Rationality & Recognisability

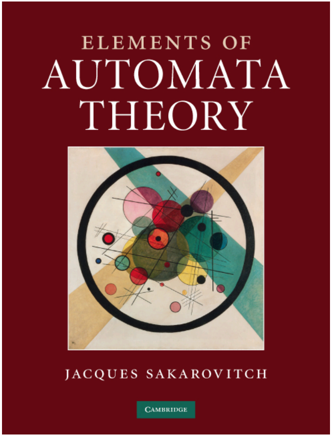
An introduction to weighted automata theory

Tutorial given at post-WATA 2014 Workshop

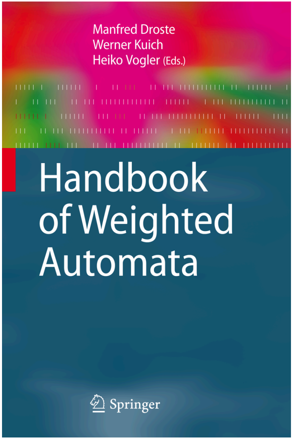
Jacques Sakarovitch

CNRS / Telecom ParisTech

Based on



Chapter III



Chapter 4

The presentation is also much inspired by joint works with

Sylvain Lombardy (Univ. Bordeaux)

entitled

- *On the equivalence and conjugacy of weighted automata, CSR 2006*,
  the journal version is still under prepapration.
- *The validity of weighted automata, CIAA 2012 & IJAC 2013*.
- Vaucanson 2 (2010–2014),
  a platform for computing with weighted automata.

Outline of the tutorial

1. The model
2. Rationality
3. Recognisability

Part I

The model of weighted automata

Outline of Part I

- Models of computation

  for computer science anf for the rest of the world

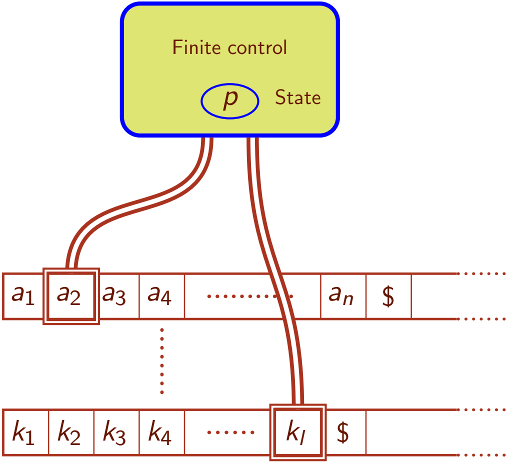
- 1-way Turing machines are equivalent to finite automata

- Once the finite automaton model is well-established,

  it is generalised to weighted automata

- Weigthed automata are the linear algebra of computer science

A touch of general system theory

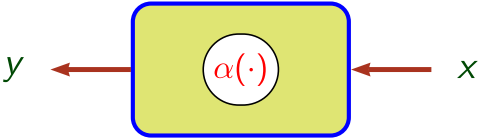


Paradigm of a machine for the computer scientists

A touch of general system theory

Paradigm of a machine for the rest of the world

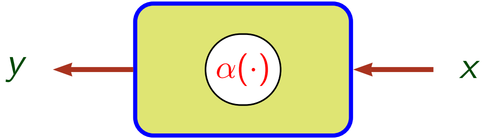
A touch of general system theory



$$y = \alpha(x)$$

Paradigm of a machine for the rest of the world

A touch of general system theory

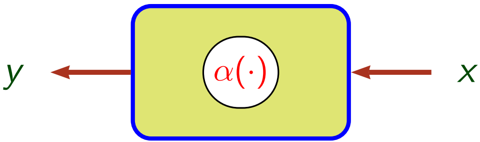


$$y = \alpha(x)$$

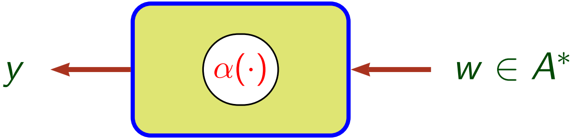
$$x \in \mathbb{R}^n\,, \quad y \in \mathbb{R}^m$$

Paradigm of a machine for the rest of the world

Getting back to computer science

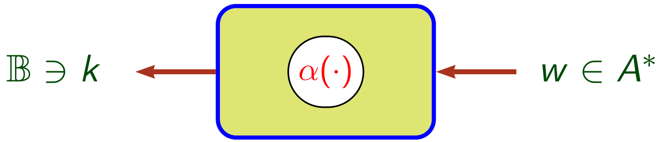


Getting back to computer science



The input belongs to a *free monoid* A^*

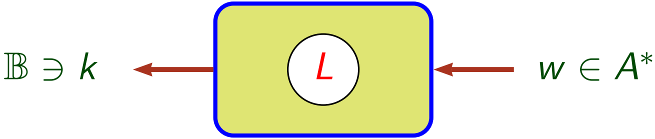
Getting back to computer science



The input belongs to a *free monoid* A^*

The output belongs to the *Boolean semiring* $\mathbb{B}$

Getting back to computer science



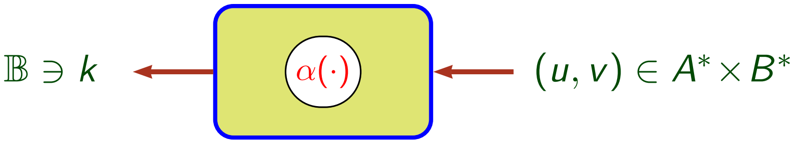
$$L \subseteq A^*$$

The input belongs to a *free monoid* A^*

The output belongs to the *Boolean semiring* $\mathbb{B}$

The function realised is *a language*

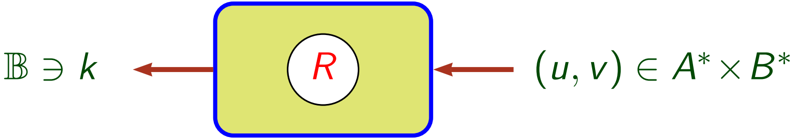
Getting back to computer science



The input belongs to a *direct product of free monoids* $A^* \times B^*$

The output belongs to *the Boolean semiring* $\mathbb{B}$

Getting back to computer science



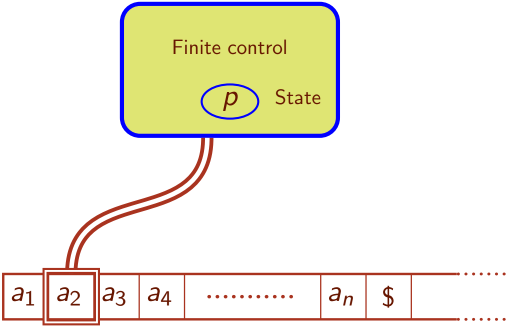
$$R \subseteq A^* \times B^*$$

The input belongs to a *direct product of free monoids* $A^* \times B^*$

The output belongs to *the Boolean semiring* $\mathbb{B}$

The function realised is *a relation between words*

The simplest Turing machine

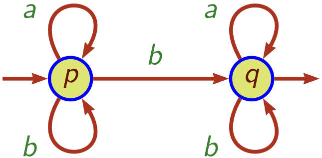


Direction of movement of the read head

The 1-way 1-tape Turing Machine (1W 1T TM)

