Radix cross-sections for length morphisms

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Abstract

We prove that the radix cross-section of a rational set for a length morphism, and more generally for a rational function from a free monoid into $\mathbb{N}$, is rational, a property that does not hold any more if the image of the function is a subset of a free monoid with two or more generators.

PROCEEDINGS SHORT VERSION[1]

The purpose of this paper is to give a positive answer to a problem left open in an old paper by the second author ([11]) and to prove the following property, a refinement of the Cross-Section Theorem ([3]):

Proposition 1. *The radix cross-section of a rational set for a length morphism is rational.*

By 'rational set' we mean *rational set of a free monoid* A^* and by 'length morphism', a morphism from A^* into $\mathbb{N}$, or, which is the same, into $\{x\}^*$, the one generator free monoid. Let us take for instance the alphabet $A = \{a, b\}$, the morphism $\theta \colon A^* \to \{x\}^*$ defined by $a\theta = x^2$ and $b\theta = x^3$ and the rational set $R = (b\,a^*)^*$. The *lexicographic cross-section* of R for θ is

$$1 + b\,a^*(1 + b) \ ,$$

the set of words in R such that each one is the smallest in the lexicographic order in its class modulo the map equivalence of θ (this smallest element exists, even if the lexicographic order is not a well-ordering, since every class is finite). The *radix cross-section* of R for θ is

$$1 + b\,(1 + a + a^2)\,b^* \ ,$$

the set of words in R obtained if we replace the lexicographic order by the *radix order* (sometimes called *shortlex* or *length-lexicographic* order), which is a well-ordering. That the lexicographic cross-section of a rational set is rational follows from results that are recalled in the next section. That the radix cross-section of a rational set is rational is thus established in this paper.

As for the Cross-Section Theorem, Proposition 1 has a dual, and equivalent, formulation which is better suited for both proof and generalisation.

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[1]A longer version with full proofs is available on the web page of the authors.

Theorem 1. *The radix uniformisation of a rational relation from $\{x\}^*$ into A^* is a rational function.*

If $\alpha\colon A^* \to \{x\}^*$ is a morphism, and R a rational set of A^*, the image of the radix uniformisation of the inverse of the composition of the intersection by R with α is the radix cross-section of R for α and Proposition 1 directly follows from Theorem 1. The proof of the latter heavily relies on results and constructions presented by the first author in his thesis ([9]) and hardly publicised yet ([8]).

Before going to this proof we want to sketch here, we first recall a series of statements developed from the Cross-Section Theorem. On the one hand, they bring some light on the meaning of Theorem 1 by describing similarities and differences with these statements. On the other hand, they lead rather naturally to the notion of (rational) relation 'with bounded length-discrepancy with respect to a given ratio', rbld-relation for short. The proof is then split in two main steps. We first remark that every transducer $\mathcal{T}$ from A^* into B^* can be mapped onto an automaton $\mathcal{A}$ over A^* with multiplicity in the min-plus semiring $\mathcal{N}$ (a *min-plus automaton*) by replacing the output by their length. We call $\mathcal{A}$ the *min-pus projection* of $\mathcal{T}$. The core of that part of the proof amounts to establish that the *minimal length selection* of a relation realised by a transducer $\mathcal{T}$ is realised by an *immersion* in the product of $\mathcal{T}$ with a min-plus automaton $\mathcal{A}$ which realises its *min-plus projection*, under the condition that $\mathcal{A}$ be unambiguous. The second step boils down to the effective decomposition of *unary* min-plus automata into domain-disjoint deterministic automata, which, in the case of unary relations, insures both that the min-plus projection is unambiguous and the minimal length selection is a rbld-relation.

1 The Cross-Section Theorem

In his treatise [3], Eilenberg defines the 'Rational Cross-Section Property' (RCSP): a map $\alpha\colon A^* \to E$ has the RCSP if for every rational set R of A^* there exists a rational set T which is a set of representatives for the trace on R of the map equivalence of α, that is, $T \subseteq R$, $T\alpha = R\alpha$, and α is injective on T — the property being that such a T can be chosen *rational*.

Theorem 2 (Eilenberg [3]). *Every morphism $\alpha\colon A^* \to B^*$ has the RCSP.*

As already quoted above, this statement can be given a *dual*, and immediately equivalent, formulation. A *uniformisation* of a relation $\theta\colon A^* \to B^*$ is a function $\tau\colon A^* \to B^*$ which has the same domain as θ and whose graph is contained in the graph of θ. The dual statement then reads:

Theorem 3. *Every rational relation $\theta\colon A^* \to B^*$ has a uniformisation which is an unambiguous rational function.*

From now on and for sake of simplicity, we speak of uniformisation only (but in some examples).

