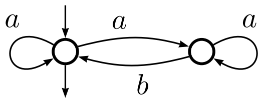
Lecture I — Exercises

Unless stated otherwise, the alphabet A is $A = \{a, b\}$.

1. **Semiring structure.** Is $\mathbb{M} = \langle\, \mathbb{N}, \mathsf{max}, +, 0, 0 \,\rangle$ a semiring?

2. **Positive semiring.** Give an example of a semiring in which the sum of any two non-zero elements is non-zero but which *is not* positive. [*Hint: consider a sub-semiring of* $\mathbb{N}^{2\times 2}$.]

3. **Example of $\mathbb{N}$-automaton.** (a) Compute the coefficient of $a^3 b\, a^2 b\, a$ in the series realised by the $\mathbb{N}$-automaton:



(b) Give the general formula for the coefficient of every word of A^*.

4. **Examples of $\mathbb{N}\mathsf{min}$, $\mathbb{N}\mathsf{max}$-automata.** Let $\mathcal{E}_1$ be the $\mathbb{N}\mathsf{min}$-automaton over $\{a\}^*$ shown in Fig. 1 (a) and $\mathcal{E}_2$ the $\mathbb{N}\mathsf{max}$-automaton shown in the same figure. Similarly, let $\mathcal{E}_3$ and $\mathcal{E}_4$ be the $\mathbb{N}\mathsf{min}$ and $\mathbb{N}\mathsf{max}$-automata shown in Fig. 1 (b).

Give a formula for $\langle\!\langle \mathcal{E}_1 |, a^n \rangle$, $\langle\!\langle \mathcal{E}_2 |, a^n \rangle$, $\langle\!\langle \mathcal{E}_3 |, a^n \rangle$, and $\langle\!\langle \mathcal{E}_4 |, a^n \rangle$.

(a) The automata $\mathcal{E}_1$ and $\mathcal{E}_2$ (b) The automata $\mathcal{E}_3$ and $\mathcal{E}_4$

Figure 1: Four 'tropical' automata

5. **A $\mathbb{Z}$-automaton.** Build a $\mathbb{Z}$-automaton $\mathcal{D}_1$ such that $\langle\!\langle \mathcal{D}_1 |, w \rangle = |w|_a - |w|_b$, for every w in A^*.

6. **Support of $\mathbb{Z}$-automata.** Give an example of a $\mathbb{Z}$-automaton $\mathcal{A}$ such that the inclusion $\mathsf{supp}\ (|\mathcal{A}|) \subseteq |\mathsf{supp}\ \mathcal{A}|$ is strict.

7. **Automata construction.** Let $\underline{a^*}$ be the characteristic $\mathbb{N}$-series of a^* : $\underline{a^*} = \sum_{n\in\mathbb{N}} a^n$. Give an 'automatic' proof (that is, by means of automata constructions) for:

$$(\underline{a^*})^2 = \sum_{n\in\mathbb{N}} (n+1)\, a^n \quad .$$

8. **Shortest run and $\mathbb{N}\mathsf{min}$-automata.** Build a $\mathbb{N}\mathsf{min}$-automaton $\mathcal{F}_1$ such that, for every w in A^*, $\langle\!\langle \mathcal{F}_1 |, w \rangle$ is the minimal length of runs of 'a' 's in w, that is, if $w = a^{n_0}\, b\, a^{n_1}\, b \cdots a^{n_{k-1}}\, b\, a^{n_k}$, then $\langle\!\langle \mathcal{F}_1 |, w \rangle = \mathsf{min}\{n_0, n_1, \ldots, n_k\}$.

9. **Identification of a $\mathbb{Q}$-automaton.** Show that the final function of the $\mathbb{Q}$-automaton $\mathcal{Q}_2$ over $\{a\}^*$ depicted on the right in Figure 2 (where every transition is labelled by $a\,|\,1$) can be specified in such a way the result is equivalent to $\mathcal{Q}_1$ depicted on the left.

