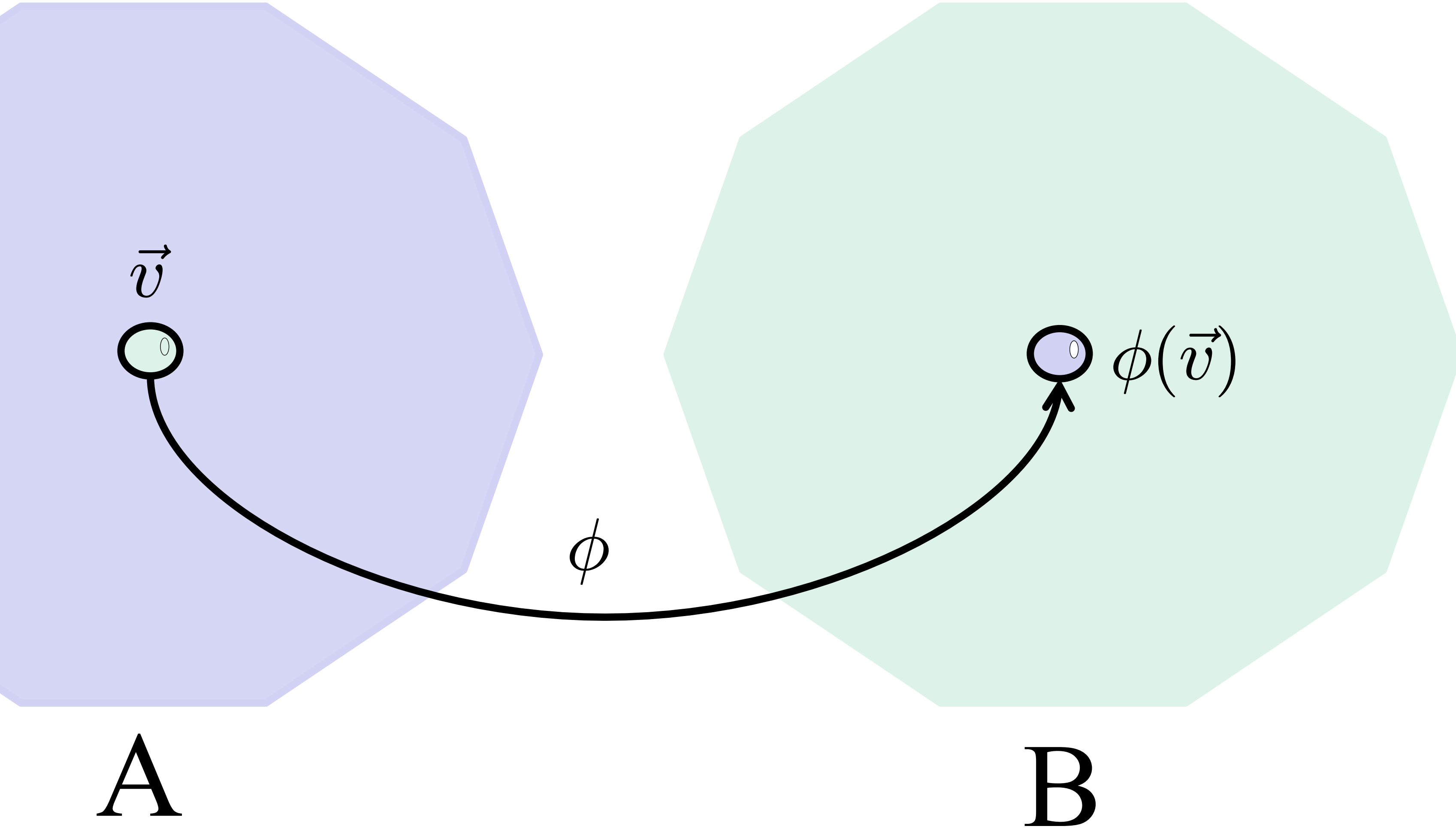
$$\phi : \; A \rightarrow B$$

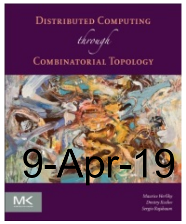
continuous

