The sequentialisation of automata and transducers

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CNRS / Université Denis-Diderot and Telecom ParisTech

Joint work with *Sylvain Lombardy*, Université de Bordeaux

Survey Lecture at the International Workshop

Weighted Automata: Theory and Applications

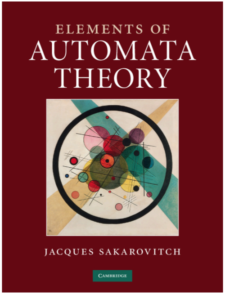
Leipzig, 22 May 2018

Based on the results presented in the survey paper:

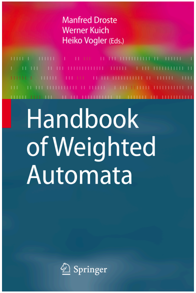
- Sequential ? *Theoret. Computer Sci.* **359** (2006)

with S. Lombardy

and described in the general framework set up in:

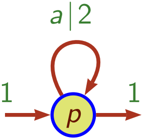


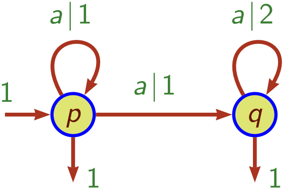
Chapter III



Chapter 4

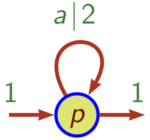
$\mathcal{A}_1$

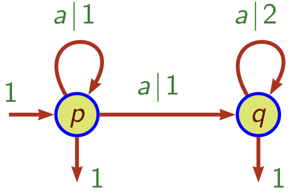




$\mathcal{A}_2$

$\mathcal{A}_1$





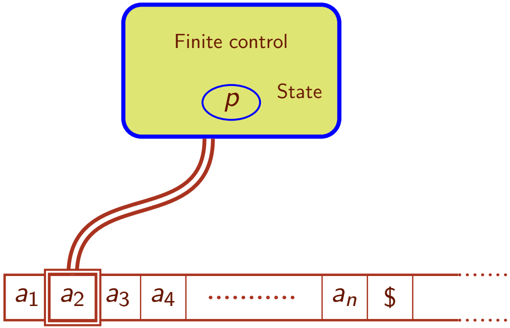
$\mathcal{A}_2$

$$s_1 = \sum_{n \in \mathbb{N}} 2^n a^n$$

Part I

Some views on the weighted automaton model

A touch of general system theory



Paradigm of a machine for the computer scientists

A touch of general system theory

Paradigm of a machine for the rest of the world

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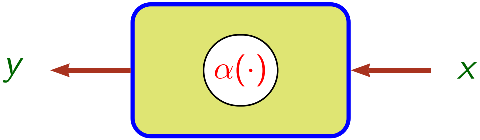

$$y = \alpha(x)$$

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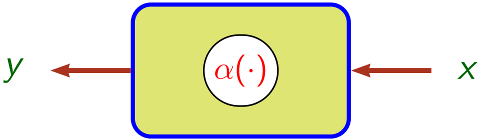

$$y = \alpha(x)$$

$$x \in \mathbb{R}^n\,, \quad y \in \mathbb{R}^m$$

Paradigm of a machine for the rest of the world

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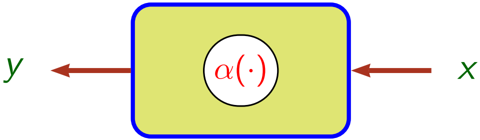
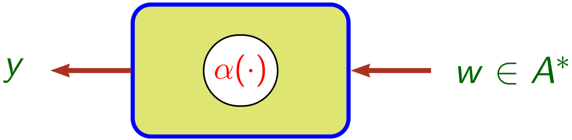


$$y = \alpha(x)$$

Getting back to computer science

A touch of general system theory

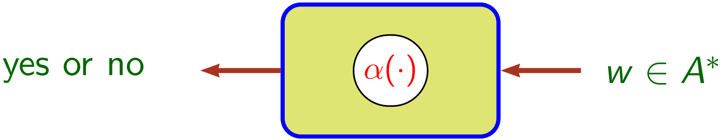


$$y = \alpha(x)$$

The input belongs to a *free monoid* A^*

Getting back to computer science

A touch of general system theory



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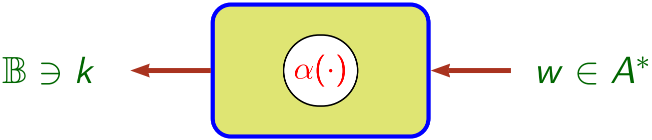

$$y = \alpha(x)$$

The input belongs to a *free monoid* A^*

The output belongs to the *Boolean semiring* $\mathbb{B}$

Getting back to computer science

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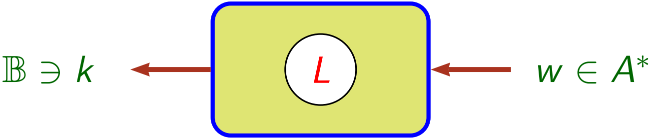

$$L \subseteq A^*$$

The input belongs to a *free monoid* A^*

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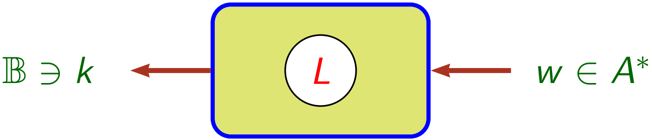
The function realised is *a language*

Getting back to computer science

A touch of general system theory



$$L \subseteq A^*$$

The input belongs to a *free monoid* A^*

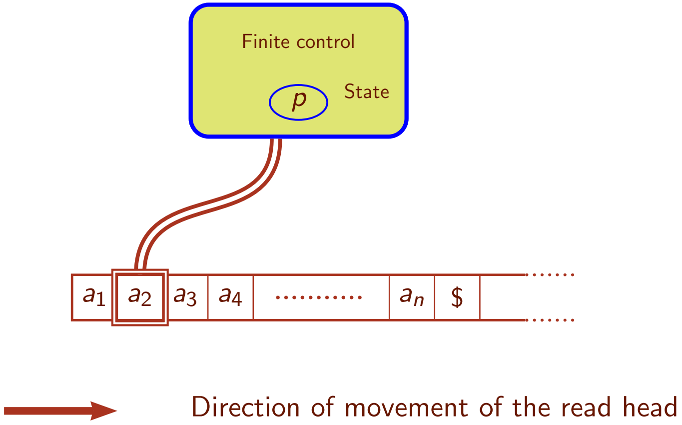
The output belongs to the *Boolean semiring* $\mathbb{B}$

The function realised is *a language*,

that is, the set of words that are accepted by the machine

Getting back to computer science

The simplest Turing machine

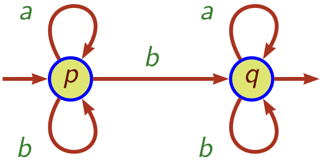


The 1-way 1-tape Turing Machine (1W 1T TM)

Getting back to computer science

The simplest Turing machine is equivalent to finite automata

$\mathcal{B}_1$



$$L(\mathcal{B}_1) = \{w \in A^* \mid w \in A^* b A^*\} = \{w \in A^* \mid |w|_b \geqslant 1\}$$

Remarkable features of the finite automaton model

Decidable equivalence (decidable inclusion)

Closure under complement

Canonical automaton for a given language
(minimal deterministic automaton)

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Decidable equivalence (decidable inclusion)

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Based on

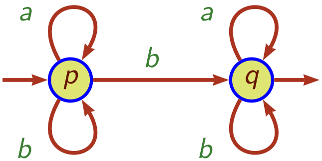
Theorem
Every finite automaton is equivalent to a deterministic one.

And what about the case of

weighted finite automata?

The weighted automaton model

$\mathcal{B}_1$



$$\xrightarrow{1} p \xrightarrow{b} p \xrightarrow{a} p \xrightarrow{b} q \xrightarrow{1}$$

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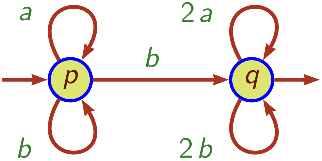
$|\mathcal{B}_1| \colon w \longmapsto |w|_b$ $\qquad$ $|\mathcal{B}_1| \colon A^* \longrightarrow \mathbb{N}$ $\qquad$ $|\mathcal{B}_1| \in \mathbb{N}\langle\langle A^* \rangle\rangle$

$$|\mathcal{B}_1| = b + a\,b + \quad b\,a + 2\,b\,a + a\,a\,b + \quad a\,b\,a + \cdots + 2\,b\,a\,b + \cdots$$

The weighted automaton model

$\mathcal{C}_1$



$$\xrightarrow{1} p \xrightarrow{b} p \xrightarrow{a} p \xrightarrow{b} q \xrightarrow{1}$$

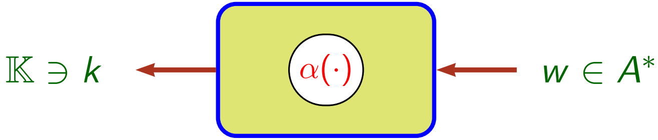
$$\xrightarrow{1} p \xrightarrow{b} q \xrightarrow{2\,a} q \xrightarrow{2\,b} q \xrightarrow{1}$$

- Weight of a path c: *product* of the weights of transitions in c
- Weight of a word w: *sum* of the weights of paths with label w