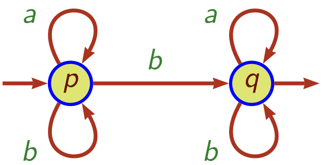
The simplest Turing machine is equivalent to finite automata

$\mathcal{B}_1$



The simplest Turing machine is equivalent to finite automata

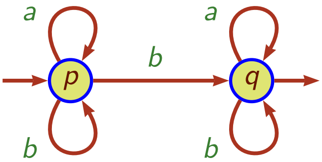
$\mathcal{B}_1$



$b\,a\,b \quad \in A^*$

The simplest Turing machine is equivalent to finite automata

$\mathcal{B}_1$



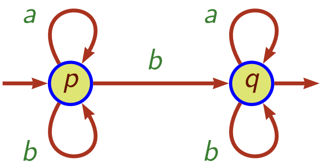
$b\,a\,b \quad \in A^*$

$$\longrightarrow p \xrightarrow{b} p \xrightarrow{a} p \xrightarrow{b} q \longrightarrow$$

$$\longrightarrow p \xrightarrow{b} q \xrightarrow{a} q \xrightarrow{b} q \longrightarrow$$

The simplest Turing machine is equivalent to finite automata

$\mathcal{B}_1$



$L(\mathcal{B}_1) \subseteq A^*$

$b\,a\,b \quad \in A^*$

$$\rightarrow p \xrightarrow{b} p \xrightarrow{a} p \xrightarrow{b} q \rightarrow$$

$$\rightarrow p \xrightarrow{b} q \xrightarrow{a} q \xrightarrow{b} q \rightarrow$$

$$L(\mathcal{B}_1) = \{w \in A^* \mid w \in A^* b A^*\} = \{w \in A^* \mid |w|_b \geqslant 1\}$$

Rational (or regular) languages

Languages accepted (or recognized) by finite automata

=

Languages described by rational (or regular) expressions

=

Languages defined by MSO formulae

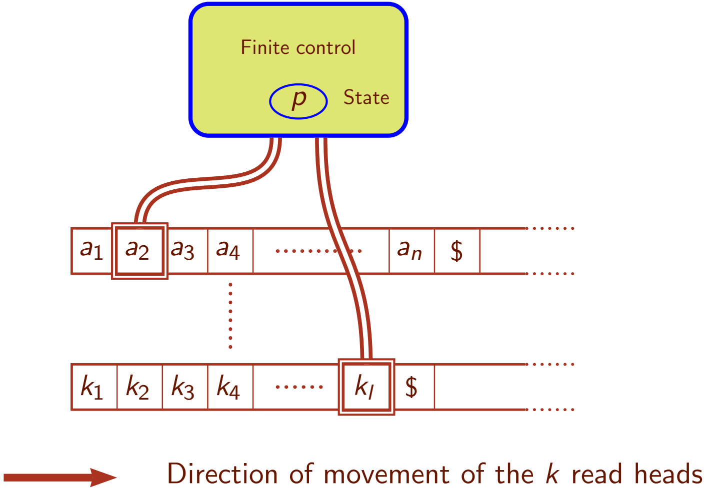
Remarkable features of the finite automaton model

Decidable equivalence (decidable inclusion)

Closure under complement

Canonical automaton (minimal deterministic automaton)

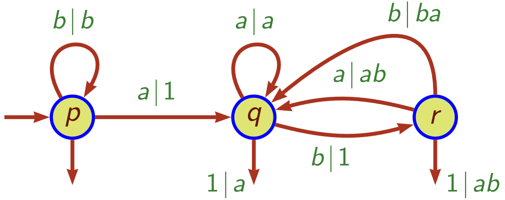
The 1W kT Turing machine



The 1-way k-tape Turing Machine (1W kT TM)

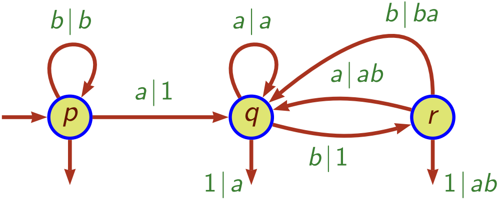
The 1W kT Turing machine is equivalent to finite transducers

$\mathcal{G}_1$



The 1W kT Turing machine is equivalent to finite transducers

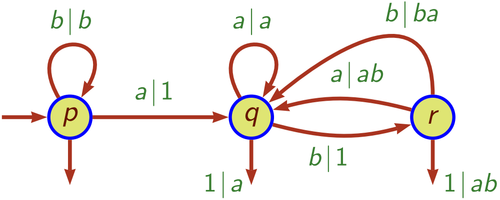
$\mathcal{G}_1$



$$\longrightarrow p \xrightarrow{a|1} q \xrightarrow{b|1} r \xrightarrow{b|b\,a} q \xrightarrow{1|a}$$

The 1W kT Turing machine is equivalent to finite transducers

$\mathcal{G}_1$

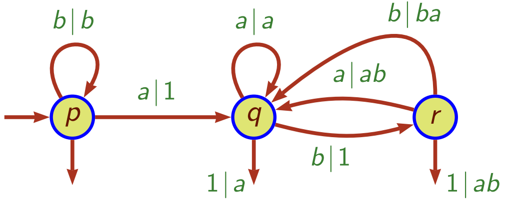


$$\longrightarrow p \xrightarrow{a|1} q \xrightarrow{b|1} r \xrightarrow{b|b\,a} q \xrightarrow{1|a}$$

$$(a\,b\,b,\, b\,a\,a) \quad \in \quad |\mathcal{G}_1|$$

The 1W kT Turing machine is equivalent to finite transducers

$\mathcal{G}_1$



$$\rightarrow p \xrightarrow{a|1} q \xrightarrow{b|1} r \xrightarrow{b|b\,a} q \xrightarrow{1|a}$$

$$(a\,b\,b,\, b\,a\,a) \quad \in \quad |\mathcal{G}_1|$$

$$|\mathcal{G}_1| \subseteq A^* \times B^*$$

Features and shortcomings of the finite transducer model

Closure under composition

Closure of Chomsky classes under rational relations

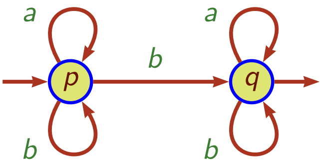
Interesting subclasses of rational relations