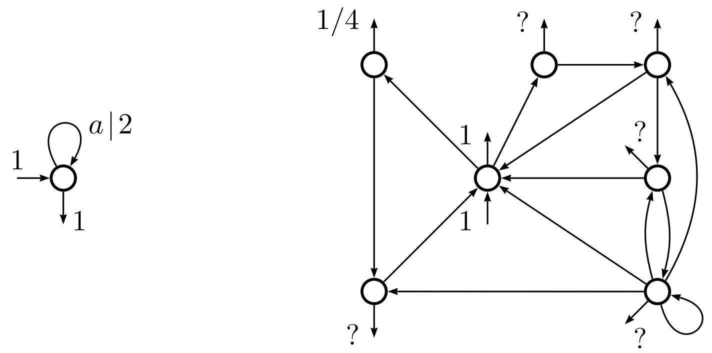


Figure 2: Two $\mathbb{Q}$-automata

10. **Ambiguous automata.** Show that it is decidable whether a Boolean automaton is unambiguous or not. [*Hint: Note that this is* not *a result nor a proof on weighted automata but on Boolean automata. It is put here in view of Example 49.*]

11. **Representation with finite image.** Let s be a $\mathbb{K}$-recognisable series of A^*, realised by a representation $\langle I, \mu, T \rangle$ of dimension Q. Show that if $\mu(A^*)$ is a finite submonoid of $\mathbb{K}^{Q \times Q}$, then, for every k in $\mathbb{K}$ the set $s^{-1}(k) = \{w \in A^* \mid \langle s, w \rangle = k\}$ is a recognisable language of A^*.

12. **Support of $\mathbb{Z}$-rational series.** (a) Give an example of a $\mathbb{Z}$-rational series over A^* whose support is not a recognisable language of A^*.

(b) Give an example of a $\mathbb{Z}$-rational series over A^* which is an $\mathbb{N}$-series (that is, all coefficients are non-negative) and which is not an $\mathbb{N}$-rational series over A^*.

13. **Support of $\mathbb{Z}$-rational series.** (a) Prove that the support of an $\mathbb{N}$-rational series over A^* is a recognisable language of A^*.

(b) Let s be in $\mathbb{N}\mathrm{Rec}\, A^*$. Prove that for any k in $\mathbb{N}$, the sets

$$s^{-1}(k) = \{w \in A^* \mid \langle s, w \rangle = k\} \quad \text{and} \quad s^{-1}(k + \mathbb{N}) = \{w \in A^* \mid \langle s, w \rangle \geqslant k\}$$

are recognisable languages of A^*.

(c) Give an example of a $\mathbb{Z}$-rational series s over A^* such that there exists an integer z such that $s^{-1}(z)$ is not a recognisable language of A^*.

14. **Support of $\mathbb{Z}\mathsf{min}$-rational series.** (a) Let s be a $\mathbb{N}\mathsf{min}$-rational series over A^*. Prove that for any k in $\mathbb{N}$, the sets

$$s^{-1}(k) = \{w \in A^* \mid \langle s, w \rangle = k\} \qquad \text{and} \qquad s^{-1}(k + \mathbb{N}) = \{w \in A^* \mid \langle s, w \rangle \geqslant k\}$$

are recognisable languages of A^*.

(b) Give an example of a $\mathbb{Z}\mathsf{min}$-rational series s over A^* such that there exists an integer z such that $s^{-1}(z)$ is not a recognisable language of A^*.

15. **Recognisable series in direct product of free monoids.** Let $\mathbb{K}$ be a commutative semiring. The two semirings $\mathbb{K}\langle\langle A^* \rangle\rangle$ and $\mathbb{K}\langle\langle B^* \rangle\rangle$ are canonically subalgebras of $\mathbb{K}\langle\langle A^* \times B^* \rangle\rangle$; the injection is induced by

$$u \mapsto (u, 1_{B^*}) \qquad \text{and} \qquad v \mapsto (1_{A^*}, v) \ ,$$

for all u in A^* and all v in B^*. Modulo this identification, a product $(k\,u)\,(h\,v)$ is written $k\,h\,(u,v)$ and the extension by linearity of this notation gives the following definition.

