Test — Solution

In the following, A denotes the alphabet $A = \{a, b\}$, $\mathbb{N}$, the set of non negative integers. If w is in A^*, we denote by $|w|_a$ the number of a's in w.

1 .— *Finite image relations.*

Recall that a relation α is said to be finite image *if* $\mathsf{Im}\,\alpha$ *is a finite set (and not if the image $\alpha(w)$ is finite for every w).*

Show that a finite image functional rational relation is sequential.

If the relation $\alpha\colon A^* \to A^*$ is finite image, let $\mathsf{Im}\,\alpha = \{w_1, w_2, \ldots, w_k\}$ and write $\mathbf{k} = \{1, 2, \ldots, k\}$. If α is a rational relation, so is α^{-1} and $\alpha^{-1}(w_j) = K_j$ is a rational subset of $\mathbb{K}_j$ for every j. The $\mathbb{K}_j$ are pairwise disjoint since α is functional. Every $\mathbb{K}_j$ is accepted by a *complete deterministic* automaton $\mathcal{A}_j = \langle\, Q_j, i_j, \delta_j, T_j \,\rangle$. Based on the product of the $\mathcal{A}_j$, we build a *deterministic* automaton which recognises all the K_j simultaneously :

$$\mathcal{A} = \langle\, Q, i, \delta, U_1, U_2, \ldots, U_k \,\rangle , \qquad \text{with :}$$
$$Q = \prod_{j \in \mathbf{k}} Q_j , \quad i = (i_1, i_2, \ldots, i_k) , \qquad \text{et} \quad U_j = \prod_{h \in \mathbf{k}, h \neq j} Q_h \times T_j \ .$$

For every u in A^*, it holds : $\delta(i, u) \in U_j \Leftrightarrow u \in K_j$.

We transform $\mathcal{A}$ into a transducer $\mathcal{T}$ by adding the output 1_{A^*} on every transition of $\mathcal{A}$ and by defining the final function U by : $U(q) = w_j$ if and only if $q \in U_j$; $\mathcal{T}$ is sequential since $\mathcal{A}$ is deterministic and $U\colon Q \to A^*$ functional.

2 .— *Commutative image.*

Let $\alpha\colon A^* \to \mathbb{N}^2$ the commutative image *map, i.e.* $\alpha(w) = (|w|_a, |w|_b)$.

Show that the equivalence map of α, i.e. the relation $\alpha^{-1} \circ \alpha\colon A^* \to A^*$ which associates with every word w of A^* all the words of A^* which have the same number of a's and the same number of b's as w, is not a rational relation.

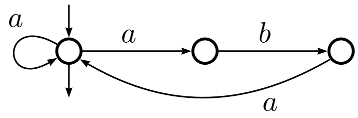
Let $L = (a\,b)^*$, a rational language. It holds :

$$[\alpha^{-1} \circ \alpha](L) = \{w \in A^* \mid |w|_a = |w|_b\} \quad ,$$

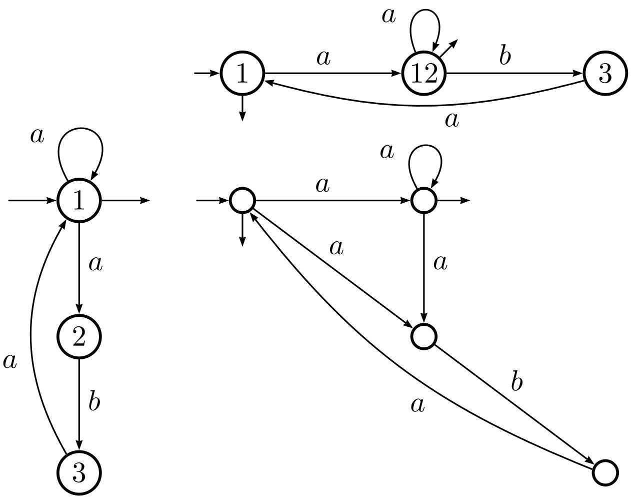
which is not a rational language, hence $\alpha^{-1} \circ \alpha$ is not a rational relation.