1.1 Lexicographic and radix uniformisations

Building a uniformisation amounts to a *choice function*: the choice of a representative in the image of every element of the domain. The foregoing statements insure that the corresponding choice can be made in such a way that the whole function is rational, but tell nothing on how this choice is made. Even worse, the constructive proofs given to these statements are likely to produce uniformisation functions which will depend upon the representation chosen for the relation and the arbitrary choices made in the corresponding algorithms. The next step in the theory is then to characterise choice functions which yield rational uniformisation or conversely to characterise rational relations for which natural choice functions are rational.

Definition 1. Let $\theta\colon A^* \to B^*$ be a relation. We call *radix uniformisation* of θ, denoted θ_{rad}, the function from A^* to B^* obtained by choosing for every f in A^* the smallest element of $f\theta$ in the radix order. As the radix order is a well-order, such a smallest element always exists and $\mathsf{Dom}\,\theta_{\mathsf{rad}} = \mathsf{Dom}\,\theta$.

We call *lexicographic selection* of θ, denoted θ_{lex}, the function from A^* to B^* obtained by choosing for every f in A^* the smallest element of $f\theta$ in the lexicographic order. As the lexicographic order is not a well-order, such a smallest element may not exist, $\mathsf{Dom}\,\theta_{\mathsf{lex}} \subseteq \mathsf{Dom}\,\theta$ and θ_{lex} is not necessarily a uniformisation.

We call *minimal-length selection* of θ, denoted θ_{ml}, the *relation* from A^* to B^* obtained by choosing for every f in A^* the elements of $f\theta$ of minimal length. Obviously $\mathsf{Dom}\,\theta_{\mathsf{ml}} = \mathsf{Dom}\,\theta$ and it holds

$$\theta_{\mathsf{rad}} = (\theta_{\mathsf{ml}})_{\mathsf{lex}} \quad . \tag{1}$$

There are rational relations whose neither radix nor lexicographic uniformisation is rational.

Example 1. Let $A = \{a < b < c\}$ be an ordered alphabet and $\theta\colon A^* \to A^*$ defined by $(a^n\, b^m)\theta = \{a^n\, b, a^m\, c\}$. It is immediate that

$$(a^n\, b^m)\theta_{\mathsf{rad}} = \begin{cases} a^n\, b & \text{if } n \leqslant m\,, \\ a^m\, c & \text{otherwise,} \end{cases} \quad \text{and} \quad (a^n\, b^m)\theta_{\mathsf{lex}} = \begin{cases} a^n\, b & \text{if } n \geqslant m\,, \\ a^m\, c & \text{otherwise.} \end{cases}$$

In [11], and by means of tedious calculations, we proved that the lexicographic cross-section of a *morphism* $\alpha\colon A^* \to B^*$ is rational, a result that has then been generalised, and given a more readable proof, by H. Johnson.

Theorem 4 ([7]). *The lexicographic selection of a deterministic rational relation is a deterministic rational function.*

The same is not true of the radix uniformisation of deterministic[2] rational relations, even of inverse morphisms, as shown by the following example (taken from [11] and that can be found in [12, Exer. V.3.7] as well).

[2]N.B. Deterministic rational relations are those relations realised by *deterministic transducers*, that is, deterministic 2-tape automata, which are distinct from *sequential transducers*, that is, transducers with deterministic underlying input automata (see [7, 12]).

Example 2. Let $\alpha\colon \{a,b,c,d,e,f\}^* \to \{x,y,z\}^*$ be defined by $a\alpha = x$, $b\alpha = y\,x\,y\,x$, $c\alpha = x\,y$, $d\alpha = y\,z$, $e\alpha = z\,y\,z\,y$, $f\alpha = z$. The radix cross-section T of $\{a,b,c,d,e,f\}^*$ for α is such that $T \cap a\,b^*(d^2)^*d = \{a\,b^n d^{2m+1} \mid m \leqslant n\}$, and is not rational.

It is thus natural to turn to an even more restricted family of rational relations.

1.2 Uniformisation of synchronous relations

Synchronous relations are those rational relations which can be realised by transducers whose transitions are labelled by *pairs* of letters. *Stricto sensu*, this definition yields *length preserving* relations only. This constraint is relaxed by allowing the replacement of a letter by a padding symbol, in either component (but under the 'padding condition', that is, no letter can appear after the padding symbol on the same component). Even if this is more than a technical trick since in particular it is not decidable whether a given relation is synchronous or not (see [5]), synchronous relations are a very natural subfamily of rational relations. They have been given a logical characterisation in [4] and are considered in many instances (automatic structures, operations on numbers, *etc.*). They form a Boolean algebra, the largest one considered so far inside the rational relations. As the lexicographic order and radix order are synchronous relations, it then follows:

Proposition 2 (see [12]). *The lexicographic selection and radix uniformisation of a synchronous relation are synchronous functions.*

Proposition 2 and its variants are commonly used to give simple proofs for statements involving rational sets and lexicographic or radix order, *e.g.* '*if K is a rational set, then the set $\mathsf{Max}\,K$ of words of K which are maximal for every length is rational*' (see [13]), or '*the radix enumeration of a rational set is a rational function*' (see [2]).