$$f : \; |A| \rightarrow |B|$$



Simplicial Approximation





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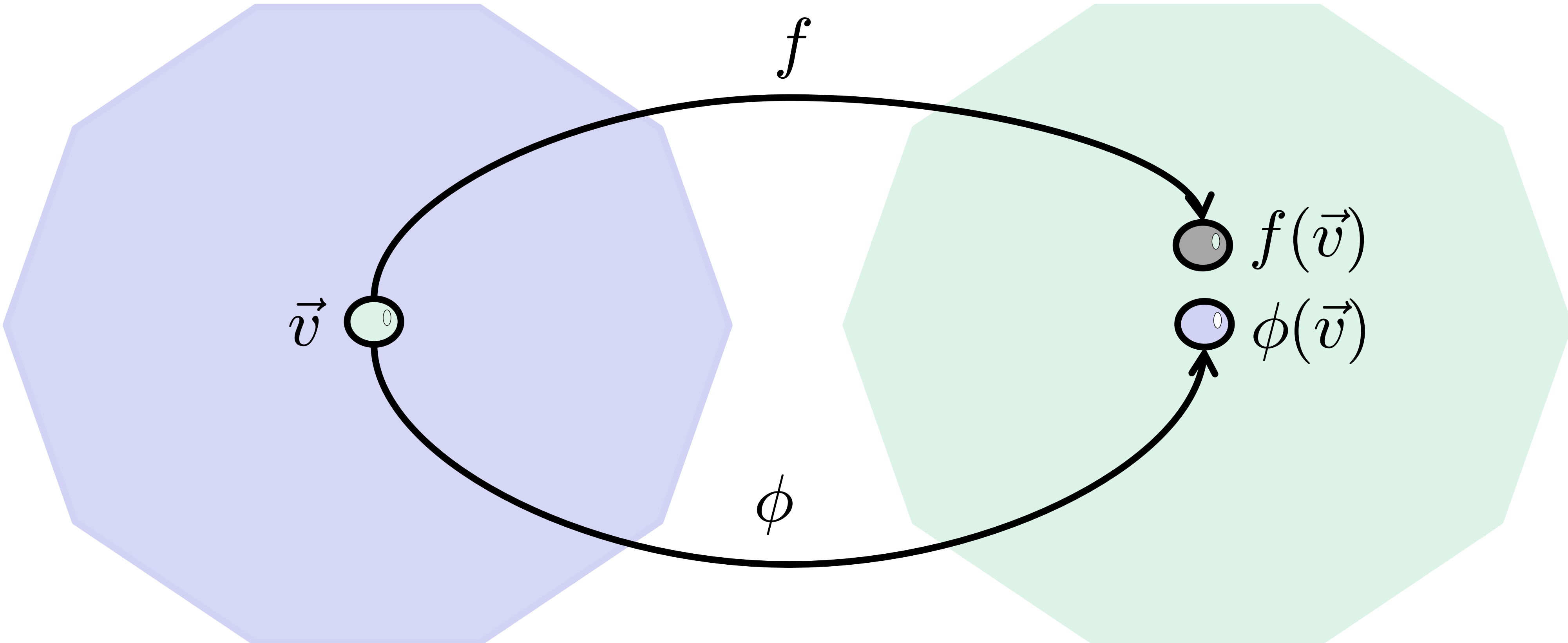
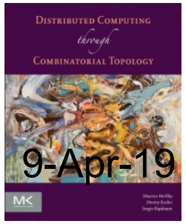
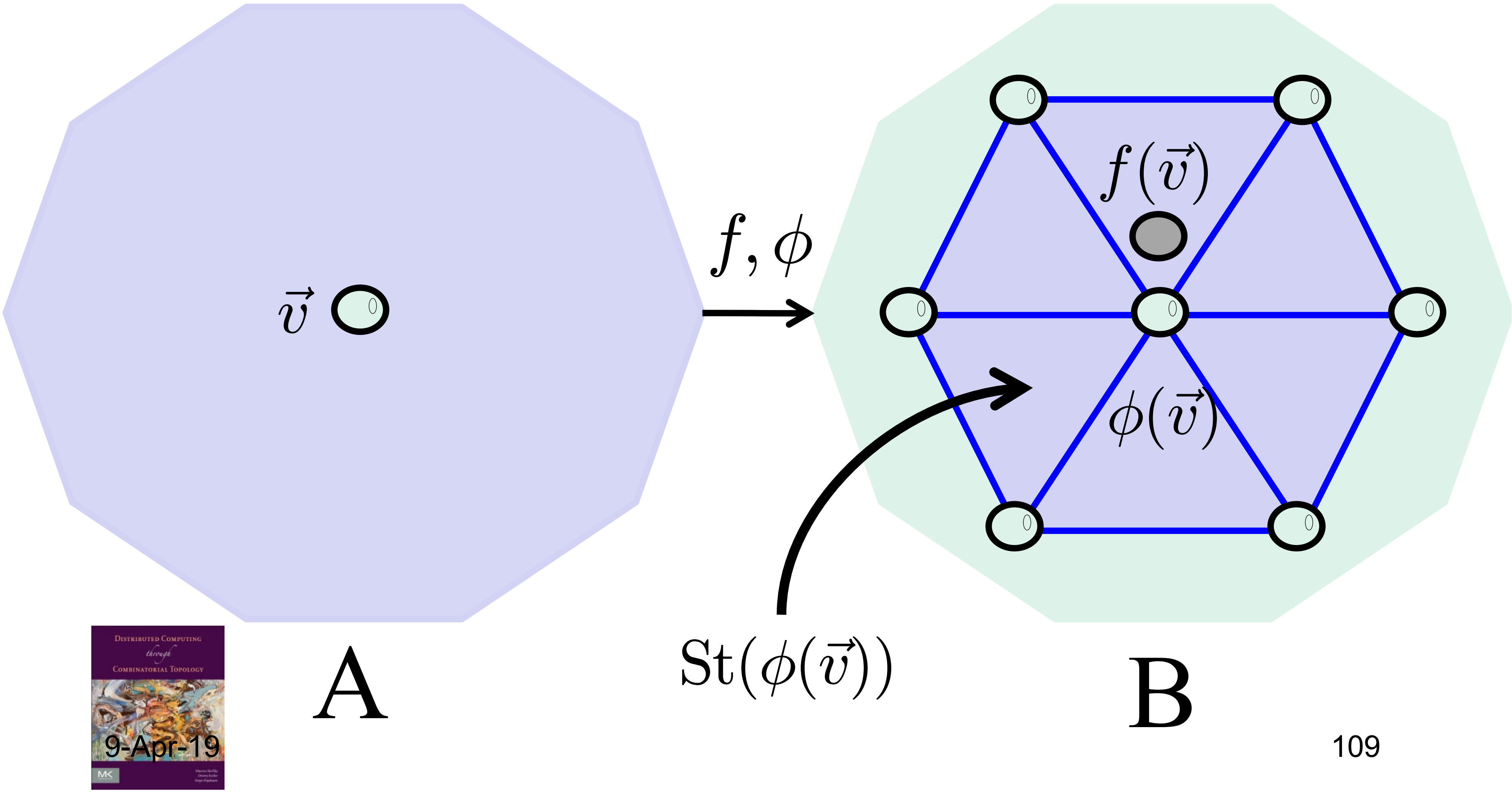