3 .— ***Coding and deciphering.***

(i) *Build the Schützenberger covering of the automaton below.*

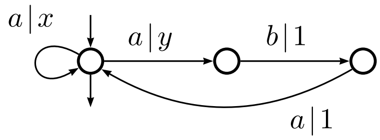


The automaton $\mathcal{A}_1$ (left, verticaly), its determinisation $\widehat{\mathcal{A}_1}$ (top, horizontaly) and its Schützenberger covering $\mathcal{S}_1$.



(ii) *Let* $\alpha\colon \{x,y\}^* \to \{a,b\}^*$ *be the morphism defined by:* $\alpha(x) = a$, $\alpha(y) = a\,b\,a$. *Show that* α *is* injective *(hence the relation* α^{-1} *is fonctional).*

The automaton $\mathcal{A}_1$ is the underlying input automaton of the transducer $\mathcal{T}_1$ which realizes α^{-1} and which is drawn below.



The morphism α is injective, and then α^{-1} is functional, if and only if the automaton $\mathcal{A}_1$ is unambiguous.

It can be seen on the figure of the former question that the projection of $\mathcal{S}_1$ onto $\widehat{\mathcal{A}_1}$ is In-bijective, that is, is a co-covering. The successful computations of $\widehat{\mathcal{A}_1}$ are then in 1-to-1 correspondence with those of $\mathcal{S}_1$, and then with those of $\mathcal{A}_1$ since $\mathcal{S}_1$ is a covering of $\mathcal{A}_1$. Hence $\mathcal{A}_1$ is unambiguous as is $\widehat{\mathcal{A}_1}$.

(iii) *Give a (finite) sequential transducer that realizes* α^{-1}.

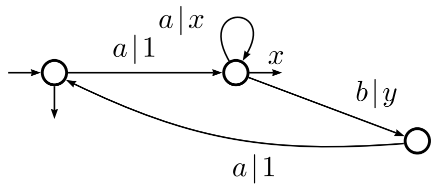
The representation corresponding to the real-time transducer $\mathcal{T}_1$ is

$$I_1 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} , \quad \mu_1(a) = \begin{pmatrix} x & y & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} , \quad \mu_1(b) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} , \quad T_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} .$$

The sequentialisation process applied to this representation leads to the following computations :

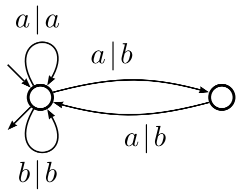
$$I_1 \cdot \mu_1(a) = \begin{pmatrix} x & y & 0 \end{pmatrix}, \quad I_1 \cdot \mu_1(a\,a) = \begin{pmatrix} x\,x & x\,y & 0 \end{pmatrix} = x \begin{pmatrix} x & y & 0 \end{pmatrix},$$
$$I_1 \cdot \mu_1(a\,b) = \begin{pmatrix} 0 & 0 & y \end{pmatrix} = y \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \cdot \mu_1(a) = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} = I_1,$$
$$I_1 \cdot \mu_1(b) = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \cdot \mu_1(a) = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}.$$

Which yields the sequential transducer $\mathcal{T}_2$ below, whose underlying input automaton is naturally equal to $\widehat{\mathcal{A}_1}$.



4 .— ***Factor replacing.***

(i) *Let $\alpha\colon A^* \to A^*$ be the relation realized by the synchronous transducer below.*



(a) What is the image of the word $a\,b\,a\,a\,b\,b$ by α ?

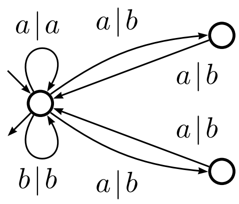
(b) Describe the relation α.

(c) Give a transducer which realizes $\alpha \circ \alpha$.

(a) $\alpha(a\,b\,a\,a\,b\,b) = \{a\,b\,a\,a\,b\,b, a\,b\,b\,b\,b\,b\}$.

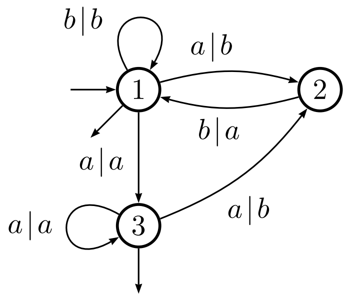
(b) The relation α associates with every word u of A^* the set of words that are obtained by replacing in u an arbitrary number (and possibly zero) factors $a\,a$ (without overlapping) by factors $b\,b$.