$|\mathcal{C}_1| \colon w \longmapsto \langle \overline{w} \rangle_2$ $\qquad |\mathcal{C}_1| \colon A^* \longrightarrow \mathbb{N}$ $\qquad |\mathcal{C}_1| \in \mathbb{N}\langle\langle A^* \rangle\rangle$

$$|\mathcal{C}_1| = b + a\,b + 2\,b\,a + 3\,b\,a + a\,a\,b + 2\,a\,b\,a + \cdots + 5\,b\,a\,b + \cdots$$

The system theory view of weighted automata



The input belongs to a *free monoid* A^*

The output belongs to the *semiring* $\mathbb{K}$

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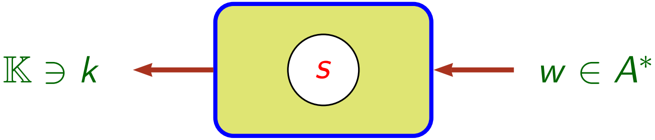

$s\colon A^* \to \mathbb{K}$

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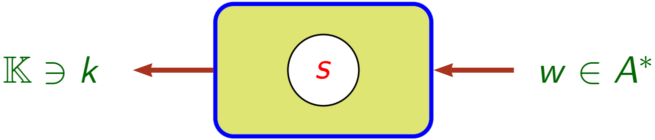


$s\colon A^* \to \mathbb{K}$ $\qquad\qquad$ $s \in \mathbb{K}\langle\langle A^* \rangle\rangle$

The input belongs to a *free monoid* A^*

The output belongs to the *semiring* $\mathbb{K}$

The function realised is *a function from* A^* to $\mathbb{K}$,

that is, *a series* in $\mathbb{K}\langle\langle A^* \rangle\rangle$

Series play the role of languages

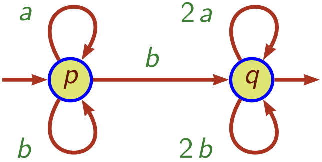
$\mathbb{K}\langle\langle A^* \rangle\rangle$ **plays the role of** $\mathfrak{P}(A^*)$

Richness of the model of weighted automata

$\mathbb{B}$	'classic' automata
$\mathbb{N}$	'usual' counting
$\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$	numerical multiplicity
$\langle\, \mathbb{Z} \cup +\infty, \min, + \,\rangle$	tropical automata
$\langle\, \mathbb{Z}, \min, \max \,\rangle$	fuzzy automata
$\mathfrak{P}(B^*) = \mathbb{B}\langle\langle B^* \rangle\rangle$	transducers
$\mathbb{N}\langle\langle B^* \rangle\rangle$	weighted transducers
$\mathfrak{P}(F(B))$	pushdown automata
$\mathfrak{P}(M)$	register automata, M-automata

Automata are matrices

$\mathcal{C}_1$



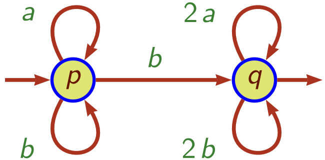
$$\mathcal{C}_1 = \langle\, I_1, E_1, T_1 \,\rangle = \left\langle \begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} a+b & b \\ 0 & 2\,a+2\,b \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle \quad .$$

Traversal of a graph corresponds to *matrix multiplication*

$$E_1{}^* = \sum_{n\in\mathbb{N}} E_1{}^n \qquad\qquad \left|\mathcal{C}_1\right| = I_1 \cdot E_1{}^* \cdot T_1 \quad .$$

Automata over free monoids are representations

$\mathcal{C}_1$



$$\mathcal{C}_1 = \langle\, I_1, E_1, T_1 \,\rangle = \left\langle \begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} a+b & b \\ 0 & 2\,a+2\,b \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle \; .$$

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} a + \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} b$$

$\mu_1 \colon A^* \to \mathbb{K}^{2\times 2}$ $\qquad \mu_1(a) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} , \quad \mu_1(b) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$

Automata over free monoids are representations

$\mathbb{K}$ semiring $\qquad$ A^* free monoid

$\mathbb{K}$-representation

Q finite $\qquad$ $\mu\colon A^* \to \mathbb{K}^{Q \times Q}$ $\qquad$ morphism

(I, μ, T) $\qquad$ $I \in \mathbb{K}^{1 \times Q}$ $\qquad$ $\mu\colon A^* \to \mathbb{K}^{Q \times Q}$ $\qquad$ $T \in \mathbb{K}^{Q \times 1}$

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(I, μ, T) realises (recognises) $s \in \mathbb{K}\langle\langle A^*\rangle\rangle$

$$\forall w \in A^* \qquad \langle s, w\rangle = I \cdot \mu(w) \cdot T$$

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(I, μ, T) realises (recognises) $s \in \mathbb{K}\langle\langle A^*\rangle\rangle$

$$\forall w \in A^* \qquad \langle s, w\rangle = I \cdot \mu(w) \cdot T$$

$$E = \sum_{a\in A} \mu(a)\, a$$

$\langle\, I, E, T\,\rangle$ $\quad$ $\mathbb{K}\mathrm{WA}\,(A^*) \Longleftrightarrow \mathbb{K}\mathrm{Rep}\,(A^*)$ $\quad$ (I, μ, T)

Definitions

A series over A^* is ($\mathbb{K}$-)*rational* or ($\mathbb{K}$-)*recognisable*

if it is realised by

a *finite* ($\mathbb{K}$-)*automaton* or a ($\mathbb{K}$-)*representation*

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$$\begin{tikzcd} \mathbb{K}\mathrm{Rat}\, A^* \arrow[r, Leftrightarrow] & \mathbb{K}\,\mathsf{RatE}\, A^* \arrow[r, shift left, Rightarrow, "\text{standard}"] & \mathbb{K}\mathsf{WA}\,(A^*) \arrow[l, shift left, Rightarrow, "\text{elimination}"] \arrow[r, Leftrightarrow, "\text{key lemma}"] & \mathbb{K}\mathrm{Rec}\, A^* \end{tikzcd}$$

Definitions

A finite ($\mathbb{K}$-)automaton is *sequential*

if its *support* is a *deterministic* Boolean automaton

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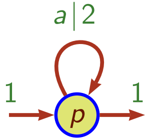
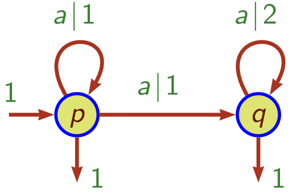
$\mathcal{A}_1$





$\mathcal{A}_2$

Definitions

A finite ($\mathbb{K}$-)automaton is *sequential*

if its *support* is a *deterministic* Boolean automaton

A series over A^* is *sequential*

if it is realized by a finite *sequential* automaton

or by a *row-monomial* representation

The problem

Is it decidable whether a given rational series

is sequential or not ?

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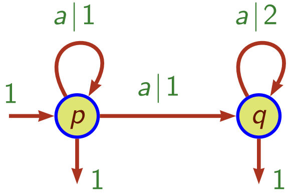

$\mathcal{A}_2$

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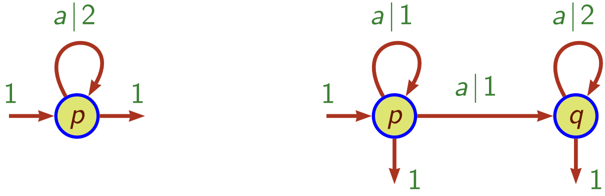


$\mathcal{A}_2$

A word on terminology

Most probably, what I call

sequential automaton

is what you call

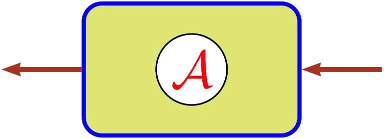
deterministic automaton.

Part II

The common sequentialisation algorithm

First step: the general determinisation procedure

$I \cdot \mu(w) \cdot T$



$w \in A^*$

$\mathcal{A} = (I, \mu, T)$ $\qquad$ $\mu \colon A^* \longrightarrow \mathbb{K}^{Q \times Q}$

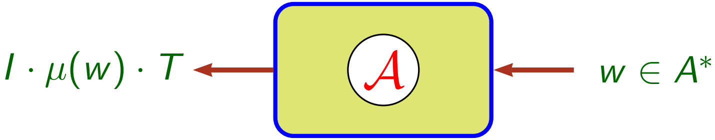
First step: the general determinisation procedure

$$I \cdot \mu(w) \cdot T \longleftarrow \boxed{\mathcal{A}} \longleftarrow w \in A^*$$

$\mathcal{A} = (I, \mu, T)$ $\qquad$ $\mu \colon A^* \longrightarrow \mathbb{K}^{Q \times Q}$

$\mathbb{K}^{1 \times Q}$ *state space* $\qquad$ I initial state

First step: the general determinisation procedure

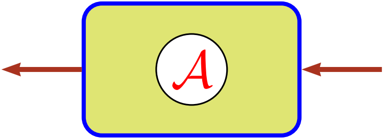


$\mathcal{A} = (I, \mu, T)$ $\qquad$ $\mu\colon A^* \longrightarrow \mathbb{K}^{Q \times Q}$

$\mathbb{K}^{1 \times Q}$ *state space* $\qquad$ I initial state

$I \cdot \mu(w)$ state after reading w

First step: the general determinisation procedure

$I \cdot \mu(w) \cdot T$  $w \in A^*$

$\mathcal{A} = (I, \mu, T)$ $\qquad\qquad$ $\mu \colon A^* \longrightarrow \mathbb{K}^{Q \times Q}$