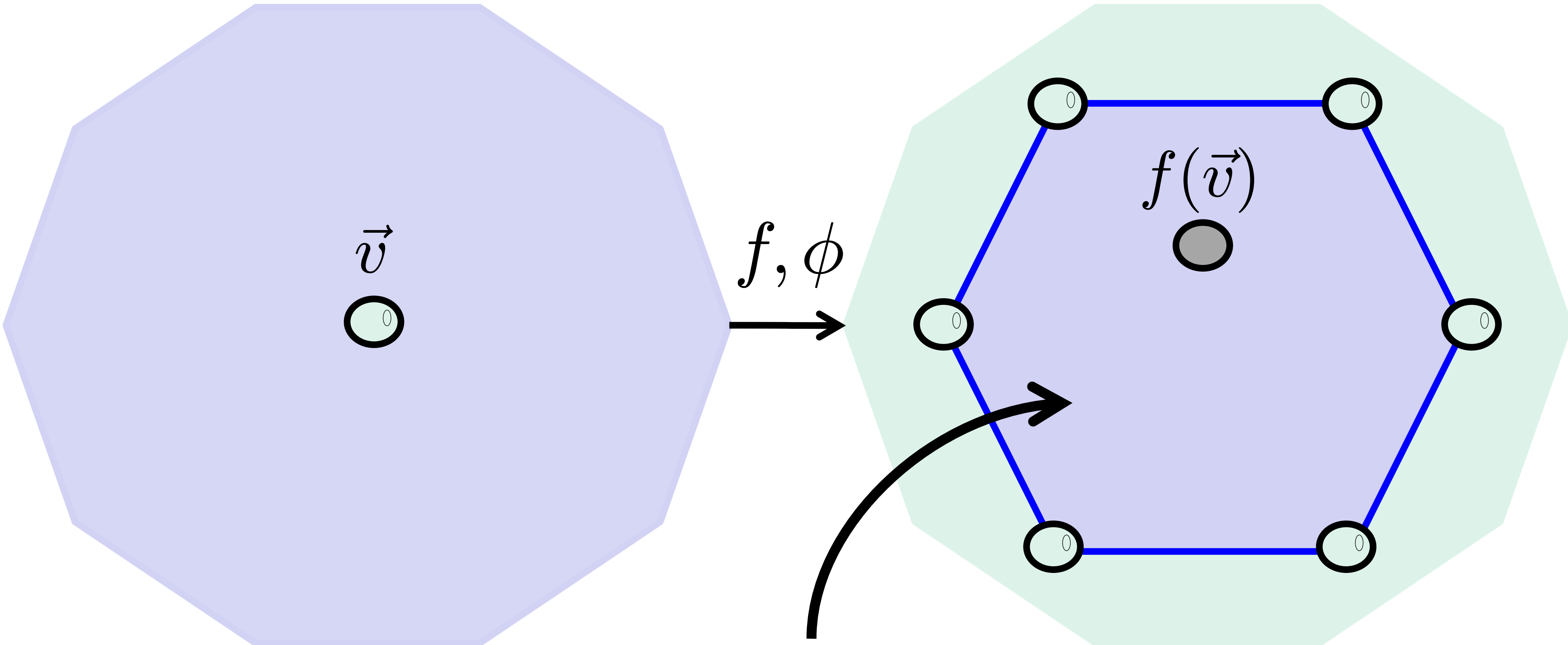
A

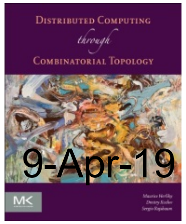
B

Simplicial Approximation



Simplicial Approximation





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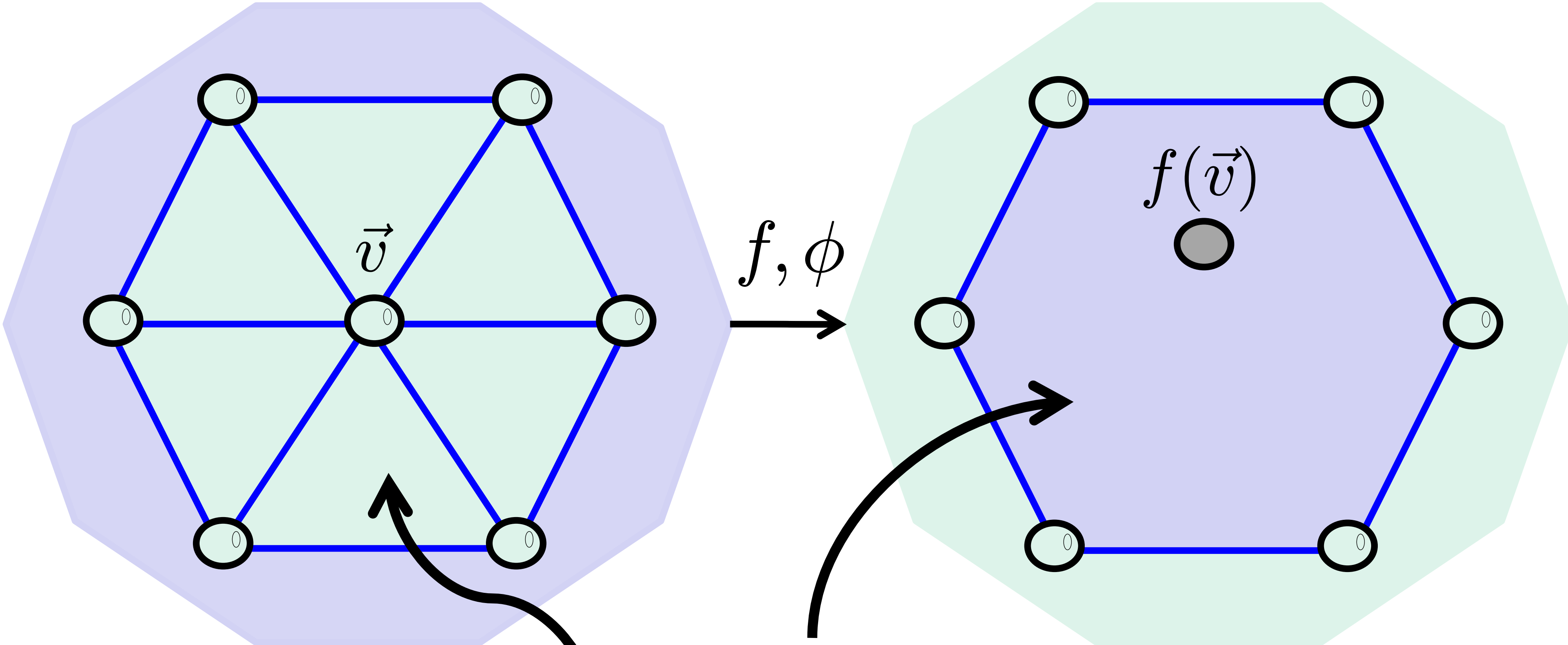