(c) The definition of α itself shows that $\alpha \circ \alpha = \alpha$ and the transducer $\mathcal{T}_1$ given above answers the question. The computation of the composition of $\mathcal{T}_1$ by itself yields another transducer below that is equivalent to $\mathcal{T}_1$.



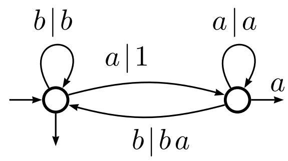
(ii) *Let $\beta\colon A^* \to A^*$ be the (functional) relation which replace* every *factor $a\,b$ of a word by a factor $b\,a$ (which does not prevent the result to contain still factors $a\,b$). For instance : $\beta\,(a\,b\,a\,a\,b\,b) = b\,a\,a\,b\,a\,b$.*
Give a synchronous transducer which realizes β.

Let 1 be the initial state of such a transducer. If a 'b' is read, 'b' is output and the transducer stays in the same state. Reading an 'a' on the contrary opens two possibilities, represented by two distinct states, respectively state 2 and 3 : either this 'a' is followed by a 'b', inwhich case a 'b' is output and the following 'b' will output an 'a', or this 'a' is followed by an 'a', or it is the last letter of the word, in which case an 'a' is output. From state 2, one can read a 'b' only and go to state 1. From state 3, one can read an 'a' only and this yields the same dilemna as before. This behaviour is realised by the transducer $\mathcal{T}_2$ below.



(iii) *(a) Give a sequential transducer which realizes β.*
(b) Give a sequential transducer which realizes $\beta \circ \beta$.

(a) One can build the same kind of reasoning as above. From the initial state, reading a 'b' outputs a 'b' and the transducer stays in the same state. Reading an 'a' moves the transducer in a state that keeps the memory of that 'a' and outputs the empty word. In that state, reading an 'a' proves that the preceding 'a' is not followed 'b' and thus yields the output of an 'a', while the transducer stays in the same state. If the word ends in that state, the 'a' that is kept by the state has to be output : it is the role of the final function. If a 'b' is read, since the preceding letter is an 'a', a factor '$a\,b$' is read, '$b\,a$' is output and the transducer goes back to the initial state, which gives the transducer $\mathcal{T}_3$ below.



It is also possible to apply the sequentialisation process to the representation corresponding to transducer $\mathcal{T}_2$:

$$I_2 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \quad \mu_2(a) = \begin{pmatrix} 0 & b & a \\ 0 & 0 & 0 \\ 0 & b & a \end{pmatrix}, \quad \mu_2(b) = \begin{pmatrix} b & 0 & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} .$$

It then holds :

$$I_2 \cdot \mu_2(a) = \begin{pmatrix} 0 & b & a \end{pmatrix}, \quad I_2 \cdot \mu_2(b) = \begin{pmatrix} b & 0 & 0 \end{pmatrix} = b \begin{pmatrix} 1 & 0 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & b & a \end{pmatrix} \cdot \mu_2(a) = \begin{pmatrix} 0 & ab & aa \end{pmatrix} = a \begin{pmatrix} 0 & b & a \end{pmatrix},$$
$$\begin{pmatrix} 0 & b & a \end{pmatrix} \cdot \mu_2(b) = \begin{pmatrix} ba & 0 & 0 \end{pmatrix} = b \begin{pmatrix} 1 & 0 & 0 \end{pmatrix},$$

and we get the transducer $\mathcal{T}_3$ again.

(b) As $\mathcal{T}_3$ is not subnormalised, it is necessary to use the composition of representations. The representation corresponding to transducer $\mathcal{T}_3$ is :

$$I_3 = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad \mu_3(a) = \begin{pmatrix} 0 & 1 \\ 0 & a \end{pmatrix}, \quad \mu_3(b) = \begin{pmatrix} b & 0 \\ ba & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 1 \\ a \end{pmatrix} .$$

The composition of this representation by itself gives :

$$I_3 \cdot \mu_3(I_3) = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}, \quad \mu_3(T_3) \cdot T_3 = \begin{pmatrix} 1 \\ a \\ a \\ aa \end{pmatrix},$$

$$[\mu_3 \circ \mu_3](a) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & a \end{pmatrix}, \quad [\mu_3 \circ \mu_3](b) = \begin{pmatrix} b & 0 & 0 & 0 \\ ba & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & ba & 0 & 0 \end{pmatrix}.$$

And this representation corresponds to the following sequential transducer :