Non closure under complement

Undecidable equivalence

Automata versus languages

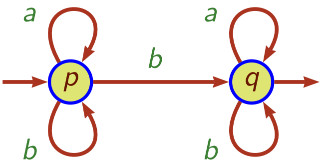
$\mathcal{B}_1$



$L(\mathcal{B}_1) \subseteq A^*$

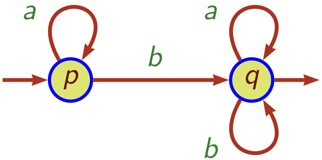
Automata versus languages

$\mathcal{B}_1$



$L(\mathcal{B}_1) \subseteq A^*$

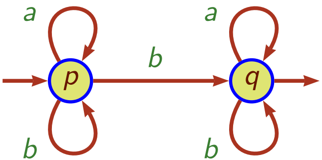
$\mathcal{B}'_1$



$L(\mathcal{B}'_1) \subseteq A^*$

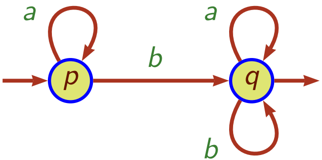
Automata versus languages

$\mathcal{B}_1$



$L(\mathcal{B}_1) \subseteq A^*$

$\mathcal{B}_1'$

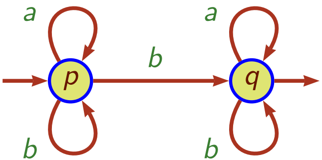


$L(\mathcal{B}_1') \subseteq A^*$

$$L(\mathcal{B}_1) = L(\mathcal{B}_1') = \left\{ w \in A^* \,\middle|\, |w|_b \geqslant 1 \right\}$$

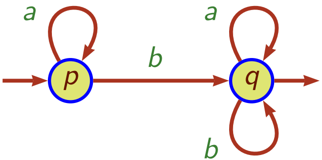
Automata versus languages

$\mathcal{B}_1$



$L(\mathcal{B}_1) \subseteq A^*$

$\mathcal{B}_1'$

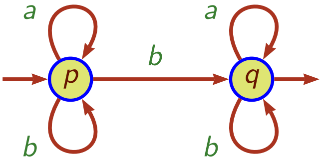


$L(\mathcal{B}_1') \subseteq A^*$

$$L(\mathcal{B}_1) = L(\mathcal{B}_1') = \left\{ w \in A^* \;\middle|\; |w|_b \geqslant 1 \right\} = A^* b A^*$$

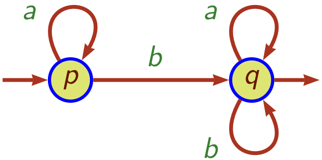
Automata versus languages

$\mathcal{B}_1$



$L(\mathcal{B}_1) \subseteq A^*$

$\mathcal{B}'_1$



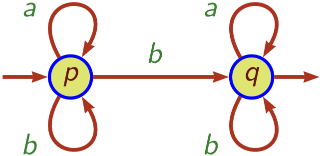
$L(\mathcal{B}'_1) \subseteq A^*$

Counting the number of successful computations

$|\mathcal{B}_1| : \; b\,a\,b \quad \longmapsto \quad 2$ $\qquad$ $|\mathcal{B}'_1| : \; b\,a\,b \quad \longmapsto \quad 1$

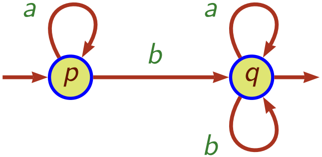
Automata versus languages

$\mathcal{B}_1$



$L(\mathcal{B}_1) \subseteq A^*$

$\mathcal{B}'_1$

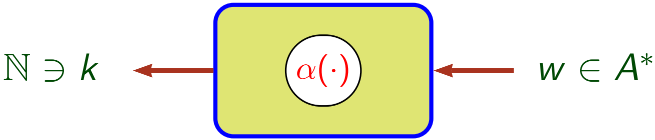


$L(\mathcal{B}'_1) \subseteq A^*$

Counting the number of successful computations

$|\mathcal{B}_1| : \ w \longmapsto |w|_b$ $\qquad |\mathcal{B}'_1| : \ w \longmapsto 1$

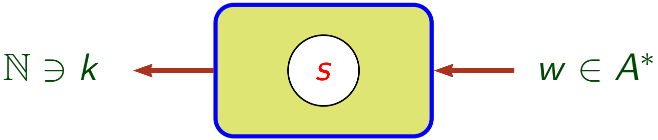
A new automaton model



The input belongs to a *free monoid* A^*

The output belongs to the *integer semiring* $\mathbb{N}$

A new automaton model



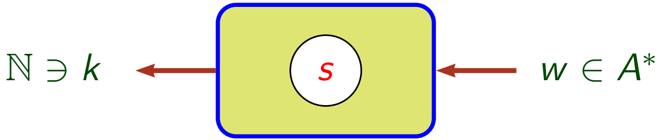
$s\colon A^* \to \mathbb{N}$

The input belongs to a *free monoid* A^*

The output belongs to the *integer semiring* $\mathbb{N}$

The function realised is *a function from* A^* to $\mathbb{N}$

A new automaton model



$s\colon A^* \to \mathbb{N}$ $\qquad$ $s \in \mathbb{N}\langle\langle A^* \rangle\rangle$

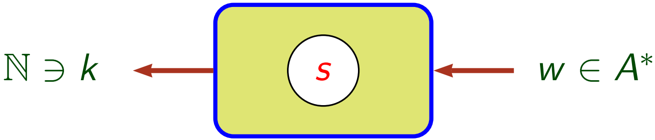
The input belongs to a *free monoid* A^*

The output belongs to the *integer semiring* $\mathbb{N}$

The function realised is *a function from* A^* to $\mathbb{N}$

we call it *a series*

A new automaton model



$s\colon A^* \to \mathbb{N}$ $\qquad$ $s \in \mathbb{N}\langle\langle A^* \rangle\rangle$

$$s_1 = b + a\,b + b\,a + 2\,b\,b + a\,a\,b + \cdots + 2\,b\,b\,a + 3\,b\,b\,b + \cdots$$

The input belongs to a *free monoid* A^*

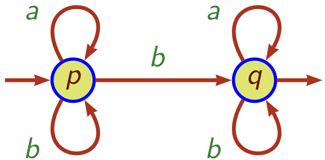
The output belongs to the *integer semiring* $\mathbb{N}$

The function realised is *a function from* A^* to $\mathbb{N}$

we call it *a series*

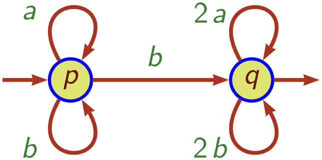
The weighted automaton model

$\mathcal{B}_1$



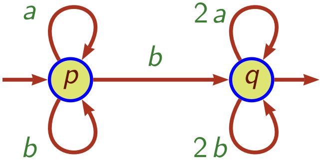
The weighted automaton model

$\mathcal{C}_1$



The weighted automaton model

$\mathcal{C}_1$

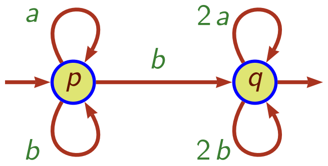


$$\xrightarrow{1} p \xrightarrow{b} p \xrightarrow{a} p \xrightarrow{b} q \xrightarrow{1}$$

$$\xrightarrow{1} p \xrightarrow{b} q \xrightarrow{2\,a} q \xrightarrow{2\,b} q \xrightarrow{1}$$