Definition 1. Let s be in $\mathbb{K}\langle\langle A^*\rangle\rangle$ and t be in $\mathbb{K}\langle\langle B^*\rangle\rangle$. The *tensor product* of s and t, written $s \otimes t$, is the series of $\mathbb{K}\langle\langle A^* \times B^*\rangle\rangle$ defined by:

$$\forall(u,v) \in A^* \times B^* \qquad \langle s \otimes t, (u,v)\rangle = \langle s, u\rangle\,\langle t, v\rangle \ .$$

On the other hand, $\mathbb{K}$-*recognisable series* over a non-free monoid M are defined, exactly as the $\mathbb{K}$-recognisable series over a free monoid, as the series realised by a $\mathbb{K}$-representation $\langle\, I, \mu, T\,\rangle$, where μ is a morphism from M into $\mathbb{K}^{Q \times Q}$.

Establish:

Proposition 2. *A series s of $\mathbb{K}\langle\langle A^* \times B^*\rangle\rangle$ is recognisable if and only if there exists a finite family $\{r_i\}_{i \in I}$ of series of $\mathbb{K}\mathrm{Rec}\, A^*$ and a finite family $\{t_i\}_{i \in I}$ of series of $\mathbb{K}\mathrm{Rec}\, B^*$ such that*

$$s = \sum_{i \in I} r_i \otimes t_i \ .$$

16. **Distance on the semirings of series.**

A *distance* on any set S is a map $\mathbf{d}\colon S \times S \to \mathbb{R}_+$ with the three properties: for all x, y and z in S it holds:

(i) *symmetry*: $\mathbf{d}(x,y) = \mathbf{d}(y,x)$;

(ii) *positivity*: $\mathbf{d}(x,y) = 0 \Leftrightarrow x = y$;

(iii) *triangular inequality*: $\mathbf{d}(x,z) \leqslant \mathbf{d}(x,y) + \mathbf{d}(y,z)$.

If (iii) is replaced by the stronger property:

(iv) *triangular inequality*: $\mathbf{d}(x,z) \leqslant \max(\mathbf{d}(x,y), \mathbf{d}(y,z))$,

then $\mathbf{d}$ is said to be an *ultrametric distance*.

(a) Show that the function defined on S by

$$\forall x,y \in S \qquad \mathbf{d}(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise} \end{cases}$$

is an ultrametric distance. We call it the *discrete distance* on S.

Classically, a sequence $\left(s_n\right)_{n \in \mathbb{N}}$ of elements of S *converges* to s in S for the distance $\mathbf{d}$ if:

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n > N \qquad \mathbf{d}(s_n, s) < \varepsilon \ .$$

In this way, a distance $\mathbf{d}$ defines a *topology* on S.

(b) Show that if S is equipped with the discrete distance, the only convergent sequences are the ultimately stationnary sequences.

Two distances on S are *equivalent* if the same sequences converge, that is, $\mathbf{d}$ and $\mathbf{d}'$ are equivalent if for any sequence $s = \left(s_n\right)_{n \in \mathbb{N}}$, s converges for $\mathbf{d}$ if and only if it converges for $\mathbf{d}'$.

(c) Show that one can always assume that *a distance is bounded by* 1, that is, if $\mathbf{d}$ is a distance on S, the function $\mathbf{f}$ defined by

$$\forall x,y \in S \qquad \mathbf{f}(x,y) = \inf\{\mathbf{d}(x,y), 1\}$$

is a distance, equivalent to $\mathbf{d}$.