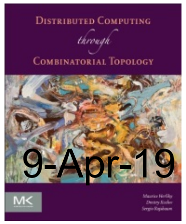

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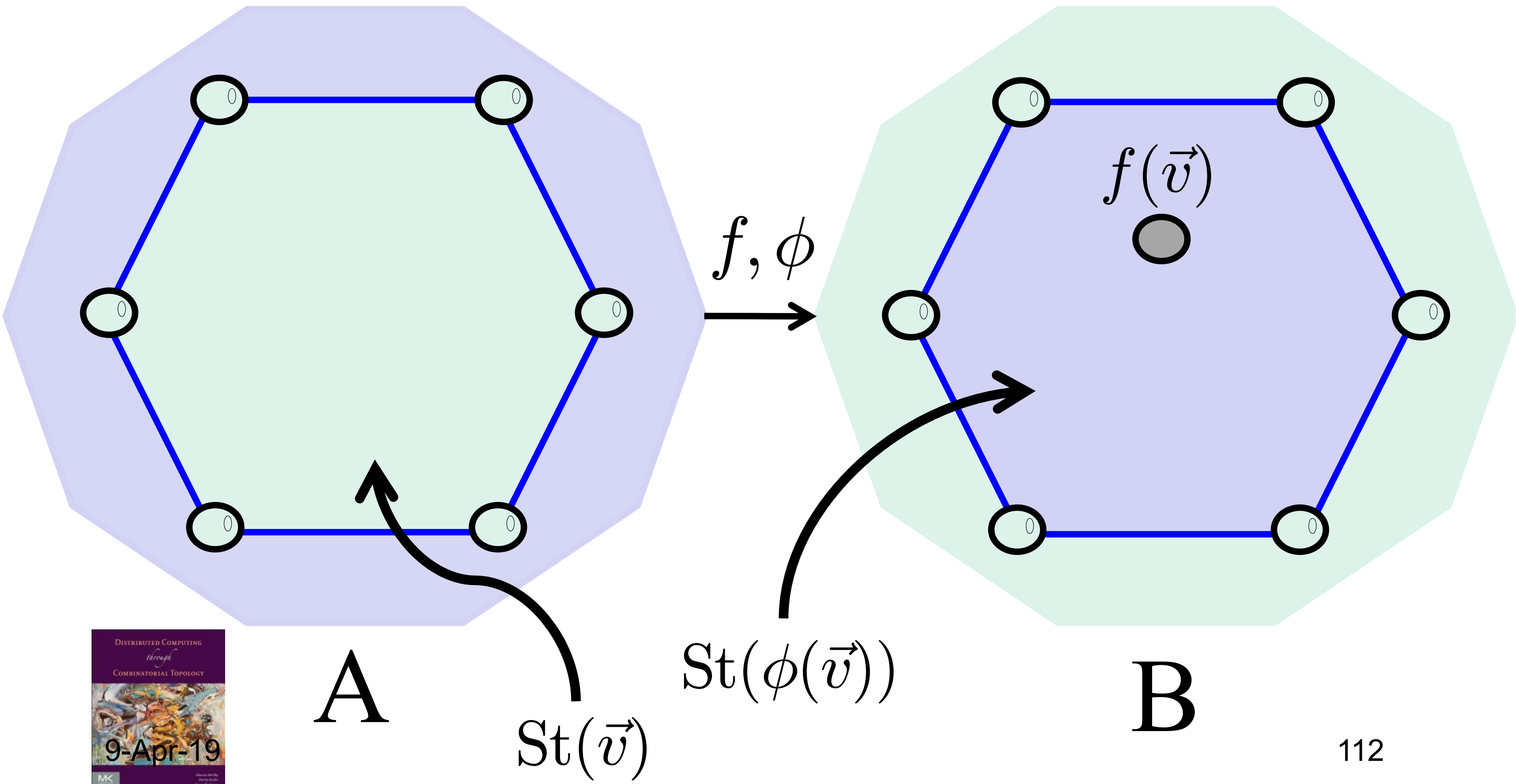