The weighted automaton model

$\mathcal{C}_1$



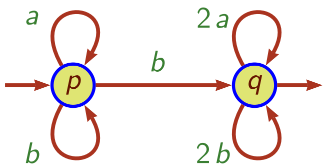
$$\xrightarrow{1} p \xrightarrow{b} p \xrightarrow{a} p \xrightarrow{b} q \xrightarrow{1}$$

$$\xrightarrow{1} p \xrightarrow{b} q \xrightarrow{2\,a} q \xrightarrow{2\,b} q \xrightarrow{1}$$

- Weight of a path c: *product* of the weights of transitions in c
- Weight of a word w: *sum* of the weights of paths with label w

The weighted automaton model

$\mathcal{C}_1$



$$\xrightarrow{1} p \xrightarrow{b} p \xrightarrow{a} p \xrightarrow{b} q \xrightarrow{1}$$

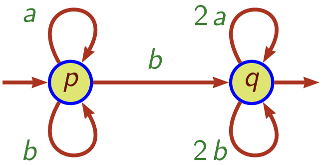
$$\xrightarrow{1} p \xrightarrow{b} q \xrightarrow{2\,a} q \xrightarrow{2\,b} q \xrightarrow{1}$$

- Weight of a path c: *product* of the weights of transitions in c
- Weight of a word w: *sum* of the weights of paths with label w

$$b\,a\,b \quad \longmapsto \quad 1 + 4 = 5$$

The weighted automaton model

$\mathcal{C}_1$



$$\xrightarrow{1} p \xrightarrow{b} p \xrightarrow{a} p \xrightarrow{b} q \xrightarrow{1}$$

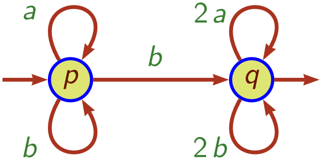
$$\xrightarrow{1} p \xrightarrow{b} q \xrightarrow{2\,a} q \xrightarrow{2\,b} q \xrightarrow{1}$$

- Weight of a path c: *product* of the weights of transitions in c
- Weight of a word w: *sum* of the weights of paths with label w

$$b\,a\,b \quad \longmapsto \quad 1 + 4 = 5 \quad = \langle 101 \rangle_2$$

The weighted automaton model

$\mathcal{C}_1$



$|\mathcal{C}_1| \in \mathbb{N}\langle\langle A^* \rangle\rangle$

$$\xrightarrow{1} p \xrightarrow{b} p \xrightarrow{a} p \xrightarrow{b} q \xrightarrow{1}$$

$$\xrightarrow{1} p \xrightarrow{b} q \xrightarrow{2\,a} q \xrightarrow{2\,b} q \xrightarrow{1}$$

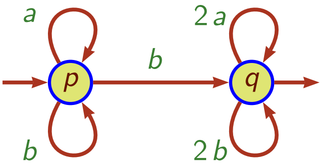
- Weight of a path c: *product* of the weights of transitions in c
- Weight of a word w: *sum* of the weights of paths with label w

$b\,a\,b \quad \longmapsto \quad 1 + 4 = 5$

$|\mathcal{C}_1| \colon A^* \longrightarrow \mathbb{N}$

The weighted automaton model

$\mathcal{C}_1$



$|\mathcal{C}_1| \in \mathbb{N}\langle\langle A^* \rangle\rangle$

$$\xrightarrow{1} p \xrightarrow{b} p \xrightarrow{a} p \xrightarrow{b} q \xrightarrow{1}$$

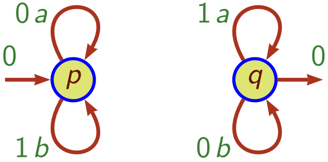
$$\xrightarrow{1} p \xrightarrow{b} q \xrightarrow{2\,a} q \xrightarrow{2\,b} q \xrightarrow{1}$$

- Weight of a path c: *product* of the weights of transitions in c
- Weight of a word w: *sum* of the weights of paths with label w

$$|\mathcal{C}_1| = b + a\,b + 2\,b\,a + 3\,b\,b + a\,a\,b + 2\,a\,b\,a + \cdots + 5\,b\,a\,b + \cdots$$

The weighted automaton model (2)

$\mathcal{L}_1$



$|\mathcal{L}_1| \in \mathbb{Z}\mathrm{min}\langle\langle A^* \rangle\rangle$

$$\xrightarrow{0} p \xrightarrow{1\,b} p \xrightarrow{0\,a} p \xrightarrow{1\,b} p \xrightarrow{0}$$

$$\xrightarrow{0} q \xrightarrow{0\,b} q \xrightarrow{1\,a} q \xrightarrow{0\,b} q \xrightarrow{0}$$

- Weight of a path c:

  product, that is, the sum, of the weights of transitions in c
- Weight of a word w:

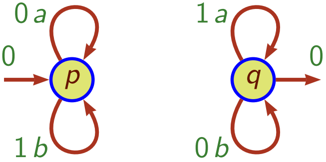
  sum, that is, the min of the weights of paths with label w

$b\,a\,b \longmapsto \min(1+0+1, 0+1+0) = 1$

$|\mathcal{L}_1| \colon A^* \longrightarrow \mathbb{Z}\mathrm{min}$

The weighted automaton model (2)

$\mathcal{L}_1$



$|\mathcal{L}_1| \in \mathbb{Z}\text{min}\langle\langle A^* \rangle\rangle$

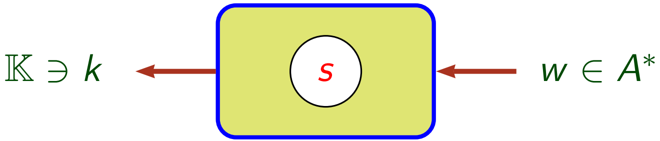
$$\xrightarrow{0} p \xrightarrow{1\,b} p \xrightarrow{0\,a} p \xrightarrow{1\,b} p \xrightarrow{0}$$

$$\xrightarrow{0} q \xrightarrow{0\,b} q \xrightarrow{1\,a} q \xrightarrow{0\,b} q \xrightarrow{0}$$

- Weight of a path c:
  product, that is, the sum, of the weights of transitions in c
- Weight of a word w:
  sum, that is, the min of the weights of paths with label w

$$|\mathcal{C}_1| = 0\,1_{A^*} + 0\,a + 0\,b + 1\,a\,b + 1\,b\,a + 0\,b\,b + \cdots + 1\,b\,a\,b + \cdots$$

The weighted automaton model (system theory mode)



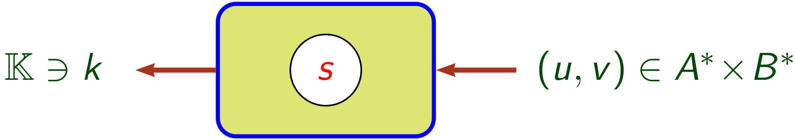
$$s \colon A^* \to \mathbb{K} \qquad\qquad s \in \mathbb{K}\langle\langle A^* \rangle\rangle$$