(d) Let **d** and **d'** be two distances on S. Show that if there exist two constant C and D in $\mathbb{R}_+ \setminus \{0\}$ such that

$$\forall x, y \in S \qquad C\,\mathbf{d}\,(x,y) \leqslant \mathbf{d'}\,(x,y) \leqslant D\,\mathbf{d}\,(x,y)$$

then **d** and **d'** are equivalent. Is this condition necessary for **d** and **d'** be equivalent?

Let $\mathbb{K}$ be a semiring. For s and t in $\mathbb{K}\langle\langle A^*\rangle\rangle$, let $\mathbf{e}(s,t)$ be the *gap between s and t*, defined as the minimal length of words on which s and t are different:

$$\mathbf{e}(s,t) = \mathsf{min}\,\{n \in \mathbb{N} \mid \exists w \in A^*, \quad |w| = n \text{ and } \langle s, w\rangle \neq \langle t, w\rangle\} \quad .$$

The gap is a generalisation of the notion of *valuation* of a series. The valuation $\mathbf{v}(s)$ of s in $\mathbb{K}\langle\langle A^*\rangle\rangle$ is defined by:

$$\mathbf{v}(s) = \mathbf{e}(s,0) = \mathsf{min}\,\{|w| \mid \langle s, w\rangle \neq 0\} = \mathsf{min}\,\{|w| \mid w \in \mathsf{supp}\ s\} \quad .$$

Conversely, and if $\mathbb{K}$ is a *ring*, $\mathbf{e}(s,t) = \mathbf{v}(s-t)$.

(e) Show that the map defined by

$$\forall s, t \in \mathbb{K}\langle\langle A^*\rangle\rangle \qquad \mathbf{d'}\,(s,t) = 2^{-\,\mathbf{e}(s,t)} \tag{0.1}$$

is an ultrametric distance on $\mathbb{K}\langle\langle A^*\rangle\rangle$, bounded by 1.

(f) Let **c** be a distance on $\mathbb{K}\langle\langle A^*\rangle\rangle$, bounded by 1. Show that the map defined by

$$\forall s, t \in \mathbb{K}\langle\langle A^*\rangle\rangle \qquad \mathbf{d}\,(s,t) = \frac{1}{2}\sum_{n\in\mathbb{N}}\left(\frac{1}{2^n}\,\mathsf{max}\,\{\mathbf{c}\,(\langle s, w\rangle, \langle t, w\rangle) \mid |w| = n\}\right) \tag{0.2}$$

is a distance on $\mathbb{K}\langle\langle A^*\rangle\rangle$, bounded by 1.

(g) Show that, whatever the distance **c**, $\mathbf{d}\,(s,t) \leqslant \mathbf{d'}\,(s,t)$ holds.

(h) Show that if **c** is the discrete distance, then $\mathbf{d'}\,(s,t) \leqslant 2\,\mathbf{d}\,(s,t)$ holds, hence that (0.1) and (0.2) define two equivalent distances on $\mathbb{K}\langle\langle A^*\rangle\rangle$ if $\mathbb{K}$ is equipped with the discrete distance.

(i) Show that the topology defined by **d** on $\mathbb{K}\langle\langle A^*\rangle\rangle$ is the topology of the simple convergence.

(j) Show that if $\mathbb{K}$ is a topological semiring, then so are $\mathbb{K}^{Q\times Q}$ (Q finite) and $\mathbb{K}\langle\langle A^*\rangle\rangle$.

(k) Let $\left(s_n\right)_{n\in\mathbb{N}}$ and $\left(t_n\right)_{n\in\mathbb{N}}$ be two sequences of series in the topological semiring $\mathbb{K}\langle\langle A^*\rangle\rangle$. Verify that $\left(s_n + t_n\right)_{n\in\mathbb{N}}$ or $\left(s_n\, t_n\right)_{n\in\mathbb{N}}$ may be convergent sequences, without $\left(s_n\right)_{n\in\mathbb{N}}$ or $\left(t_n\right)_{n\in\mathbb{N}}$ being convergent sequences.