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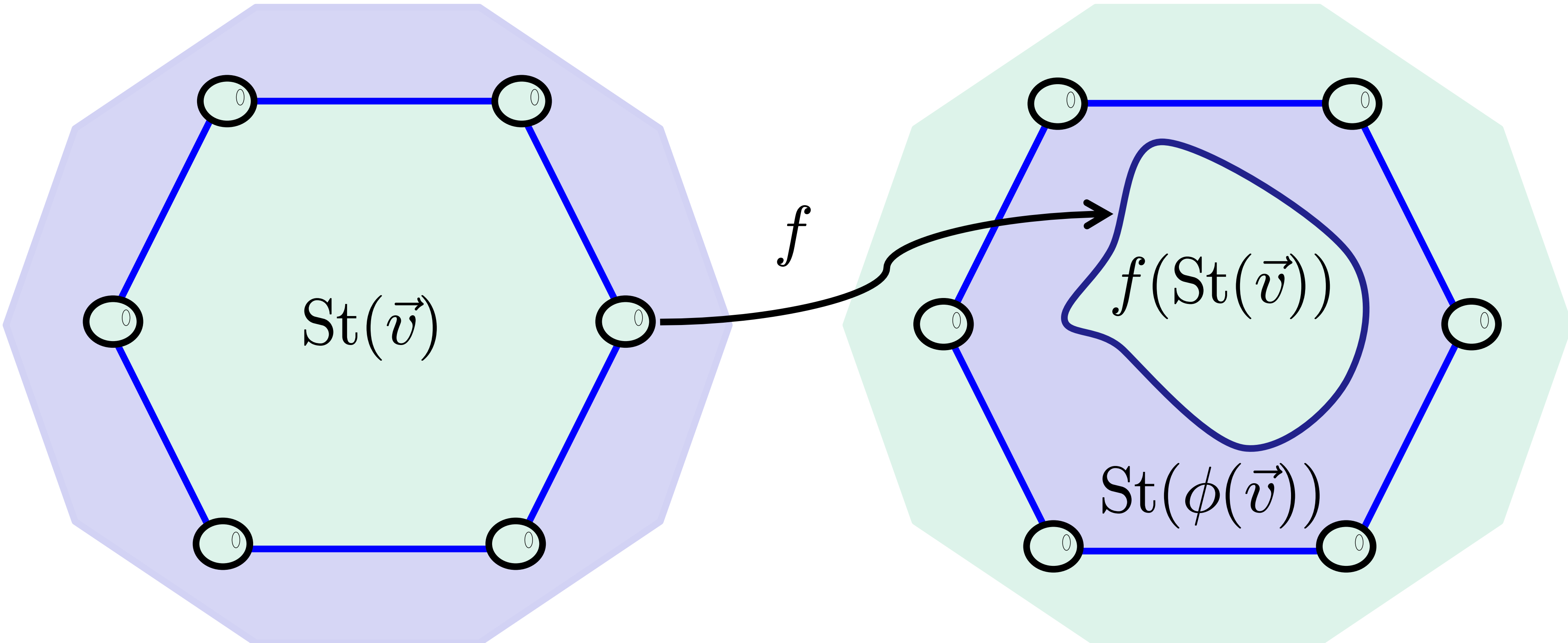
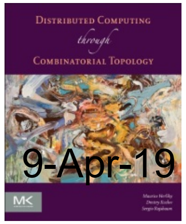