1.3 Rational ρ-bld-relations

A synchronous relation is, by definition, realised by a transducer whose transitions are labelled by pairs of letters, that is, such that the ratio between the length of the 'input' and the the length of the 'output' is fixed and equal to 1. For every rational number $\rho = \frac{p}{q}$ (p and q are co-prime integers), we define the *ρ-synchronous relations* as those rational relations that are realised by transducers whose transitions are labelled by pairs of words (f,g) where $|f| = q$ and $|g| = p$ (as above, we allow the use of a padding symbol, under the padding condition). It is straightforward to verify that the ρ-synchronous relations form a Boolean algebra. Moreover, standard constructions show that the *composition* of a ρ-synchronous relation with a (1-)synchronous relation is a ρ-synchronous relation, hence the same proof as above yield the following.

Proposition 3. *The lexicographic selection and radix uniformisation of a ρ-synchronous relation are ρ-synchronous functions.*

Let $\theta \colon A^* \to B^*$ be a relation with the property that there exists a rational number ρ and an integer k such that, for every f in A^* and every g in $f\theta$, then $\big|\rho|f| - |g|\big| \leqslant k$. If $\rho = 1$ and $k = 0$, θ is a *length preserving relation.* If $\rho = 1$ and k is arbitrary, θ has been called a *bounded length difference relation* ([5]) or *bounded length discrepancy relation* ([12]), *bld-relation* for short in any case; for arbitrary ρ (and k), let us call θ a *ρ-bld-relation.* And let us say that θ is a *rbld-relation* if there exists a ρ such that θ is a *ρ-bld-relation.*

It is not difficult to verify that a rational relation is ρ-bld if, and only if, any transducer $\mathcal{T}$ (without padding symbol!) which realises θ has the property that the label of every circuit in $\mathcal{T}$ is such that the ratio between the length of the 'input' and the the length of the 'output' is fixed and equal to ρ, a property which is thus decidable. The following result is essentially due to Eilenberg who proves it for length-preserving relations ([3]); it has been extended to bld-relations in [5], the generalisation to ρ-bld-relations can be found in [12, IV.6].

Theorem 5. *A rational ρ-bld-relation is a ρ-synchronous relation.*

Theorem 1 will thus be proved once we have established the following.

Theorem 6. *The minimal length selection of a rational relation from $\{x\}^*$ into A^* is an effectively computable finite union of domain-disjoint rational rbld-relations.*

Indeed, if $\theta_{\mathsf{ml}} = \bigcup \theta_i$ where every θ_i is ρ_i-bld, then $\theta_{\mathsf{rad}} = \bigcup (\theta_i)_{\mathsf{lex}}$ by (1) and since the θ_i are domain disjoint: θ_{rad} is then rational by Theorem 5 and Proposition 3 and since the union is finite.

2 Minimal-length selection of transducers

The construction underlying the proof of Theorem 6 is reminiscent of that of the *Schützenberger covering* that allowed us to give a new proof of the rational uniformisation theorem ([12]). There, the product of a transducer with the determinisation of its underlying input automaton was used to choose among computations with same inputs. Here, we consider the product of a transducer with a min-plus automaton that produces for every input the minimal length of its images, and then a length-difference covering of that product to select the computations that yield the outputs of minimal length.