$\mathbb{K}^{1 \times Q}$ *state space* $\qquad\qquad$ I initial state

$I \cdot \mu(w)$ state after reading w

$I \cdot \mu(w) \cdot T$ output in state $I \cdot \mu(w)$

First step: the general determinisation procedure

$$\mathcal{A} = (I, \mu, T) \qquad\qquad \mu\colon A^* \longrightarrow \mathbb{K}^{Q\times Q}$$

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$$\mathcal{A} = (I, \mu, T) \qquad\qquad \mu \colon A^* \longrightarrow \mathbb{K}^{Q\times Q}$$

μ *morphism* $\implies$ $I \cdot \mu(w\,a) = (I \cdot \mu(w)) \cdot \mu(a)$

First step: the general determinisation procedure

$$\mathcal{A} = (I, \mu, T) \qquad\qquad \mu \colon A^* \longrightarrow \mathbb{K}^{Q \times Q}$$

μ *morphism* $\Longrightarrow$ $I \cdot \mu(w\,a) = (I \cdot \mu(w)) \cdot \mu(a)$

μ defines an *action* of A^* over $\mathbb{K}^{1 \times Q}$

First step: the general determinisation procedure

$$\mathcal{A} = (I, \mu, T) \qquad\qquad \mu\colon A^* \longrightarrow \mathbb{K}^{Q\times Q}$$

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μ defines an *action* of A^* over $\mathbb{K}^{1\times Q}$

This action (with I and T) defines an automaton:

the determinisation $\widehat{\mathcal{A}}$ of $\mathcal{A}$

First step: the general determinisation procedure

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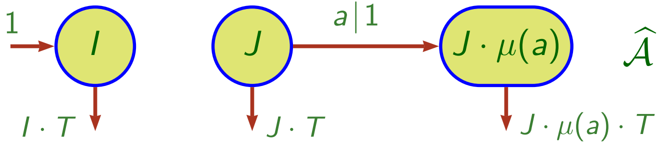
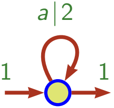
$$J = I \cdot \mu(u)$$



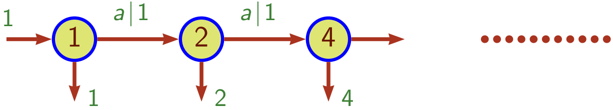
First step: the general determinisation procedure

$\mathcal{A}_1$



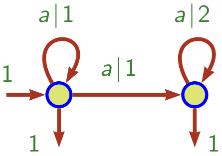
$$\mathcal{A}_1 = \big((1)\,,(2)\,,(1)\big)$$

$\widehat{\mathcal{A}_1}$



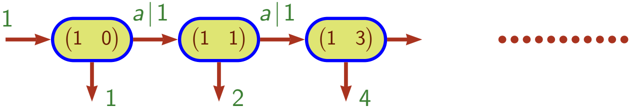
First step: the general determinisation procedure

$\mathcal{A}_2$



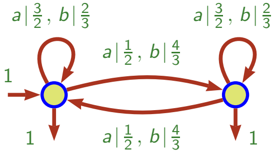
$$\mathcal{A}_2 = \left(\begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)$$

$\widehat{\mathcal{A}_2}$



First step: the general determinisation procedure

$\mathcal{A}_3$



$$\mu_3(a) = \begin{pmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{pmatrix}$$

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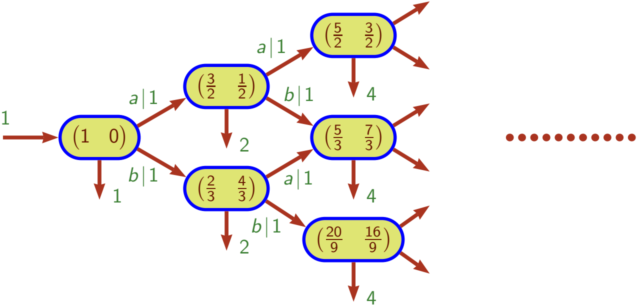
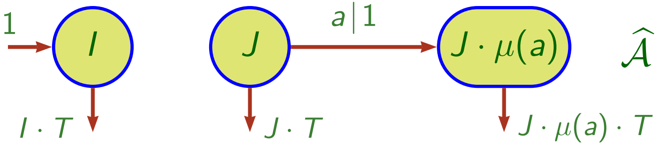
$\widehat{\mathcal{A}_3}$



First step: the general determinisation procedure

If $\mathbb{K} = \mathbb{B}$, determinisation = *subset construction*

$$J = I \cdot \mu(u)$$



$\widehat{\mathcal{A}}$

First step: the general determinisation procedure

If $\mathbb{K} = \mathbb{B}$,   determinisation $=$ *subset construction*

Determinisation yields a *deterministic automaton*

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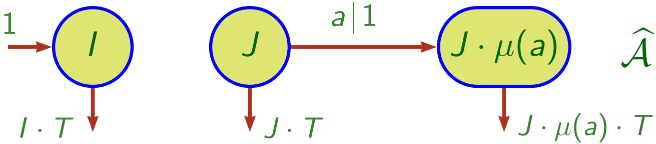


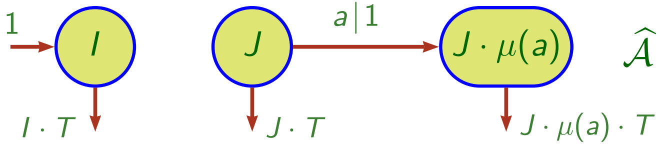
First step: the general determinisation procedure

If $\mathbb{K} = \mathbb{B}$, determinisation = *subset construction*

Determinisation yields a *deterministic automaton*

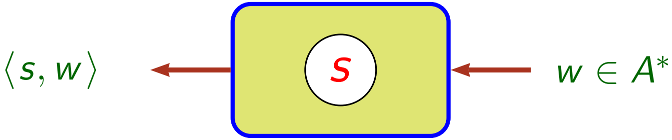
and *conversely*

$$J = I \cdot \mu(u)$$



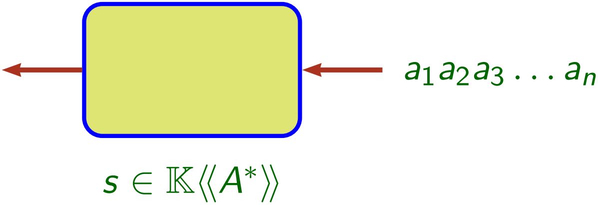
$\widehat{\mathcal{A}}$

Second step: the universal minimisation process



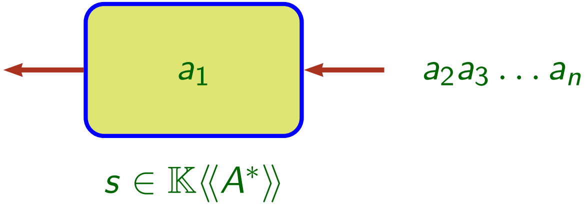
$s \in \mathbb{K}\langle\langle A^* \rangle\rangle$

Second step: the universal minimisation process



The input belongs to a free monoid A^*

Second step: the universal minimisation process



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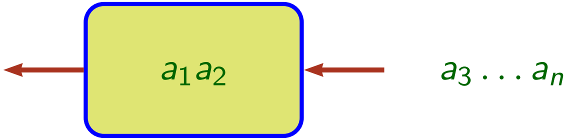

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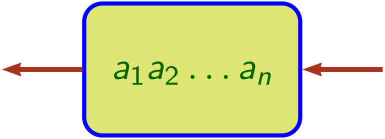
Second step: the universal minimisation process



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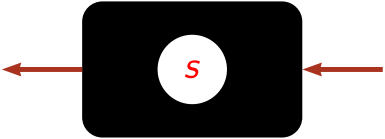
The input belongs to a free monoid A^*

Second step: the universal minimisation process



$s \in \mathbb{K}\langle\langle A^* \rangle\rangle$

The input belongs to a free monoid A^*

Second step: the universal minimisation process

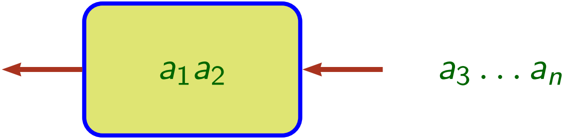
$\langle s, a_1 \ldots a_n \rangle = k$

$s \in \mathbb{K}\langle\langle A^* \rangle\rangle$

The input belongs to a free monoid A^*

The output belongs to $\mathbb{K}$

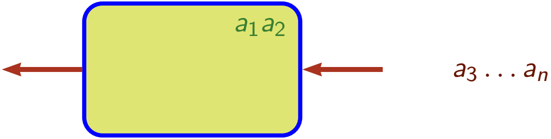
Second step: the universal minimisation process



$s \in \mathbb{K}\langle\langle A^* \rangle\rangle$

A basic construct: the quotient series

Second step: the universal minimisation process



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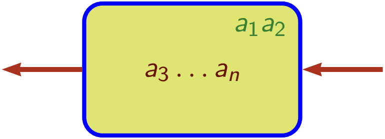

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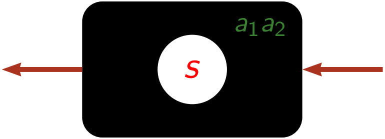
A basic construct: the quotient series

Second step: the universal minimisation process



$$s \in \mathbb{K}\langle\langle A^* \rangle\rangle$$

A basic construct: the quotient series

Second step: the universal minimisation process

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A basic construct: the quotient series