The input belongs to a *free monoid* A^*

The output belongs to a *semiring* $\mathbb{K}$

The function realised is *a function from* A^* to $\mathbb{K}$: *a series* in $\mathbb{K}\langle\langle A^* \rangle\rangle$

The weighted automaton model (sytem theory mode)



$s\colon A^* \times B^* \to \mathbb{K}$ $\qquad$ $s \in \mathbb{K}\langle\langle A^* \times B^* \rangle\rangle$

The input belongs to a *direct product of free monoids* $A^* \times B^*$

The output belongs to a *semiring* $\mathbb{K}$

The function realised is *a function from* $A^* \times B^*$ to $\mathbb{K}$:
a series in $\mathbb{K}\langle\langle A^* \times B^* \rangle\rangle$

Richness of the model of weighted automata

- $\mathbb{B}$ 'classic' automata
- $\mathbb{N}$ 'usual' counting
- $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ numerical multiplicity
- $\langle\, \mathbb{Z} \cup +\infty, \mathsf{min}, +\,\rangle$ Min-plus automata
- $\langle\, \mathbb{Z}, \mathsf{min}, \mathsf{max}\,\rangle$ fuzzy automata
- $\mathfrak{P}\,(B^*) = \mathbb{B}\langle\!\langle B^* \rangle\!\rangle$ transducers
- $\mathbb{N}\langle\!\langle B^* \rangle\!\rangle$ weighted transducers
- $\mathfrak{P}\,(F(B))$ pushdown automata

Series play the role of languages

$\mathbb{K}\langle\langle A^* \rangle\rangle$ **plays the role of** $\mathfrak{P}(A^*)$

Series play the role of relations

$\mathbb{K}\langle\!\langle A^* \times B^* \rangle\!\rangle$ **plays the role of** $\mathfrak{P}(A^* \times B^*)$

Weighted automata theory

is the linear algebra

of computer science

Part II

Rationality

Outline of Part II

- Definition of rational series
- The Fundamental Theorem of Finite Automata
  What can be computed by a finite automaton
  is exactly what can be computed by the star operation
  (together with the algebra operations)
- Morphisms of weighted automata

The semiring $\mathbb{K}\langle\langle A^* \rangle\rangle$

$\mathbb{K}$ semiring $\qquad$ A^* free monoid

$$s \in \mathbb{K}\langle\langle A^* \rangle\rangle \qquad s \colon A^* \to \mathbb{K} \qquad s \colon w \longmapsto \langle s, w \rangle$$

$$s = \sum_{w \in A^*} \langle s, w \rangle \, w$$

Point-wise addition $\qquad \langle s + t, w \rangle = \langle s, w \rangle + \langle t, w \rangle$

Cauchy product $\qquad \langle s\,t, w \rangle = \sum_{u\,v = w} \langle s, u \rangle \langle t, v \rangle$

$\{(u, v) \mid u\,v = w\}$ finite $\quad \Longrightarrow \quad$ Cauchy product well-defined

$\mathbb{K}\langle\langle A^* \rangle\rangle$ is a semiring

The semiring $\mathbb{K}\langle\langle M \rangle\rangle$

$\mathbb{K}$ semiring $\qquad$ M monoid

$$s \in \mathbb{K}\langle\langle M \rangle\rangle \qquad s \colon M \to \mathbb{K} \qquad s \colon m \longmapsto \langle s, m \rangle$$

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Cauchy product $\qquad \langle s\, t, m \rangle = \sum_{x\, y = m} \langle s, x \rangle \langle t, y \rangle$

$\forall m \ \{(x, y) \,|\, x\, y = m\}$ finite $\quad \Longrightarrow \quad$ Cauchy product well-defined

The semiring $\mathbb{K}\langle\langle M\rangle\rangle$

Conditions for $\{(x,y) \mid x\,y = m\}$ finite for all m

Definition

M is *graded* if M equipped with a length function φ

$\varphi\colon M \to \mathbb{N} \qquad \varphi(m\,m') = \varphi(m) + \varphi(m')$

$$M \text{ f.g. and graded} \implies \mathbb{K}\langle\langle M\rangle\rangle \text{ is a semiring}$$

Examples

$\mathbb{M}$ trace monoid, then $\mathbb{K}\langle\langle M\rangle\rangle$ is a semiring

$\mathbb{K}\langle\langle A^* \times B^*\rangle\rangle$ is a semiring

$F(A)$, the free group on A , is not graded

The algebra $\mathbb{K}\langle\langle M\rangle\rangle$

$\mathbb{K}$ semiring $\quad$ M f.g. graded monoid

$$s \in \mathbb{K}\langle\langle A^*\rangle\rangle \qquad s\colon M \to \mathbb{K} \qquad s\colon m \longmapsto \langle s, m\rangle$$

$$s = \sum_{m\in M} \langle s, m\rangle\, m$$

Point-wise addition $\quad$ $\langle s+t, m\rangle = \langle s, m\rangle + \langle t, m\rangle$

Cauchy product $\quad$ $\langle s\,t, m\rangle = \sum_{x\,y=m} \langle s, x\rangle \langle t, y\rangle$

External multiplication $\quad$ $\langle k\,s, m\rangle = k\,\langle s, m\rangle$

$\mathbb{K}\langle\langle M\rangle\rangle$ is an algebra

The star operation

$t \in \mathbb{K}$ $\qquad\qquad$ $t^* = \sum_{n \in \mathbb{N}} t^n$

How to define infinite sums ?

One possible solution

Topology on $\mathbb{K}$

Definition of summable families and of their sum

t^* defined $\qquad$ if $\qquad$ $\{t^n\}_{n \in \mathbb{N}}$ summable

Other possible solutions

axiomatic definition of star, $\quad$ equational definition of star

The star operation

$t \in \mathbb{K}$

$$t^* = \sum_{n \in \mathbb{N}} t^n$$

The star operation

$$t \in \mathbb{K} \qquad\qquad t^* = \sum_{n \in \mathbb{N}} t^n$$

- $\forall \mathbb{K} \qquad (0_{\mathbb{K}})^* = 1_{\mathbb{K}}$
- $\mathbb{K} = \mathbb{N} \qquad \forall x \neq 0 \qquad x^*$ not defined.
- $\mathbb{K} = \mathcal{N} = \mathbb{N} \cup \{+\infty\} \qquad \forall x \neq 0 \qquad x^* = \infty$.
- $\mathbb{K} = \mathbb{Q} \qquad (\frac{1}{2})^* = 2$ with the natural topology, $(\frac{1}{2})^*$ is undefined with the discrete topology.