2.1 The min-plus projection

All automata or transducers that we consider in this paper are 'real-time', that is, every transition is labelled with a letter (in automata) or by a pair of words whose first component is a letter (in transducers). As a consequence, they can equally be described as 'tuples' or as 'representations'. A $\mathbb{K}$-automaton $\mathcal{A}$ over A^* is described either as $\mathcal{A} = \langle\, Q, A, \mathbb{K}, E, I, T\,\rangle$ where E is the set of transitions of the form (p, a, k, q) and I and T are functions from Q into $\mathbb{K}$ or as a triple $\langle\, I, \mu, T\,\rangle$, where μ is a morphism from A^* into the monoid of matrices $\mathbb{K}^{Q\times Q}$ and I (*resp.* T) is a row (*resp.* column) vector of $\mathbb{K}^Q$. The automaton $\mathcal{A}$ *realises* the $\mathbb{K}$-rational series $|\mathcal{A}| = \sum_{f\in A^*}(I\cdot f\mu\cdot T)\,f$. A transducer $\mathcal{T}$ over A^* with outputs in B^* and set of states Q is described either as $\mathcal{T} = \langle\, Q, A, B^*, E, I, T\,\rangle$ where E is the set of transitions of the form (p,a,L,q) with L in $\mathrm{Rat}\,B^*$ and I and T are functions from Q into $\mathrm{Rat}\,B^*$, or as a triple $\langle\, I,\mu,T\,\rangle$, where μ is a morphism from A^* into the monoid of matrices $(\mathrm{Rat}\,B^*)^{Q\times Q}$ and I (*resp.* T) is a row (*resp.* column) vector of $(\mathrm{Rat}\,B^*)^Q$. The image of a word f in A^* by $\mathcal{T}$ is $f|\mathcal{T}| = I\cdot f\mu\cdot T$ and $|\mathcal{T}|$ the relation realised by $\mathcal{T}$ is rather seen as a series over A^* with coefficients in $\mathfrak{P}(B^*)$ (in $\mathrm{Rat}\,B^*$ in fact) than as a (rational) subset of $A^*\times B^*$.

The *min-plus semiring* is denoted with $\mathcal{N} = \langle\,\mathbb{N}\cup\{+\infty\},\mathsf{min},+,+\infty,0\,\rangle$. For any alphabet B, the set $\mathfrak{P}(B^*)$ equipped with union and product is also a semiring and the map $\psi\colon \mathfrak{P}(B^*)\to\mathcal{N}$ defined by $R\psi = \mathsf{min}\{|f| \mid f\in R\}$ for every $R\subseteq B^*$ is a semiring morphism, which we call the *min-plus projection*.

Let $\mathcal{T} = \langle\, I,\mu,T\,\rangle$ be a transducer, $\tau = |\mathcal{T}|$ the relation it realises and $\sigma = \tau\psi$. Then, by definition, for every f in A^*,

$$f\sigma = \mathsf{min}\{|u| \mid u\in f\tau\} \ .$$

Moreover, if $\mathcal{A} = \mathcal{T}\psi$ is the image of $\mathcal{T}$ by ψ, that is, the weight of every transition of $\mathcal{A}$ is obtained by taking the image by ψ of the output of the corresponding transition in $\mathcal{T}$, then σ is realised by $\mathcal{A}$ (as ψ is a semiring morphism). A transducer and its min-plus projection are shown below, at Figure 1.

2.2 Product of a transducer by a min-plus automaton

Let us first define *min-plus transducers*, that is, transducers where every word of the output of a transition is weighted by an element of $\mathcal{N}$, or, which is equivalent, transducers which are realised by representations in $\mathcal{N}\mathrm{Rat}\,B^*$. Let us then note that both $\mathrm{Rat}\,B^*$ and $\mathcal{N}$ are subsemirings of $\mathcal{N}\mathrm{Rat}\,B^*$, by assigning a weight $0 = 1_{\mathcal{N}}$ to every element of any subset of B^* on one hand, and by considering every element k in $\mathcal{N}$ as the monomial $k\,\varepsilon$ on the other hand (in this context where there are 0's and 1's from $\mathcal{N}$ going around, we rather denote the empty word as ε). Moreover, we remark that after these embeddings, every element of $\mathrm{Rat}\,B^*$ commutes with every element of $\mathcal{N}$ within $\mathcal{N}\mathrm{Rat}\,B^*$ (which is not itself a commutative semiring).

It follows then that every transducer $\mathcal{T}$ from A^* to B^* and every $\mathcal{N}$-automaton $\mathcal{A}$ may be seen as min-plus transducers, that their *product* $\mathcal{T}\otimes\mathcal{A}$ is well-defined and is a min-plus transducer described by the *tensor product* of their representations.

By Schützenberger's Theorem, the product of $\mathcal{T}$ by $\mathcal{A}$ realises the Hadamard product of their respective behaviours (see [1, 12]). Hence the following.

Proposition 4. *Let $\mathcal{T}$ be a transducer and $\mathcal{A}$ a $\mathcal{N}$-automaton which realises $|\mathcal{T}|\psi$. Then $\mathcal{U} = \mathcal{T}\otimes\mathcal{A}$ is the $\mathcal{N}$-transducer which associates to every word f of A^* the following series:*

$$f|\mathcal{U}| = \sum\{k\,u \mid u\in f|\mathcal{T}|\,,\quad k = \mathsf{min}\{|v| \mid v\in f|\mathcal{T}|\}\} \ .$$

Example 3. Figure 1 shows the product of a transducer $\mathcal{T}_1$ (vertical, left) with its min-plus projection (horizontal, above).