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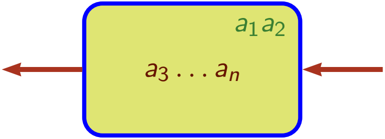
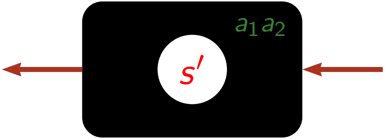


$$s' \in \mathbb{K}\langle\langle A^* \rangle\rangle$$

A basic construct: the quotient series

Second step: the universal minimisation process



$$s' \in \mathbb{K}\langle\langle A^* \rangle\rangle$$

A basic construct: the quotient series

Second step: the universal minimisation process

$$k = \langle s', a_3 \ldots a_n \rangle = \langle s, a_1 a_2 a_3 \ldots a_n \rangle$$

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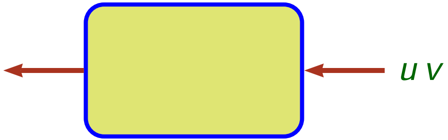
$$k = \langle s', a_3 \ldots a_n \rangle = \langle s, a_1a_2a_3 \ldots a_n \rangle$$

$$s' = [a_1a_2]^{-1}s$$

The series s' is *the quotient* of s by a_1a_2

A basic construct: the quotient series

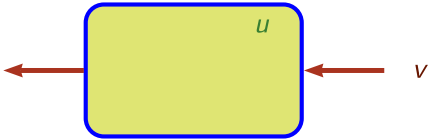
Second step: the universal minimisation process



$s \in \mathbb{K}\langle\langle A^* \rangle\rangle$

A basic construct: the quotient series

Second step: the universal minimisation process



A basic construct: the quotient series

Second step: the universal minimisation process

$$k = \langle s', v \rangle = \langle s, u\,v \rangle$$

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The series s' is *the quotient* of s by u

A basic construct: the quotient series

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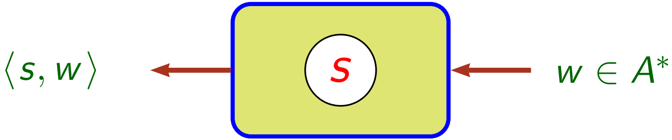
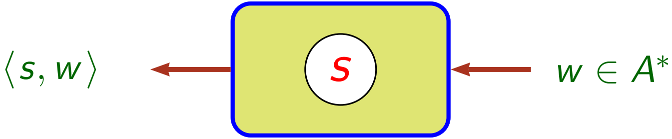


$\mathbf{Q}_s = \{u^{-1}s \mid u \in A^*\}$ set of quotients of s

Second step: the universal minimisation process

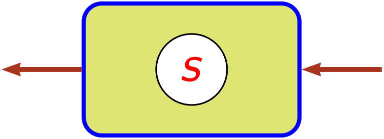


$\mathbf{Q}_s = \{u^{-1}s \mid u \in A^*\}$ set of quotients of s

$$\mathbf{Q}_{s_1} = \{2^n s_1 \mid n \in \mathbb{N}\}$$

Second step: the universal minimisation process

$\langle s, w \rangle$



$w \in A^*$

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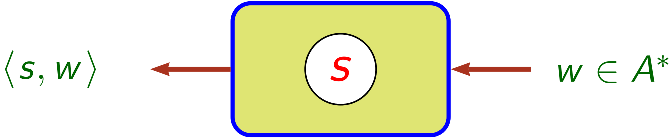
Theorem (Schützenberger–Fliess–Jacob)

A series s is recognisable iff $\mathbf{Q}_s$ is contained in a finitely generated stable submodule of $\mathbb{K}\langle\langle A^* \rangle\rangle$

Second step: the universal minimisation process



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Theorem (Myhill-Nerode)

A language L is recognisable iff $\mathbf{Q}_L$ is finite

Second step: the universal minimisation process

Associativity in A^* $\Longrightarrow$ $(u\,v)^{-1}s = v^{-1}\,[u^{-1}s]$

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This action defines, for every s ,

a deterministic automaton:

the minimal deterministic automaton $\mathcal{A}_s$ of s

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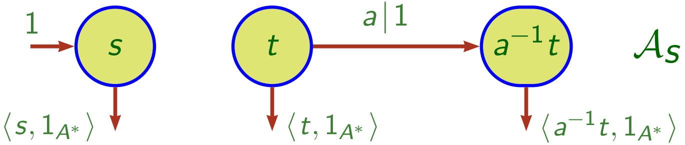
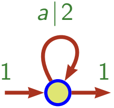


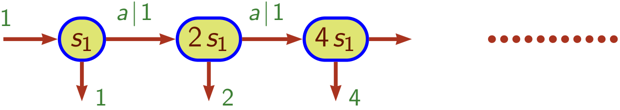
Second step: the universal minimisation process

$\mathcal{A}_1$



$$\mathcal{A}_1 = \big((1)\,,(2)\,,(1)\big)$$

$\mathcal{A}_{s_1}$



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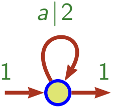

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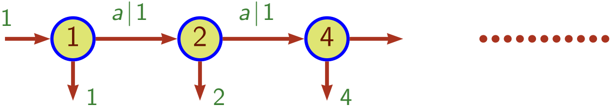
$\widehat{\mathcal{A}_1}$



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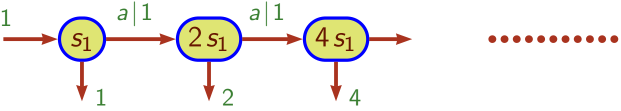
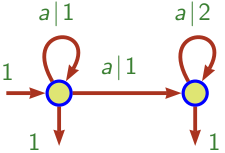


Second step: the universal minimisation process

$\mathcal{A}_2$



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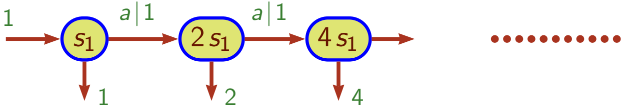
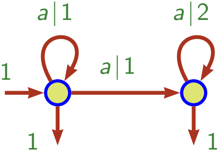


Second step: the universal minimisation process

$\mathcal{A}_2$



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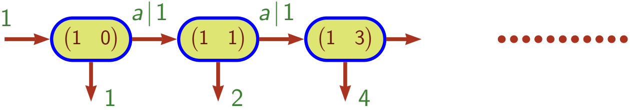
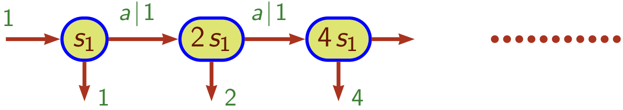
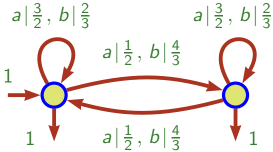


$\mathcal{A}_{s_1}$



Second step: the universal minimisation process

$\mathcal{A}_3$



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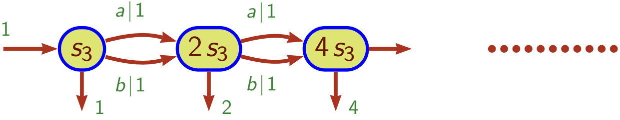
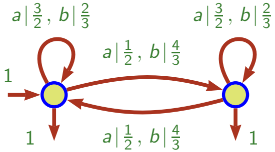


Second step: the universal minimisation process

$\mathcal{A}_3$



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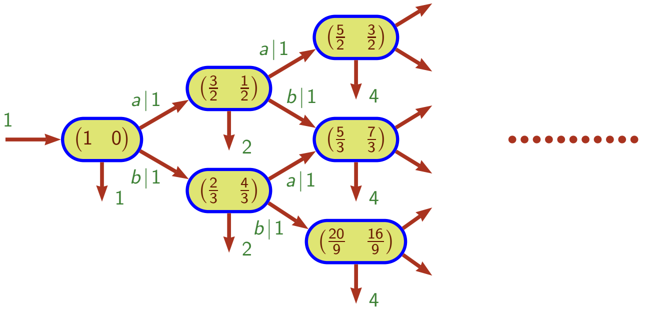
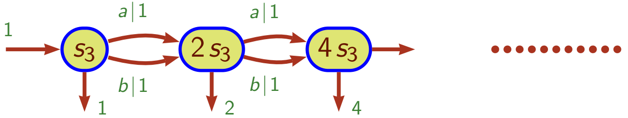


$\mathcal{A}_{s_3}$



Second step: the universal minimisation process

$$\begin{tikzcd} \mathcal{A}_3 \arrow[d, dotted, "\textit{determinisation}"] \\ \widehat{\mathcal{A}_3} \arrow[d, "(\mathbb{K})\textit{-quotient}"] \\ \mathcal{A}_{S_3} \end{tikzcd}$$

Third step: characterisation of sequentiality

Theorem (Schützenberger–Fliess–Jacob)

A series s *is recognisable iff* $\mathbf{Q}_s$ *is contained in a stable finitely generated submodule of* $\mathbb{K}\langle\langle A^* \rangle\rangle$

Third step: characterisation of sequentiality

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Definition

$\ell \subseteq \mathbb{K}\langle\langle A^* \rangle\rangle$ is a *line* if $\ell = \{ k\, r \mid k \in \mathbb{K} \}$ for a given $r \in \mathbb{K}\langle\langle A^* \rangle\rangle$

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Proposition

A series s is sequential iff $\mathbf{Q}_s$ is contained in a stable finite set of lines of $\mathbb{K}\langle\langle A^* \rangle\rangle$