The star operation

$$t \in \mathbb{K} \qquad\qquad t^* = \sum_{n \in \mathbb{N}} t^n$$

In any case

$$t^* = 1_{\mathbb{K}} + t\, t^*$$

Star has the same flavor as the inverse

If $\mathbb{K}$ is a ring

$$t^* \, (1_{\mathbb{K}} - t) = 1_{\mathbb{K}}$$

$$\frac{1_{\mathbb{K}}}{1_{\mathbb{K}} - t} = 1_{\mathbb{K}} + t + t^2 + \cdots + t^n + \cdots$$

Star of series

$s \in \mathbb{K}\langle\langle A^* \rangle\rangle$ When is $s^* = \sum_{n \in \mathbb{N}} s^n$ defined ?

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Topology on $\mathbb{K}$ yields topology on $\mathbb{K}\langle\langle A^* \rangle\rangle$

The simple convergence topology on $\mathbb{K}\langle\langle A^* \rangle\rangle$

Topology on $\mathbb{K}$ given by a *distance* **c** $\quad$ $\mathbf{c}\colon \mathbb{K} \times \mathbb{K} \to \mathbb{R}_+$

The simple convergence topology on $\mathbb{K}\langle\langle A^*\rangle\rangle$

Topology on $\mathbb{K}$ given by a *distance* $\mathbf{c}$ $\qquad \mathbf{c}\colon \mathbb{K}\times\mathbb{K}\to\mathbb{R}_+$

- symmetry: $\mathbf{c}(x,y)=\mathbf{c}(y,x)$
- positivity: $\mathbf{c}(x,y)>0$ if $x\neq y$ and $\mathbf{c}(x,x)=0$
- triangular inequality: $\mathbf{c}(x,y)\leqslant\mathbf{c}(x,z)+\mathbf{c}(y,z)$

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A sequence $\{k_n\}_{n \in \mathbb{N}}$ of elements of $\mathbb{K}$ *converges toward* k

$k = \lim_{n \to +\infty} k_n \qquad \forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geqslant N \qquad \mathbf{c}(k_n, k) \leqslant \varepsilon$

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Remark

Always assume $\mathbf{c}(x, y) \leqslant 1$

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Remark

Discrete topology $\quad\quad$ $x \neq y \Rightarrow \mathbf{c}(x, y) = 1$

Converging sequences $=$ *stationnary sequences*

The simple convergence topology on $\mathbb{K}\langle\langle A^* \rangle\rangle$

▸ $\mathbb{B}$, $\mathbb{N}$, $\mathbb{Z}$,	discrete topology
▸ $\mathcal{M} = \langle\, \mathbb{N}, \min, + \,\rangle$	discrete topology
▸ $\mathbb{Q}$, $\mathbb{Q}_+$, $\mathbb{R}$, $\mathbb{R}_+$	"natural distance"

The simple convergence topology on $\mathbb{K}\langle\langle A^* \rangle\rangle$

Definition

$\{s_n\}_{n\in\mathbb{N}}$, $s_n \in \mathbb{K}\langle\langle A^* \rangle\rangle$, converges toward s iff

$$\forall w \in A^* \ \langle s_n, w \rangle \text{ converges toward } \langle s, w \rangle \text{ in } \mathbb{K}.$$

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The simple convergence topology on $\mathbb{K}\langle\langle A^* \rangle\rangle$
is defined by a distance $\mathbf{d}$:

If $\mathbb{K}$ is equipped with the discrete topology:

$$\mathbf{e}(s,t) = \min\{n \in \mathbb{N} \mid \exists w \in A^* \quad |w| = n \quad \text{and} \quad \langle s, w \rangle \neq \langle t, w \rangle\} \ ,$$

$$\mathbf{d}(s,t) = 2^{-\mathbf{e}(s,t)}$$

The simple convergence topology on $\mathbb{K}\langle\langle A^* \rangle\rangle$

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is defined by a distance $\mathbf{d}$:

If $\mathbb{K}$ is equipped with the topology defined by the distance $\mathbf{c}$:

$$\mathbf{d}(s,t) = \frac{1}{2} \sum_{n\in\mathbb{N}} \left(\frac{1}{2^n} \max\left\{ \mathbf{c}(\langle s, w \rangle, \langle t, w \rangle) \mid |w| = n \right\} \right) \quad .$$

The simple convergence topology on $\mathbb{K}\langle\langle A^* \rangle\rangle$

Proposition

If $\mathbb{K}$ is a topological semiring,
then $\mathbb{K}\langle\langle A^* \rangle\rangle$, equipped with the simple convergence topology,
is a topological semiring.

The simple convergence topology on $\mathbb{K}\langle\langle A^* \rangle\rangle$

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A locally finite family of series is summable.

Star of series

$s \in \mathbb{K}\langle\langle A^* \rangle\rangle$ When is $s^* = \sum_{n \in \mathbb{N}} s^n$ defined ?

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Definition
$\mathbb{K}$ strong product of two summable families summable.

Proposition
$\mathbb{K}$ *strong,* $s \in \mathbb{K}\langle\langle A^* \rangle\rangle$ s^* *is defined iff* s_0^* *is defined*

$$s^* = (s_0^* \, s_{\mathrm{p}})^* s_0^* = s_0^* (s_{\mathrm{p}} \, s_0^*)^*$$

Rational series

$\mathbb{K}\langle A^*\rangle \subseteq \mathbb{K}\langle\langle A^*\rangle\rangle$ subalgebra of polynomials

$\mathbb{K}\mathrm{Rat}\, A^*$ closure of $\mathbb{K}\langle A^*\rangle$ under

- sum
- product
- exterior multiplication
- and star

$\mathbb{K}\mathrm{Rat}\, A^* \subseteq \mathbb{K}\langle\langle A^*\rangle\rangle$ subalgebra of rational series

Fundamental theorem of finite automata

Theorem

$$s \in \mathbb{K}\mathrm{Rat}\, A^* \qquad \Longleftrightarrow \qquad \exists \mathcal{A} \in \mathbb{K}\mathsf{WA}\,(A^*) \quad s = |\mathcal{A}|$$

Fundamental theorem of finite automata

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Kleene theorem ?

Fundamental theorem of finite automata

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Kleene theorem ?