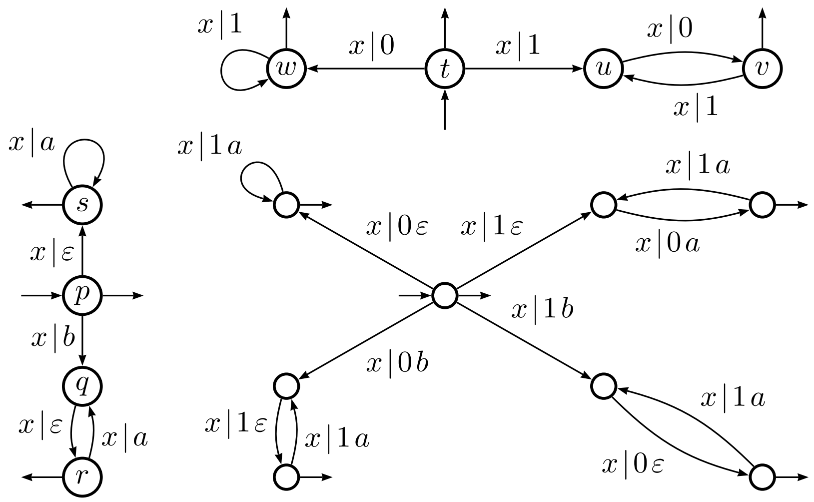


Figure 1: Product of a unary transducer with its min-plus projection

2.3 Length-difference unfolding of a min-plus transducer

Let $\mathcal{U} = \langle\, S, A, B^*, \mathcal{N}, E, I, T\,\rangle$ be a min-plus transducer. We call *length-difference unfolding* of $\mathcal{U}$ the (possibly infinite) transducer[3]

$$\mathcal{V} = \langle\, S \times \mathbb{Z}, A, B^*, E', I', T'\,\rangle \qquad \text{defined by}$$
$$E' = \big\{\big((p,z),(a,u),(q, z + |u| - k)\big) \mid \big(p,(a,k\,u),q\big) \in E, z \in \mathbb{Z}\big\}\ ,$$
$$I' = \{(i,-k) \mid i \in S,\ (i)I = k\} \quad \text{and} \quad T' = \{(t,k) \mid t \in S,\ (t)T = k\}\ .$$

The transducer $\mathcal{V}$ is an *immersion* in $\mathcal{U}$ and not a *covering* of $\mathcal{U}$: it would be one if every state (t,z) for t such that $(t)T \neq +\infty$ were final. Along a computation of $\mathcal{V}$, and at every step, the second component of the state gives the difference between the length of the word which is output and its coefficient; as in our case, the coefficient will represent the length of the shortest output for the input read sofar, the name 'length-difference' is justified. This construction is associated with the product of transducer with min-plus automata in the following statement.

Proposition 5. *Let $\mathcal{T}$ be a transducer, $\mathcal{A}$ an* unambiguous *$\mathcal{N}$-automaton which realises $|\mathcal{T}|\psi$ and $\mathcal{U} = \mathcal{T} \otimes \mathcal{A}$. Let $\mathcal{V}$ be the length-difference unfolding of $\mathcal{U}$. Then:*
(i) *$\mathcal{V}$ realises $|\mathcal{T}|_{\mathsf{ml}}$;*
(ii) *the trim part of $\mathcal{V}$ is finite.*

[3]For the sake of simplicity of notations, we suppose that the functions I and T have values in $\mathcal{N}$ and not in $\mathcal{N}\langle\!\langle B^*\rangle\!\rangle$. It would be straightforward to generalise both the definition and the proof.

Proof. For every f in $\mathsf{Dom}\,\mathcal{T}$, there exists a *unique* computation $i \xrightarrow[\mathcal{A}]{f|k} t$ labelled by f in $\mathcal{A} = \langle J, k, U \rangle$. If $(i)J = h$ and $(t)U = l$, then $f|\mathcal{A}| = h + k + l$ is the minimal length of the outputs of $\mathcal{T}$ for the input f. For every successful computation $p \xrightarrow[\mathcal{T}]{f|u} q$, there exists then a *unique* computation $(p,i) \xrightarrow[\mathcal{U}]{f|k\,u} (q,t)$ and for every x in $\mathbb{Z}$, a *unique* computation $((p,i),x) \xrightarrow[\mathcal{V}]{f|u} ((q,t),y)$ where $y = x + |u| - k$. This computation is successful if, and only if, $x = -h$ and $y = l$, and thus $|u| = h+k+l$ that is, if, and only if, u is an output of f by $\mathcal{T}$ of minimal length.