Third step: characterisation of sequentiality

Further hypothesis

$\mathbb{K}$ admits a *greatest common divisor* operation (gcd)

Third step: characterisation of sequentiality

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Examples

- $\mathbb{K} = \mathbb{N}$ $\qquad$ $\text{gcd}\,(4, 6, 12) = 2$
- $\mathbb{K} = \mathbb{N}\text{min}$ $\qquad$ $\text{gcd}\,(4, 6, 12) = \text{min}\{4, 6, 12\} = 4$
- $\mathbb{K} = \mathbb{Z}\text{min}$, $\mathbb{K} = \mathbb{F}$ $\qquad$ need for a convention
- $\mathfrak{P}\,(B^*)$ $\qquad$ has no gcd but

  $\{B^* \cup \emptyset\}$ $\qquad$ has one: the longuest common prefix

Third step: characterisation of sequentiality

Further hypothesis

$\mathbb{K}$ admits a *greatest common divisor* operation (gcd)

Notation let $\mathbb{K}$ with gcd

- $\xi \in \mathbb{K}^Q$ $\mathring{\xi} \in \mathbb{K}$ $\mathring{\xi} = \mathrm{gcd}\left(\{\xi_q \mid q \in Q\}\right)$
- $s \in \mathbb{K}\langle\langle A^*\rangle\rangle$ $\mathring{s} \in \mathbb{K}$ $\mathring{s} = \mathrm{gcd}\left(\{\langle s, w\rangle \mid w \in A^*\}\right)$
- $\xi^\sharp \in \mathbb{K}^Q$ $\xi^\sharp = \left(\mathring{\xi}\right)^{-1} \xi$ *i.e.* $\xi = \mathring{\xi}\,\xi^\sharp$
- $s \in \mathbb{K}\langle\langle A^*\rangle\rangle$ $s^\sharp = \left(\mathring{s}\right)^{-1} s$ *i.e.* $s = \mathring{s}\,s^\sharp$

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Example

$$s_1 = 1_{A^*} + 2\,a + 4\,a^2 + 8\,a^3 + \cdots + 2^n a^n + \cdots$$

Third step: characterisation of sequentiality

Further hypothesis

$\mathbb{K}$ admits a *greatest common divisor* operation (gcd)

Notation let $\mathbb{K}$ with gcd

- $\xi \in \mathbb{K}^Q$ $\quad \mathring{\xi} \in \mathbb{K} \quad \mathring{\xi} = \operatorname{gcd}\left(\{\xi_q \mid q \in Q\}\right)$
- $s \in \mathbb{K}\langle\langle A^* \rangle\rangle$ $\quad \mathring{s} \in \mathbb{K} \quad \mathring{s} = \operatorname{gcd}\left(\{\langle s, w\rangle \mid w \in A^*\}\right)$
- $\xi^\sharp \in \mathbb{K}^Q$ $\quad \xi^\sharp = \left(\mathring{\xi}\right)^{-1} \xi \quad$ *i.e.* $\quad \xi = \mathring{\xi}\, \xi^\sharp$
- $s \in \mathbb{K}\langle\langle A^* \rangle\rangle$ $\quad s^\sharp = \left(\mathring{s}\right)^{-1} s \quad$ *i.e.* $\quad s = \mathring{s}\, s^\sharp$

Example

$$s_1 = 1_{A^*} + 2\,a + 4\,a^2 + 8\,a^3 + \cdots + 2^n a^n + \cdots$$

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Third step: characterisation of sequentiality

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Convention $\qquad \mathbb{K} = \mathbb{F}$, $\mathbb{Z}$min

- $\xi \in \mathbb{K}^Q$ $\qquad$ first entry of $\xi^\sharp = 1_{\mathbb{K}}$
- $s \in \mathbb{K}\langle\langle A^* \rangle\rangle$ $\qquad \langle \xi^\sharp, 1_{A^*} \rangle = 1_{\mathbb{K}}$

Third step: characterisation of sequentiality

Definition

$s \in \mathbb{K}\langle\langle A^* \rangle\rangle$, $u \in A^*$ $\quad$ $[u^{-1}s]^\sharp$ *translation* of s by u

$\mathbf{G}_s = \left\{ [u^{-1}s]^\sharp \;\middle|\; u \in A^* \right\}$ $\quad$ set of translations of s

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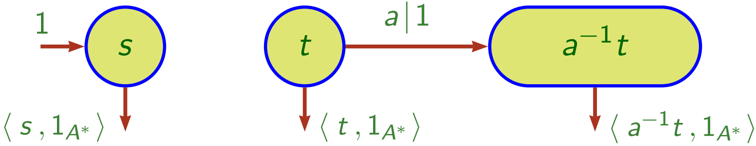

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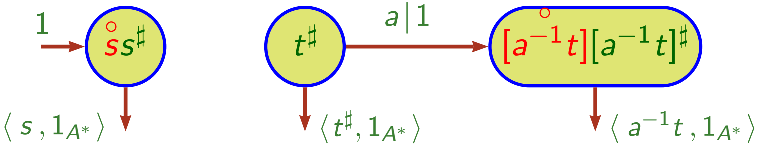

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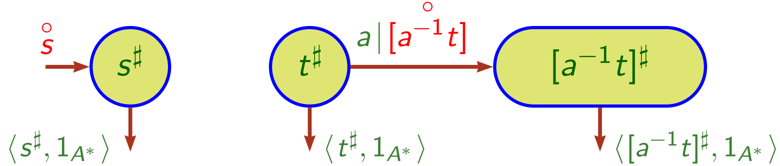

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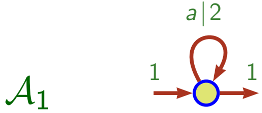
Theorem (Raney 58)

A series s is sequential iff $\mathbf{G}_s$ is finite

Third step: characterisation of sequentiality



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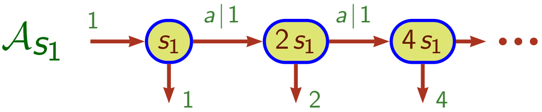

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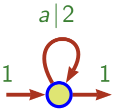

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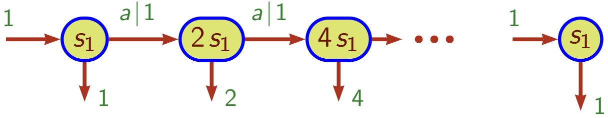

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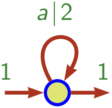

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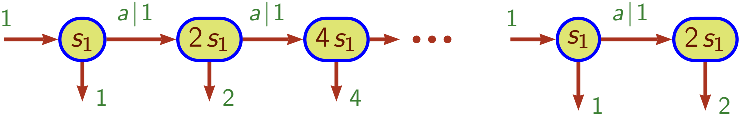

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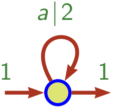

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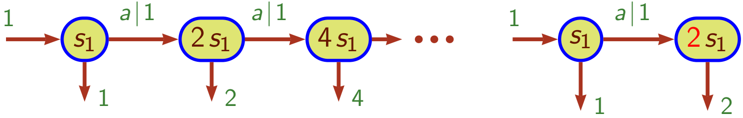

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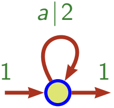

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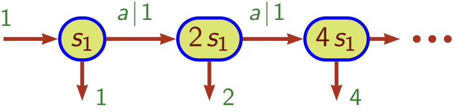

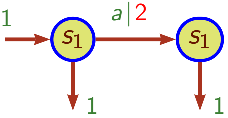

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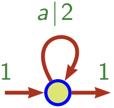

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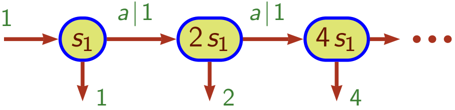

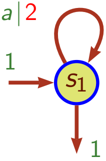

$\mathcal{D}_{s_1}$

Forth step: the sequentialisation algorithm

$$\mathcal{A} = (I, \mu, T) \qquad\qquad \mu\colon A^* \longrightarrow \mathbb{K}^{Q\times Q}$$

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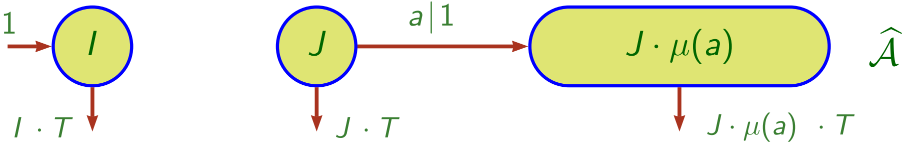

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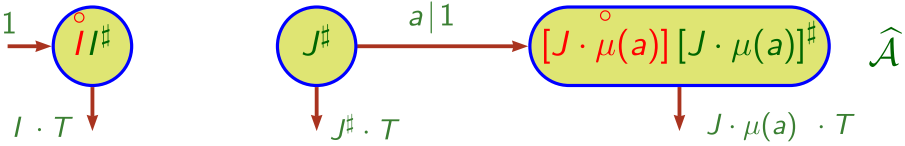

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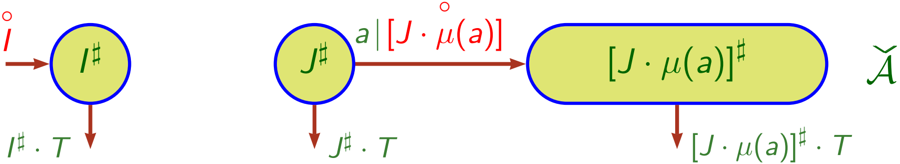