Theorem

M *finitely generated graded monoid*

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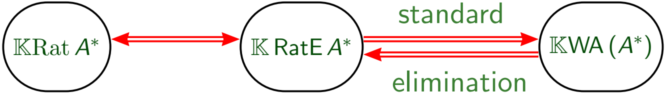
Fundamental theorem of finite automata



Standard automaton

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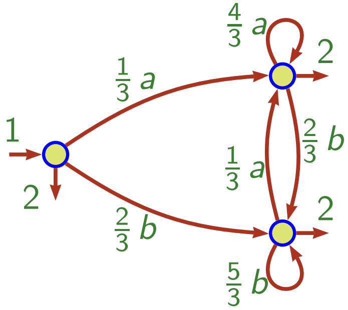


Automata are matrices

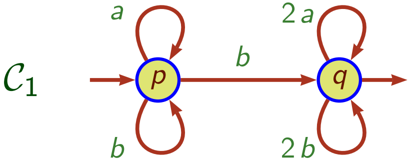
- Automata are (essentially) matrices: $\mathcal{A} = \langle\, I, E, T \,\rangle$

- Computing the behaviour of an automaton boils down
  to solving a linear system $X = E \cdot X + T$ (s)

- Solving the linear system (s) amounts to invert
  the matrix $(Id - E)$ (hence the name rational)

- The inversion of $Id - E$ is realised by
  an infinite sum $Id + E + E^2 + E^3 + \cdots$: the star of E

Automata are matrices



$$\mathcal{C}_1 = \langle\, I_1, E_1, T_1 \,\rangle = \left\langle \begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} a+b & b \\ 0 & 2\,a+2\,b \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle \ .$$

Automata are matrices

$\mathcal{A} = \langle\, I, E, T \,\rangle$ $\qquad$ $E =$ incidence matrix

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Notation

$\mathbf{wl}(x)$ = weighted label of x

In our model, e transition $\Rightarrow$ $\mathbf{wl}(e) = k\,a$

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Lemma

$$E^{n}_{p,q} = \sum \{\mathbf{wl}(c) \mid c \quad \textit{computation from } p \textit{ to } q \textit{ of length } n\}$$

Automata are matrices

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Automata are matrices

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$$E^* = \sum_{n \in \mathbb{N}} E^n$$

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$$\left|\mathcal{A}\right| = I \cdot E^* \cdot T$$

Automata are matrices

$\mathbb{K}$ semiring $\qquad$ M graded monoid

$\mathbb{K}\langle\langle M\rangle\rangle^{Q\times Q}$ is isomorphic to $\mathbb{K}^{Q\times Q}\langle\langle M\rangle\rangle$

$E \in \mathbb{K}\langle\langle M\rangle\rangle^{Q\times Q}$ $\qquad$ E proper $\Longrightarrow$ E^* defined

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Theorem

The entries of E^* *are*

in the rational closure of the entries of E

Fundamental theorem of finite automata

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Theorem
The family of behaviours of weighted automata over M

with coefficients in $\mathbb{K}$ *is rationally closed.*

The collect theorem

$$\mathbb{K}\langle\langle A^* \times B^* \rangle\rangle \text{ is isomorphic to } [\mathbb{K}\langle\langle B^* \rangle\rangle]\,\langle\langle A^* \rangle\rangle$$

Theorem

Under the above isomorphism,

$$\mathbb{K}\mathrm{Rat}\, A^* \times B^* \textit{ corresponds to } [\mathbb{K}\mathrm{Rat}\, B^*]\,\mathrm{Rat}\, A^*$$

Morphisms of automata

1. **Automata are structures.**

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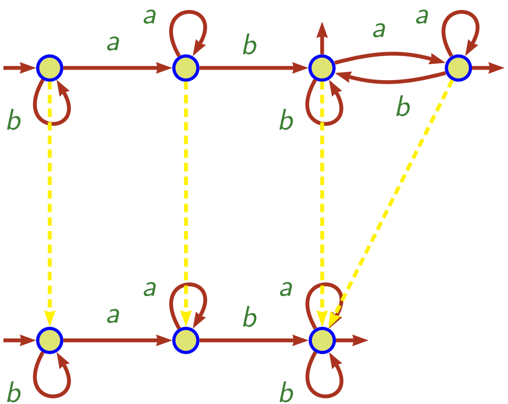
2. **Automata realise series**

   Can we find an equivalent smaller automaton?
   of minimal size?
   that respects the structure?

Morphisms of Boolean automata

Minimisation of deterministic automata



Morphisms of Boolean automata

Definition

$\mathcal{A} = \langle\, I, E, T \,\rangle$ and $\mathcal{B} = \langle\, J, F, U \,\rangle$ $\mathbb{B}$-automata of dimension Q and R

A map $\varphi\colon Q \to R$ defines a morphism $\varphi\colon \mathcal{A} \to \mathcal{B}$ if

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$$|\mathcal{A}| \subseteq |\mathcal{B}|$$

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Problem:

Find conditions such that $|\mathcal{A}| = |\mathcal{B}|$

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Local conditions

Problem:

Neither the definition, nor the solution, extend directly to $\mathbb{B}$-automata

Conjugacy of automata

Definition

Let $\mathcal{A} = \langle\, I, E, T \,\rangle$ and $\mathcal{B} = \langle\, J, F, U \,\rangle$ be two $\mathbb{K}$-automata.

$\mathcal{A}$ is conjugate to $\mathcal{B}$ if

$\exists X$ $\mathbb{K}$-matrix $\quad I\,X = J, \quad E\,X = X\,F, \quad$ and $\quad T = X\,U$

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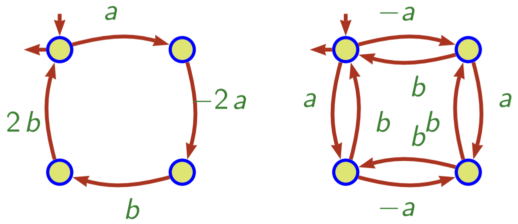
Conjugacy of automata

$$X_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \; .$$

$\mathcal{A}_1$



$\mathcal{B}_1$

$$\mathcal{A}_1 \stackrel{X_1}{\Longrightarrow} \mathcal{B}_1$$

Conjugacy of automata

$$\mathcal{C}' = \left\langle \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & z & 0 \\ 0 & 0 & z \\ 0 & 0 & 2z \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\rangle \qquad \mathcal{A}' = \left\langle \begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & z \\ 0 & 2z \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$$

$$\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & z & 0 \\ 0 & 0 & z \\ 0 & 0 & 2z \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 0 & z \\ 0 & 2z \end{pmatrix},$$

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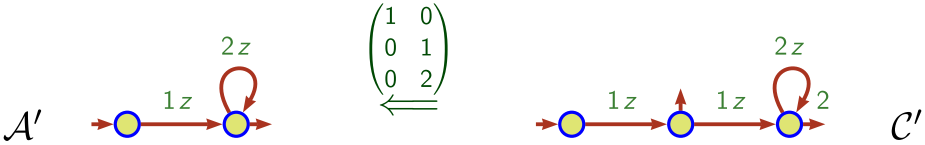

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$I\,E\,E\,T$

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$$I\,E\,E\,T = I\,E\,E\,X\,U = I\,E\,X\,F\,U = I\,X\,F\,F\,U = J\,F\,F\,U$$

and then $\quad I\,E^*\,T = J\,F^*\,U$

Morphisms of weighted automata

Definition

A map $\varphi\colon Q \to R$ defines a $(Q \times R)$-amalgamation matrix H_φ

$$\varphi_2\colon \{j, r, s, u\} \to \{i, q, t\} \qquad \text{defines} \qquad H_{\varphi_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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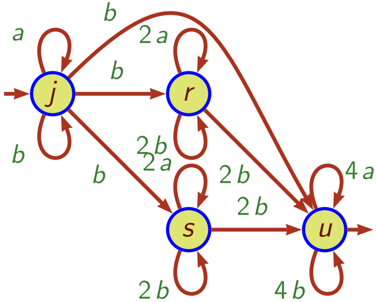
Directed notion