Let $((q,s),x)$ and $((q,s),y)$ be two accessible states in $\mathcal{V}$, with $x < y = x + n$. We prove that $((q,s),y)$ is not co-accessible in $\mathcal{V}$. Suppose, by way of contradiction, that it is: there exists a computation

$$((q,s),y) \xrightarrow[\mathcal{V}]{g|v} ((r,t),l) \qquad\qquad \text{with } ((r,t),l) \text{ final in } \mathcal{V},$$

and thus a computation $(q,s) \xrightarrow[\mathcal{U}]{g|k'\,v} (r,t)$ with $k' = |v| + y - l$. As $((q,s),x)$ is accessible, there exists a computation

$$((p,i),-h) \xrightarrow[\mathcal{V}]{f|u} ((q,s),x) \qquad\qquad \text{with } ((p,i),-h) \text{ initial in } \mathcal{V},$$

and thus a computation $(p,i) \xrightarrow[\mathcal{U}]{f|k\,u} (q,s)$ with $k = |u| - x - h$. By projection on $\mathcal{T}$, $p \xrightarrow[\mathcal{T}]{f\,g|u\,v} r$ is a successful computation and $u\,v$ is in $(f\,g)|\mathcal{T}|$. By projection on $\mathcal{A}$, $i \xrightarrow[\mathcal{A}]{f\,g|k+k'} t$ is a successful computation and $(f\,g)|\mathcal{A}| = h + k + k' + l$ (under the assumption that $\mathcal{A}$ is unambiguous), whereas $|u| + |v| = h + k + k' + l - n$, a contradiction with the hypothesis that $(f\,g)|\mathcal{A}|$ is the minimal length of the elements of $(f\,g)|\mathcal{T}|$. □

Example 4 (continued). Figure 2 shows the length-difference unfolding of the product of the transducer $\mathcal{T}_1$ with an unambiguous automaton which realises its min-plus projection.

With stronger hypotheses, we get an interesting corollary of Proposition 5.

Definition 2. We call *yield* of a circuit in a min-plus automaton the ratio between the weight of the circuit and its length.

Corollary 1. *With the hypotheses and notation of Proposition 5, and if moreover all circuits in $\mathcal{A}$ have the same yield ρ, then $\mathcal{V}$ realises a ρ-bld-relation.*

Proof. A circuit in $\mathcal{V}$ comes from a circuit in $\mathcal{U}$ which itself projects onto a circuit in $\mathcal{A}$ with yield ρ. A circuit in $\mathcal{U}$ gives rise to a circuit in $\mathcal{V}$ if, and only if, the length of its output is equal to its weight and thus the ratio between the lengths of the output and input in a circuit of $\mathcal{V}$ is fixed and equal to ρ. □

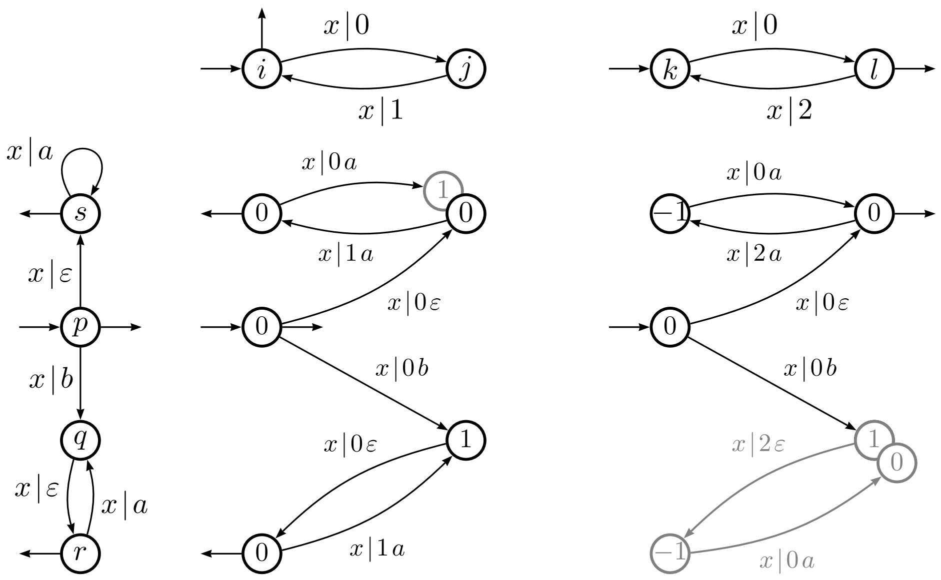


Figure 2: A length-difference unfolding for $\mathcal{T}_1$

More generally, and still with the notation of Proposition 5, if $\mathcal{A}$ is a finite union of pairwise domain disjoint min-plus automata, each of which with the property that all circuits have the same yield, then $\mathcal{V}$ realises a rbld-relation. Our last step in the proof of Theorem 6 will then be to establish that any rational relation from $\{x\}^*$ into A^* has a min-plus projection which meets this last condition.