Forth step: the sequentialisation algorithm

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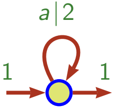

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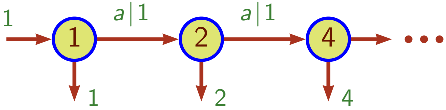
$\widehat{\mathcal{A}_1}$



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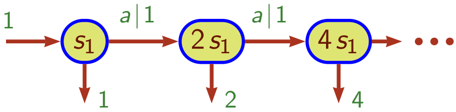

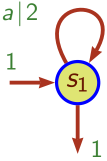

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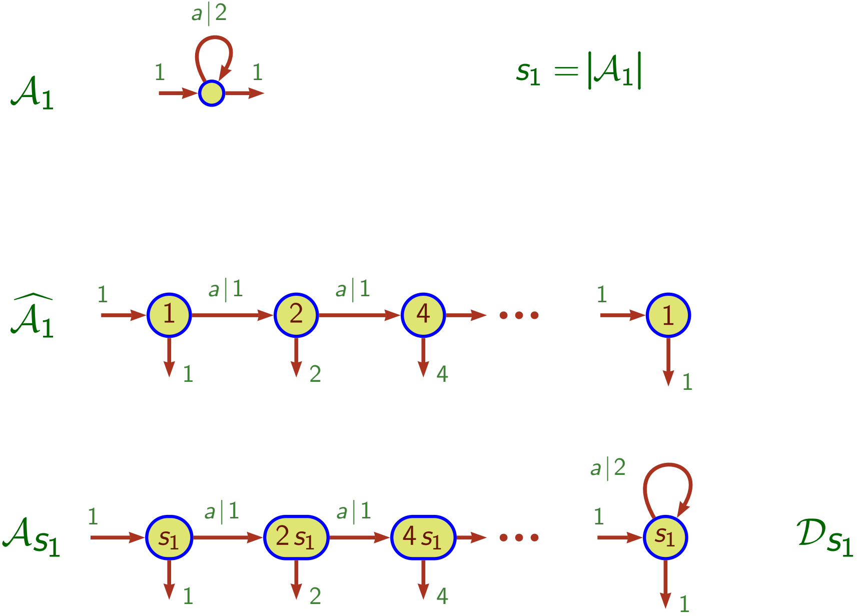

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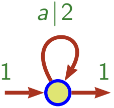

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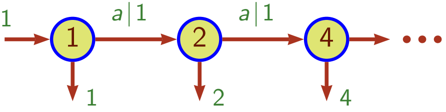

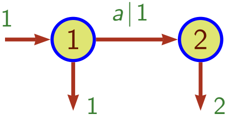

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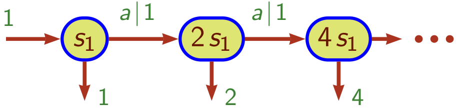

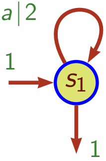

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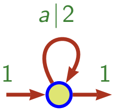

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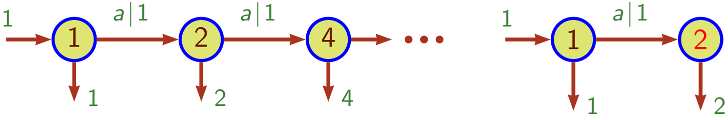

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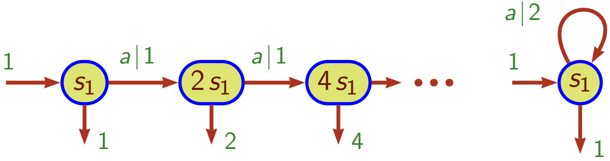

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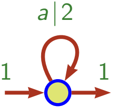

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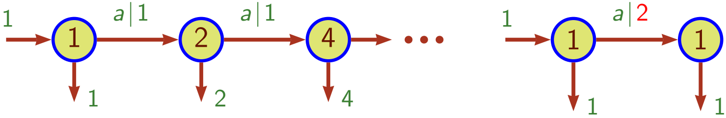

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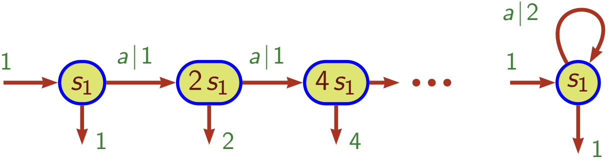

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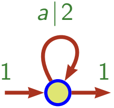

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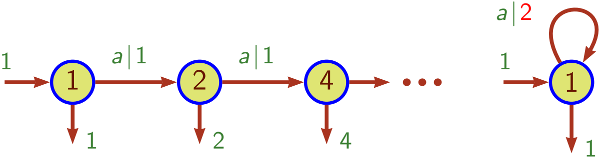
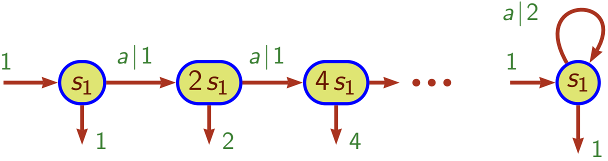


$\check{\mathcal{A}}_1$

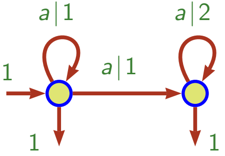
$\mathcal{A}_{s_1}$



$\mathcal{D}_{s_1}$

Forth step: the sequentialisation algorithm

$\mathcal{A}_2$



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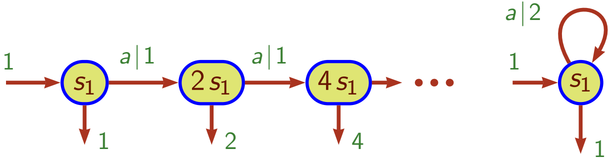
$\mathcal{A}_{s_1}$



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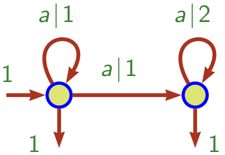
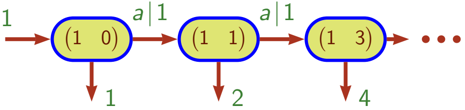
Forth step: the sequentialisation algorithm

$\mathcal{A}_2$

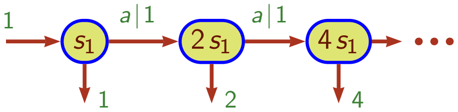


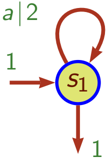
$$s_1 = |\mathcal{A}_1| = |\mathcal{A}_2|$$

$\widehat{\mathcal{A}_2}$



$\mathcal{A}_{s_1}$

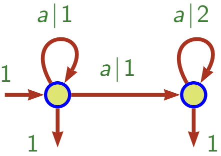




$\mathcal{D}_{s_1}$

Forth step: the sequentialisation algorithm

$\mathcal{A}_2$



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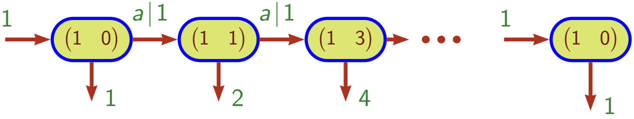

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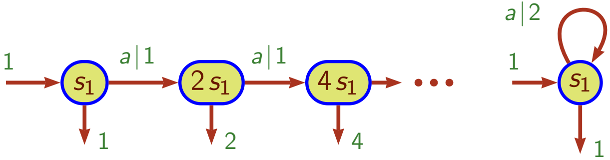

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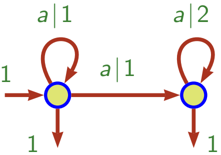

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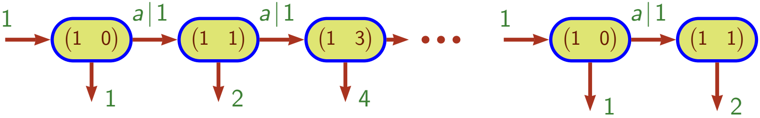

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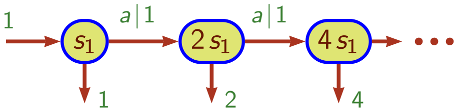