A

B



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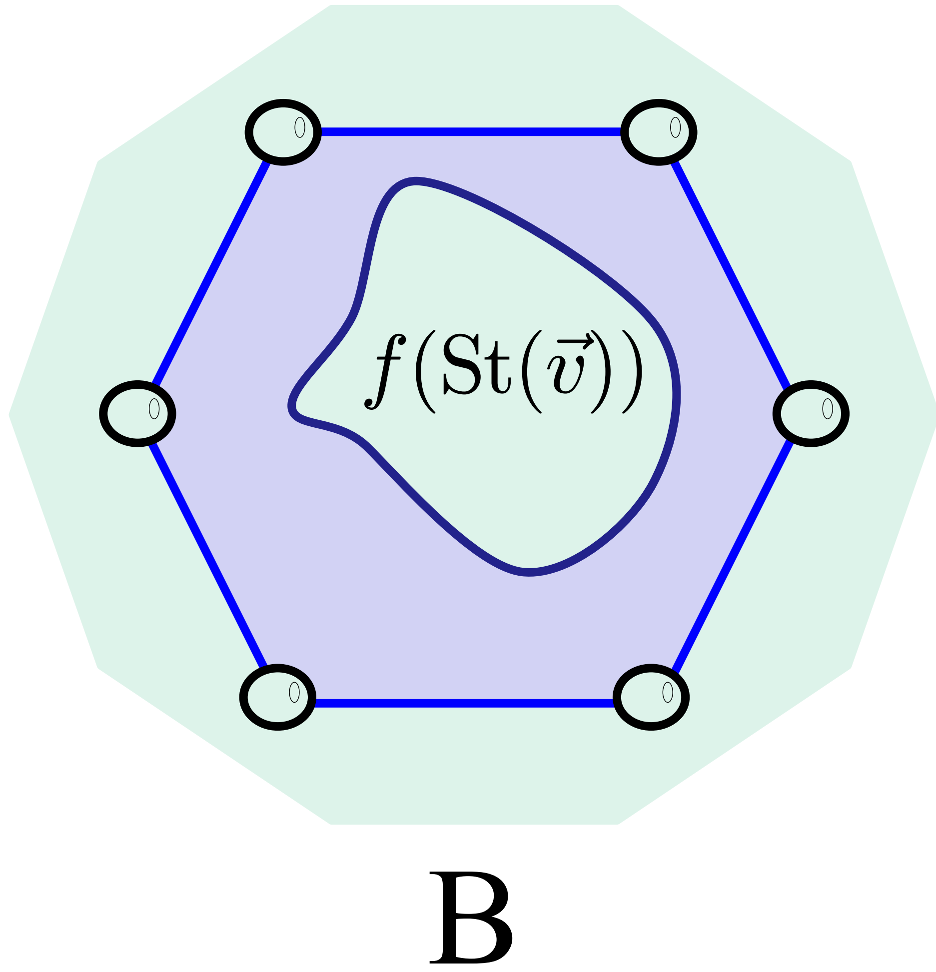
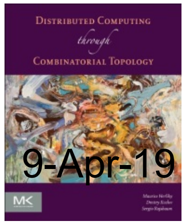
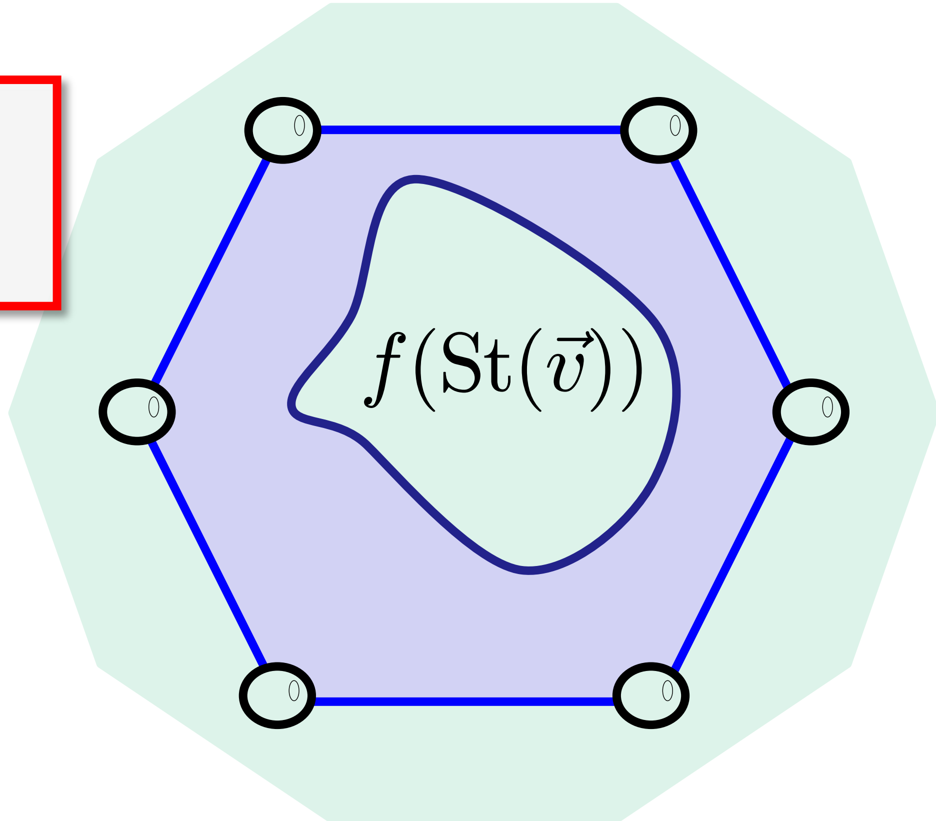