3 Sequential decomposition of min-plus unary automata

The result we have in mind has first been stated in an existential way (for any commutative dioid); we specialise it for the min-plus semiring $\mathcal{N}$. For sake of simplicity, let us call a series over $\{x\}^*$ a *unary series*, an automaton over $\{x\}^*$ a *unary automaton*. Let us say that a unary min-plus series s is *ultimately arithmetic* if it is finite or if there exist non negative integers N and r, and an integer k such that, for every n larger than N, $\langle s, x^n \rangle = nr + k$. (In $\mathcal{N}$, these series are actually *geometric*, since the sum is the multiplication law of the semiring, the product is an exponentiation in $\mathcal{N}$.) A series s is the *merge* of p series s_0, $s_1, \ldots,$ s_{p-1} if for every pair (n,i) in $\mathbb{N} \times [0;p-1]$, it holds $\langle s, x^{np+i} \rangle = \langle s_i, x^n \rangle$.

Proposition 6 ([6]). *A unary min-plus rational series is a merge of ultimately arithmetic series.*

The original proof of Proposition 6 consists in proving that the family of merges of ultimately arithmetic series are closed under rational operations. In terms of automata, it means that every unary min-plus rational series can be realised by a finite union of domain disjoint deterministic min-plus automata.

Proposition 7 ([9]). *A min-plus unary automaton can be effectively decomposed into a finite union of deterministic min-plus (unary) automata whose supports are pairwise disjoint.*

3.1 Sequentialisation of unary tight automata

In [10], we have described a folklore 'generalised sequentialisation procedure' which applies to every min-plus automata, but which does not always yield a finite (sequential) automaton, even for automata that realise a sequential series.

Definition 3. A circuit in a min-plus automaton $\mathcal{A}$ is *critical* if its yield is minimal (among the yields of all circuits in $\mathcal{A}$). The *critical part* of $\mathcal{A}$ is the union of its critical circuits. A computation in $\mathcal{A}$ is critical if it meets the critical part of $\mathcal{A}$. An automaton is *tight* if almost all words in its domain label a successful critical computation.

Remark 1. The critical part of an automaton is easily computable, since critical circuits are either primitive or composed of smaller critical circuits. It is therefore sufficient to detect primitive critical circuits.

Theorem 7 ([9]). *A unary min-plus automaton realises a sequential min-plus series if, and only if, it is tight.*

3.2 Iterative decomposition of unary min-plus automata

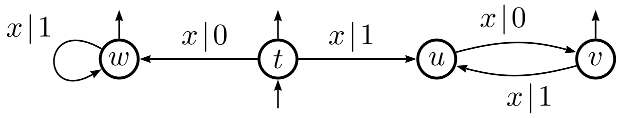
The following proposition is the core of the iterative method. The first point describes a step of the iteration, the second one guarantees that the iteration ends.

Proposition 8. *Let $\mathcal{A}$ be a min-plus (unary) automaton, ρ its minimal yield, and L the set of words which label a successful critical computation. Then:*
(i) *the restriction of $s = |\mathcal{A}|$ to L is* sequential*, realised by a deterministic automaton whose loop has a yield which is equal to ρ;*
(ii) *the restriction of s to the complement of L is realised by an automaton $\mathcal{B}$ which is an immersion in $\mathcal{A}$ and whose critical yield is strictly smaller than ρ.*

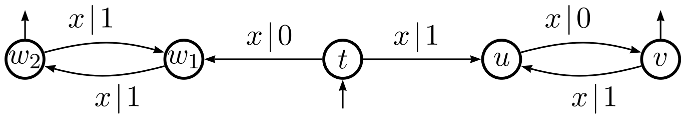
Proof. i) Let $\mathsf{supp}(\mathcal{A})$ be the support of $\mathcal{A}$ in which the critical part of $\mathcal{A}$ is tagged. Let $\mathcal{B}$ be an automaton formed by two copies of $\mathsf{supp}(\mathcal{A})$; the first one does not contain any final state and the second one does not contain any initial state; it is possible to jump from a state of the first copy to the corresponding state in the second copy if and only if this state is tagged. Therefore $\mathcal{B}$ recognises L. Thus the product $\mathcal{A} \times \mathcal{B}$ realises the restriction of s to these words.