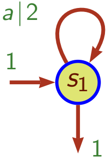

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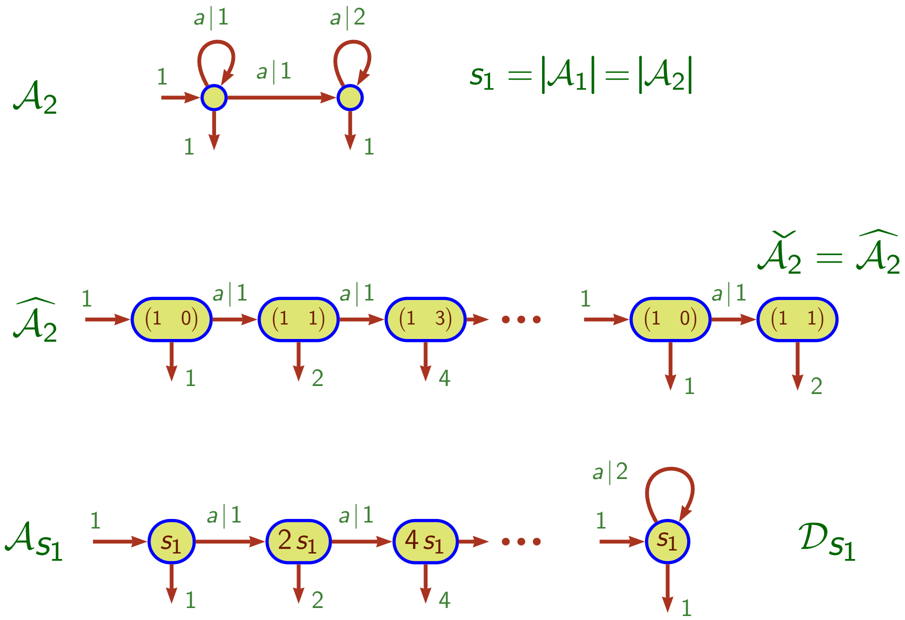
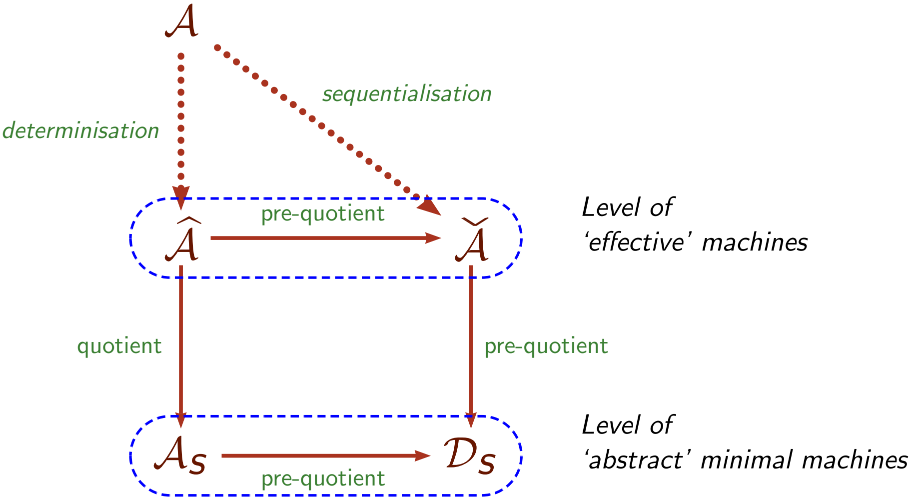


The global framework



The global framework

- The (trivial) finite case
- The field case
- The idempotent semiring case

Part III

The trivial finite case

The trivial finite case

$\mathcal{A} = (I, \mu, T) \qquad \mathbf{R}_{\mathcal{A}} = \{ I \cdot \mu(w) \mid w \in A^* \}$

$$\begin{tikzcd}
\mathcal{A} \arrow[d, dotted] \arrow[dr, dotted] & \\
\widehat{\mathcal{A}} \arrow[r, Rightarrow] \arrow[d] & \check{\mathcal{A}} \arrow[d, Rightarrow] \\
\mathcal{A}_S \arrow[r, Rightarrow] & \mathcal{D}_S
\end{tikzcd}$$

Proposition (?)

$\mathbb{K}$ *finite* $\Longrightarrow \widehat{\mathcal{A}}$ *finite*.

Example

$\mathbb{B}$, $\mathbb{Z}/n\mathbb{Z}$, $\mathbb{N}/[n = n + k]$

The trivial finite case

$$\mathcal{A} = (I, \mu, T) \qquad \mathbf{R}_{\mathcal{A}} = \{ I \cdot \mu(w) \mid w \in A^* \}$$

A semiring $\mathbb{K}$ is *locally finite*
if every finitely generated subsemiring is finite.

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Example

Fuzzy semirings: $\langle\, \mathbb{N}, \min, \max \,\rangle$, $\langle\, [0,1], \min, \max \,\rangle$

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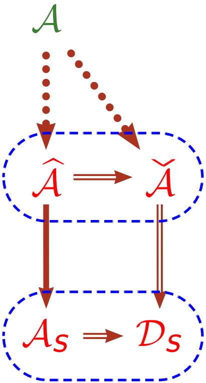

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Example

Fuzzy semirings: $\langle\, \mathbb{N}, \min, \max \,\rangle$, $\langle\, [0,1], \min, \max \,\rangle$

Counting in a locally finite semiring is not really counting.

Part IV

The field case

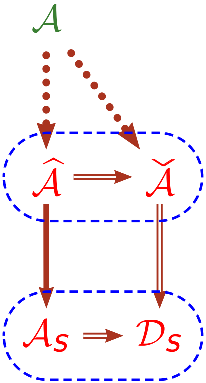
The field case

$\mathbb{K} = \mathbb{F}$ field

$\mathcal{A} = (I, \mu, T)$ $\quad \mathbf{R}_{\mathcal{A}} = \{ I \cdot \mu(w) \mid w \in A^* \}$

$s = |\mathcal{A}|$ $\quad \mathbf{Q}_s = \{ u^{-1}s \mid u \in A^* \}$

$r_s = \dim \langle \mathbf{Q}_s \rangle$ $\quad r_s$ *rank* of s



Theorem (Schützenberger 61)

The s is recognisable iff r_s is finite

The field case

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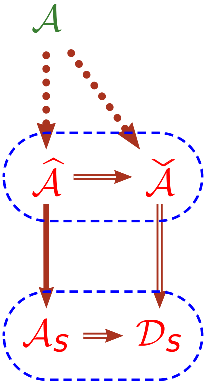

Theorem (Schützenberger 61)

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Definition

$\mathcal{A}$ is *reduced* if $\dim \langle \mathbf{R}_{\mathcal{A}} \rangle = r_s$

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A reduced representation of s is computable from any $\mathcal{A}$ realising s

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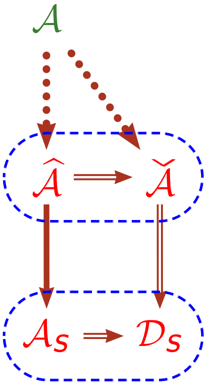

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Theorem (Schützenberger 61)

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Theorem (Reutenauer, L–S 06)

If $\mathcal{A}$ is reduced, then $\check{\mathcal{A}} = \mathcal{D}_s$

Part V

The idempotent semiring case

Idempotent semirings

Definition

$\mathbb{K}$ *idempotent* if $\quad k + k = k \quad \forall k \in \mathbb{K}$

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Example

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- Language semirings $\langle \mathfrak{P}(B^*), \cup, \cdot \rangle$, $\langle \mathrm{Rat}\, B^*, \cup, \cdot \rangle$, ...

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Theorem (Kleene–Schützenberger)

$$\mathrm{Rat}\,(A^* \times B^*) \quad \cong \quad [\mathrm{Rat}\, B^*]\mathrm{Rat}\, A^* \quad = \quad [\mathrm{Rat}\, B^*]\mathrm{Rec}\, A^*$$

A paradox

A paradox

Tropical automata and transducers are the

““ most sequentialised”” automata

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Tropical semirings, $\mathfrak{P}(B^*)$ are *very complex*,

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Theorem (from Post 36)

Equivalence of transducers is undecidable

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Equivalence of transducers is undecidable

Theorem (Krob 91)

Equivalence of tropical automata is undecidable

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$\mathfrak{P}(B^*)$ does not even have gcd !

An explanation ?

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First relief: $B^* \cup \{\emptyset\}$ has a gcd : the *longest common prefix*

Second relief:

Theorem (Schützenberger 75)

Functionality of transducers is decidable.

A quiproquo

A quiproquo

Consider for sequentialisation:

- the functional transducers
- the tropical automata

A quiproquo

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They look so similar!

A quiproquo

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A quiproquo

Consider for sequentialisation:

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What make them different? *1-valuedness*

Definition

$\mathcal{A}$ is *1-valued* if

every path labelled by a word w

has the same weight.

A quiproquo

Observation 1

Functional transducers are 1-valued, by definition

A quiproquo

Observation 1
Functional transducers are 1-valued, by definition

Observation 2
Tropical automata are not necessarily 1-valued

A quiproquo

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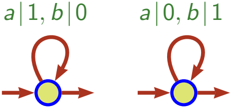

realises s_4 , $\langle s_4, w \rangle = \min\{|w|_a, |w|_b\}$

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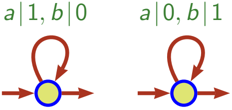


realises s_4 , $\langle s_4, w \rangle = \min\{|w|_a, |w|_b\}$

s_4 cannot be realised by a 1-valued automaton

Why is 1-valuedness so important ?

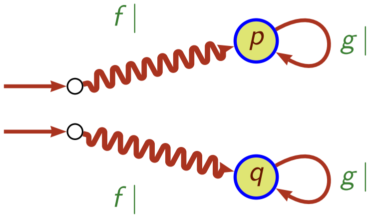
Theorem (Schützenberger 77)

Every 1-valued (finite) automaton is equivalent to an unambiguous (finite) automaton

The twinning property

The twinning property

Twin states



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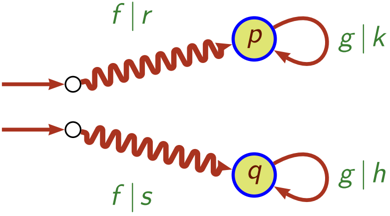
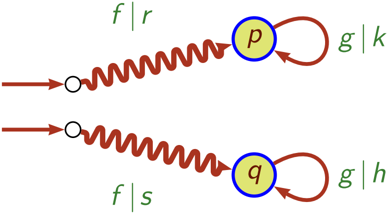


Congruent twin states

$$(r, s)^\sharp = (r\,k, s\,h)^\sharp$$

The twinning property

Twin states



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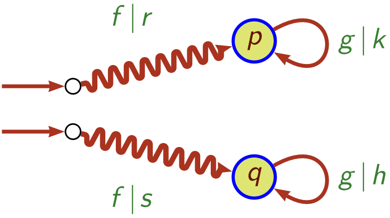

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$\mathcal{A}$ has the *twinning property* if all twin states are congruent

The twinning property

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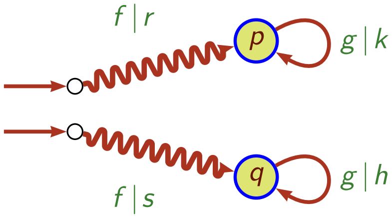

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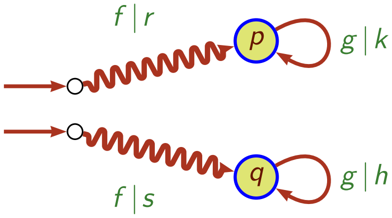
Theorem (Choffrut 77)

The twinning property is decidable.