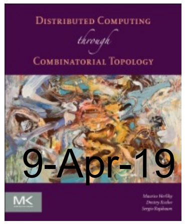


Simplicial Approximation

ϕ is a simplicial approximation of f if …



B

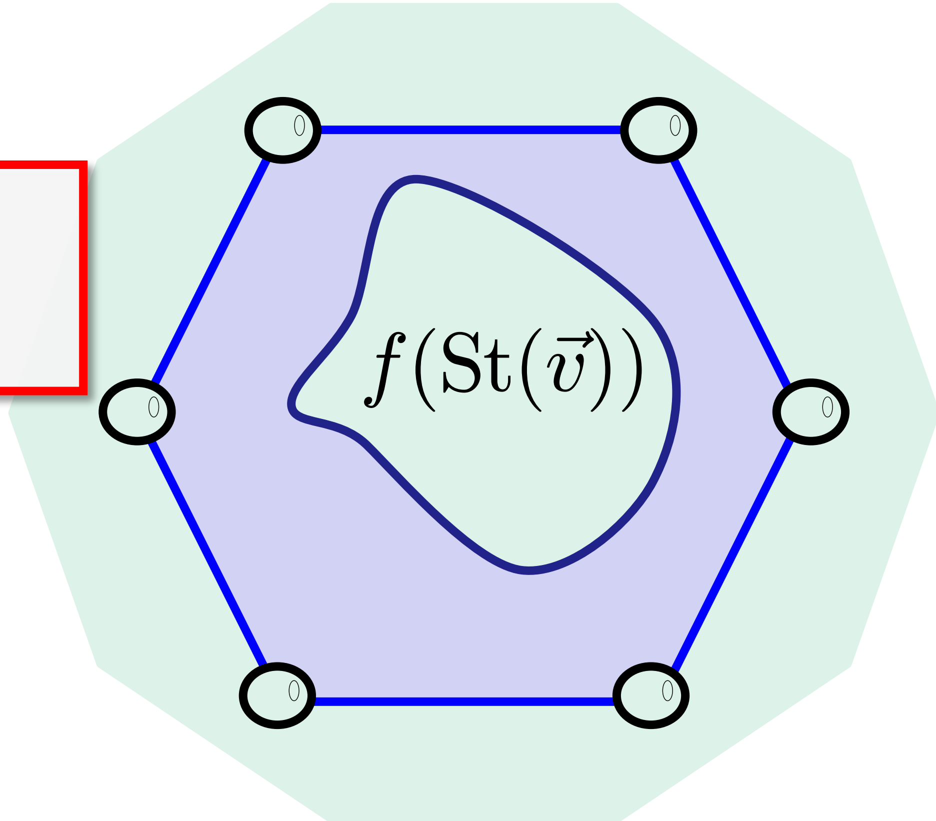


Simplicial Approximation

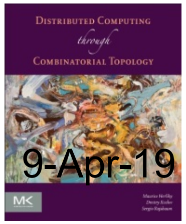
ϕ is a simplicial approximation of f if …

for every v in $\mathcal{A}$ …

$$f\left(St^0(\vec{v})\right) \subseteq St^0(\phi(\vec{v}))$$



B



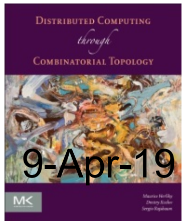
Simplicial Approximation Theorem

- Given a continuous map

$$f : |A| \to |B|$$

- there is an N such that f has a simplicial approximation

$$\phi : Bary^N A \to B$$



Simplicial Approximation Theorem

- Given a continuous map

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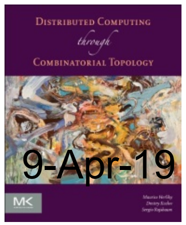
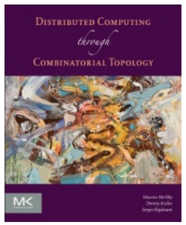
Actually holds for most other (mesh-shrinking) subdivisions….




Distributed Computing
through
Combinatorial Topology