ii) Let $\mathcal{C}$ be a deterministic automaton that recognises the complement of L. The product $\mathcal{A} \times \mathcal{C}$ realises the restriction of s to the complement of L. This product is an *immersion*, each of its circuits corresponds to a circuit of $\mathcal{A}$ with the same yield. ☐

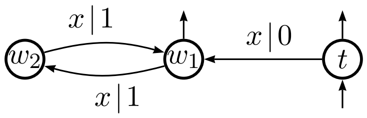
Example 5 (continued). The min-plus projection $\tau_1 = |\mathcal{T}_1|\psi$ of the transducer $\mathcal{T}_1$ is (*cf.* Figure 1):



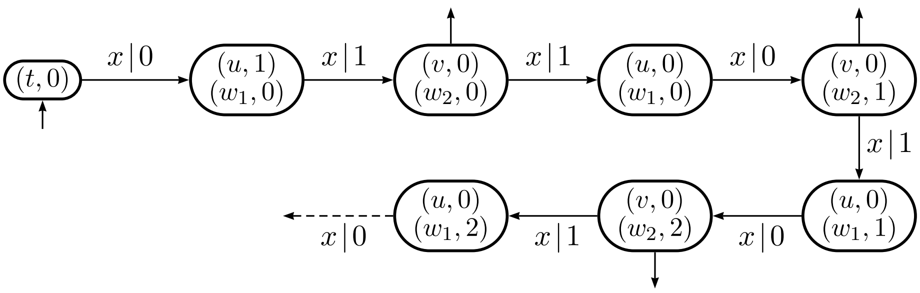
The smallest yield of circuits in this automaton is $1/2$, and the critical graph contains states u and v. The language of words which are label of computations which meet the critical graph is L_1, *the set of non empty words of even length*. The restriction of τ_1 to this language is realized by $\mathcal{A}_{1,1}$:



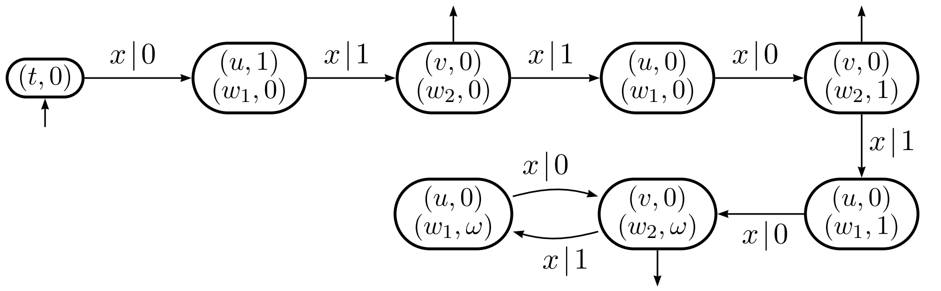
And the restriction of τ_1 to the complement of L_1 is realised by $\mathcal{A}_{1,2}$:



The automaton $\mathcal{A}_{1,2}$ is deterministic, but $\mathcal{A}_{1,1}$ has to be determinised. To tell the truth, it is readily seen that suppressing the states w_1 and w_2 in $\mathcal{A}_{1,1}$ yields a determinitistic automaton $\mathcal{D}_{1,1}$ equivalent to $\mathcal{A}_{1,1}$. But for the sake of exemplificaion, we describe the general process. The generalised sequentialisation procedure gives the automaton $\mathcal{S}_{1,1}$:



This automaton $\mathcal{S}_{1,1}$ is infinite: its states are labelled by vectors of dimension Q — where Q is the set of states of $\mathcal{A}_{1,1}$ — with entries in $\mathcal{N}$. What appears in the proof of Theorem 7 (and thus not in the present text) is that these entries may be bounded by a certain quantity $\Delta = M\,n^2 + \rho n$ where n is the dimension of the automaton to be sequentialised, M the maximum weight of transitions, and ρ the minimal yield, and that those larger than Δ may be replaced by ω which plays the role of infinity: in this way the constructed automaton is finite. In our case, $\Delta = 5^2 + 5\,\frac{1}{2} = 27,5$ and the corresponding automaton would have maore than 50 states. By other means, special to the particular case of this example, we can compute by another method a much smaller bound, and we set an entry to ω as soon as it is equal to 2. It gives the following deterministic automaton:



The minimisation of this min-plus automaton (computed in the same way as the minimisation of sequential transducers) is precisely the expected automaton $\mathcal{D}_{1,1}$: and τ_1 is realised by the disjoint union of deterministic automata $\mathcal{D}_{1,1}$ and $\mathcal{A}_{1,2}$.

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