The twinning property

Twin states



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Theorem (WK 95, BCPS 00, BCW 98, AM 03)

The twinning property is decidable in polynomial time.

Decision procedure

Proposition (Choffrut 77, Mohri 97)

$\mathcal{A}$ *has* *twinning p.* $\Longrightarrow$ $\check{\mathcal{A}}$ *finite*.

$$\begin{tikzcd}
 & \mathcal{A} \arrow[dl, dotted] \arrow[dr, dotted] & \\
\widehat{\mathcal{A}} \arrow[rr, Rightarrow] \arrow[d] & & \check{\mathcal{A}} \arrow[d, Rightarrow] \\
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Decision procedure

Proposition (Choffrut 77, Mohri 97)

$\mathcal{A}$ *has twinning p.* $\Longrightarrow \check{\mathcal{A}}$ *finite*.

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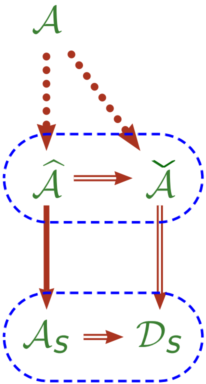
$\mathcal{A}$ *unambiguous and* $|\mathcal{A}|$ *sequential*

$\Longrightarrow \mathcal{A}$ *has twinning p.*



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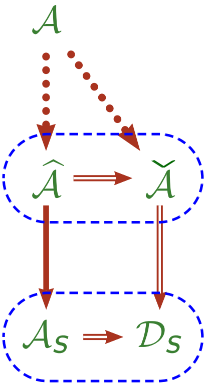

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Corollary

Sequentiality is decidable

for transducers and 1-valued tropical automata.

Beyond 1-valuedness

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Problem

Is sequentiality decidable for tropical recognisable series ?

Beyond 1-valuedness

Problem

Is sequentiality decidable for tropical recognisable series ?

Some answers in four special cases

Beyond 1-valuedness

Problem

Is sequentiality decidable for tropical recognisable series ?

Some answers in four special cases

1. Unary tropical series
2. Heap automata
3. Finitely ambiguous automata
4. Polynomialy ambiguous automata

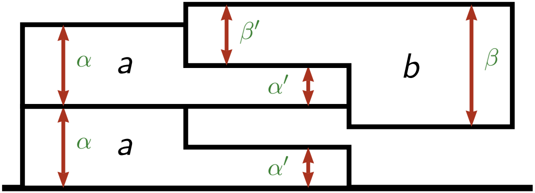
Unary tropical series

Unary tropical series

Theorem (Gaubert 94, Lombardy 01)

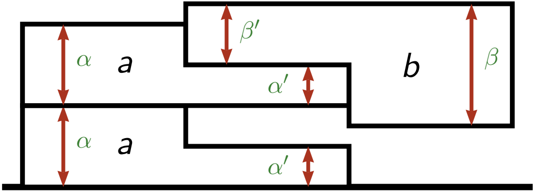
Sequentiality is decidable for tropical recognisable series

Heap automata



A heap model...

Heap automata



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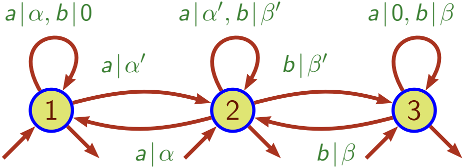

…and its heap automaton

Heap automata

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Heap automata

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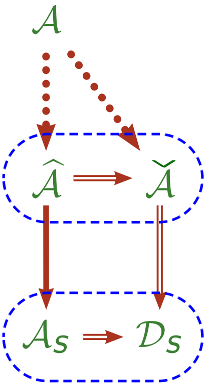

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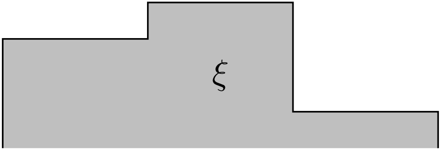

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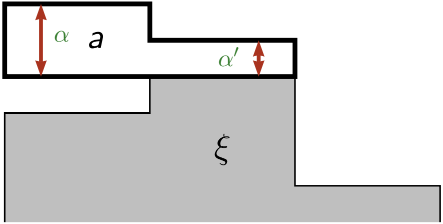

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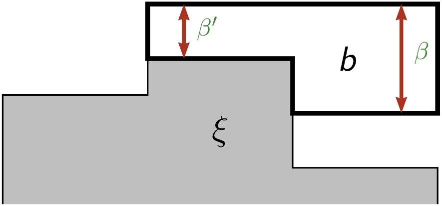

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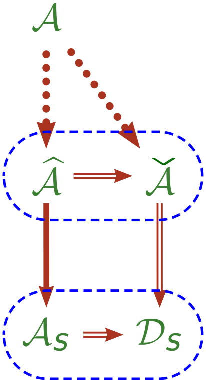

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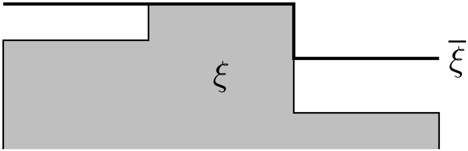

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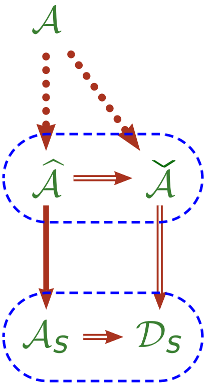

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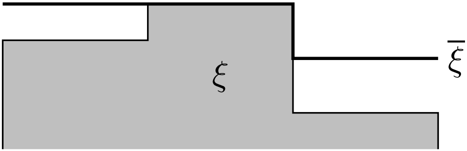

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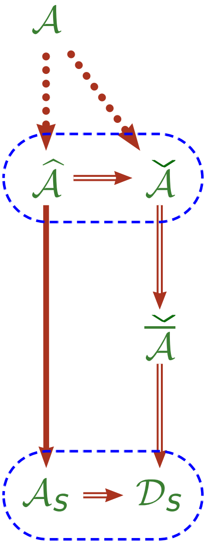

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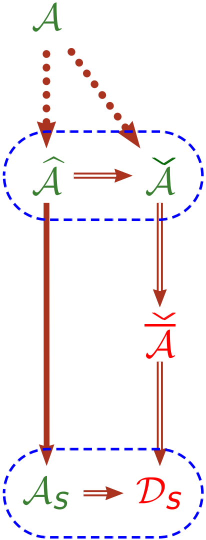

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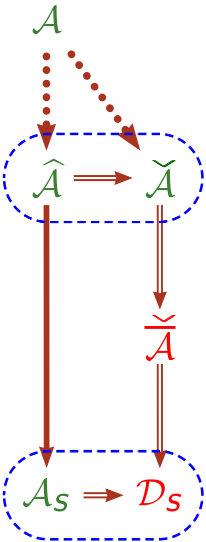

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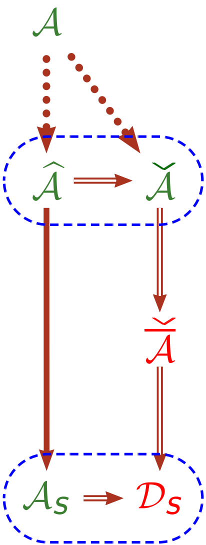

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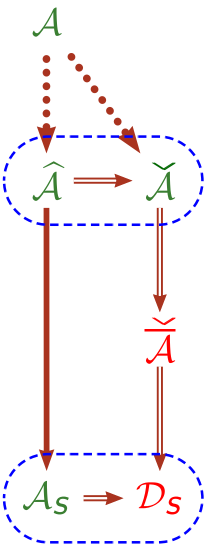

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Theorem (Mairesse and Vuillon 02)

[Besides trivial cases]

A two-letter heap automaton $\mathcal{A}$ *is sequentialisable*

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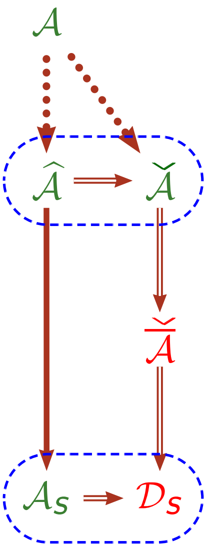

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$\mathcal{A}$ is *finitely ambiguous* if

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Proposition (Mandel Simon 77)

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Proposition (Hashiguchi Ishiguro Jimbo 02)

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Proposition (Krob 91)

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