Price to pay for the weight

Morphisms of weighted automata

$\mathcal{C}_2$



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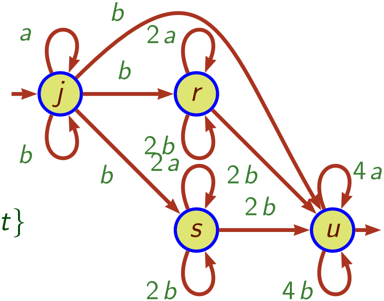

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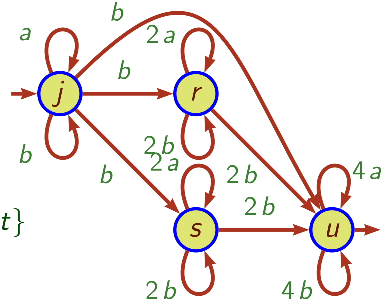

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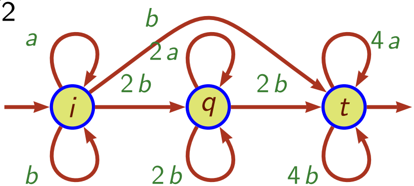

$$\mathcal{C}_2 \stackrel{H_{\varphi_2}}{\Longrightarrow} \mathcal{V}_2$$

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if $\mathcal{B}$ is conjugate to $\mathcal{A}$ by the matrix ${}^{\mathsf{t}}H_\varphi$: $\mathcal{B} \overset{{}^{\mathsf{t}}H_\varphi}{\Longrightarrow} \mathcal{A}$

$$J\ {}^{\mathsf{t}}H_\varphi = I, \qquad F\ {}^{\mathsf{t}}H_\varphi = {}^{\mathsf{t}}H_\varphi\ E, \qquad U = {}^{\mathsf{t}}H_\varphi\ T$$

$\mathcal{B}$ is a co-quotient of $\mathcal{A}$

Directed notion

Price to pay for the weight

Morphisms of weighted automata

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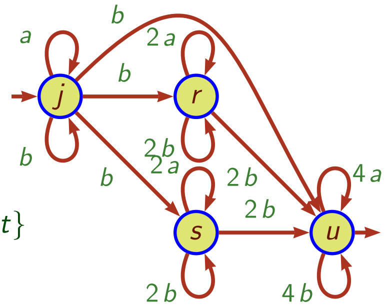

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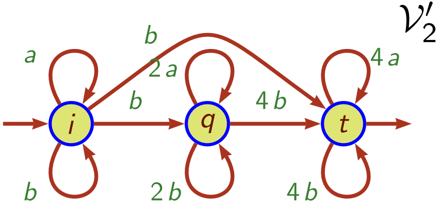
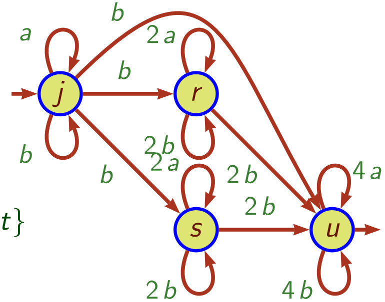


$$\mathcal{V}'_2 \xrightarrow{{}^{t}H_{\varphi_2}} \mathcal{C}_2$$

Morphisms of weighted automata

$\mathcal{C}_2$



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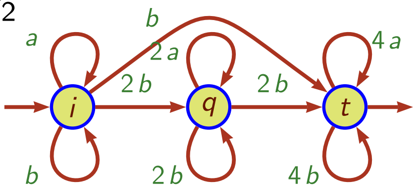
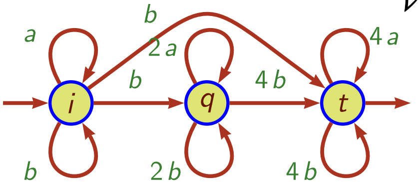


$\mathcal{V}_2'$



$\mathcal{C}_2 \xRightarrow{H_{\varphi_2}} \mathcal{V}_2$

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Theorem

Every $\mathbb{K}$-automaton has a minimal quotient that is effectively computable (by Moore algorithm).

Morphisms of weighted automata

A practical look at conjugacy by H_φ

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- Multiplying E by H_φ on the right amounts to *add columns*
- Multiplying F by H_φ on the left amounts to *duplicate lines*
- Merging states p and q realises an *Out-morphism* if
  adding columns p and q in E yields
  a matrix whose lines p and q are equal

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$$I\,H_\varphi = J, \qquad E\,H_\varphi = H_\varphi\,F, \quad \text{and} \quad T = H_\varphi\,U$$

- Multiplying E by H_φ on the right amounts to *add columns*
- Multiplying F by H_φ on the left amounts to *duplicate lines*
- Merging states p and q realises an *Out-morphism* if
  adding columns p and q in E yields
  a matrix whose lines p and q are equal
  (and if $T_p = T_q$)

Morphisms of weighted automata

$$\underbrace{\begin{pmatrix} a+b & b & b & b \\ 0 & 2a+2b & 0 & 2b \\ 0 & 0 & 2a+2b & 2b \\ 0 & 0 & 0 & 4a+4b \end{pmatrix}}_{R_2} \quad \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} R_2$$

Morphisms of weighted automata

$$\underbrace{\begin{pmatrix} a+b & b & b & b \\ 0 & 2a+2b & 0 & 2b \\ 0 & 0 & 2a+2b & 2b \\ 0 & 0 & 0 & 4a+4b \end{pmatrix}}_{R_2} \left.\vphantom{\begin{pmatrix} a \\ 0 \\ 0 \\ 0 \end{pmatrix}}\right\} R_2$$

$$\underbrace{\begin{pmatrix} a+b & 2b & b \\ 0 & 2a+2b & 2b \\ 0 & 2a+2b & 2b \\ 0 & 0 & 4a+4b \end{pmatrix}}_{Q_2} \left.\vphantom{\begin{pmatrix} a \\ 0 \\ 0 \\ 0 \end{pmatrix}}\right\} R_2$$

Morphisms of weighted automata

$$\underbrace{\begin{pmatrix} a+b & b & b & b \\ 0 & 2a+2b & 0 & 2b \\ 0 & 0 & 2a+2b & 2b \\ 0 & 0 & 0 & 4a+4b \end{pmatrix}}_{R_2} \quad \left.\vphantom{\begin{pmatrix}a\\a\\a\\a\end{pmatrix}}\right\} R_2$$

$$\underbrace{\begin{pmatrix} a+b & 2b & b \\ 0 & 2a+2b & 2b \\ 0 & 0 & 4a+4b \end{pmatrix}}_{Q_2} \quad \left.\vphantom{\begin{pmatrix}a\\a\\a\end{pmatrix}}\right\} Q_2$$

Part III

Recognisability