Trees and languages with periodic signature

presented at LATIN 2016

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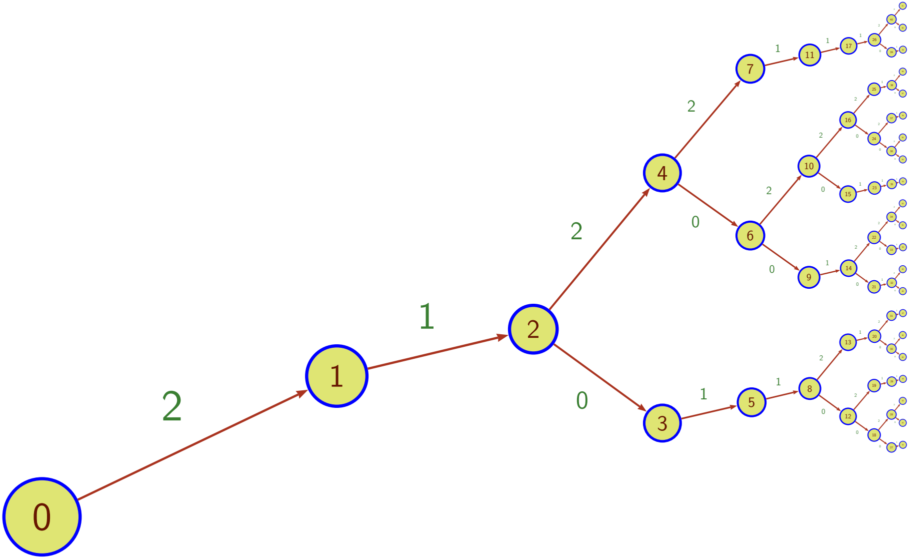
When order generates disorder

Part I

The $T_{\frac{p}{q}}$ *enigma*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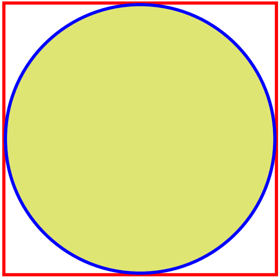
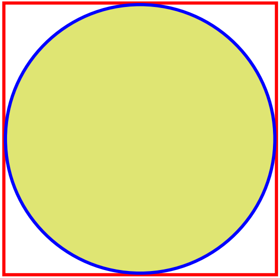
Numbers do exist



$$\frac{\pi}{4} = \frac{C}{P} = \frac{D}{S}$$

Numbers do exist



But you have to write them in order to compute with them

How are the representations in base 3 computed ?

$V = \{v_i = (3)^i \mid i \in \mathbb{N}\}$ together with $A = \{0, 1, 2\}$

Greedy algorithm $17 \in \mathbb{N}$ $3^{2+1} > 17 \geqslant 3^2$

$N_2 = 17$ $k = 2$

$N_1 = 17 - 1 \cdot 3^2 = 8$ $a_2 = 1 \in A\,,$ $3^2 > 8$

$N_0 = 8 - 2 \cdot 3^1 = 2 = a_0$ $a_1 = 2 \in A\,,$ $3^1 > 2$

$17 = 1 \cdot 3^2 + 2 \cdot 3^1 + 2 \cdot 3^0$ $\langle 17 \rangle_3 = 1\,2\,2$

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$$L_p = \{ \langle N \rangle_p \mid N \in \mathbb{N} \} = A^* \setminus 0\,A^*$$

The base 3 number system – another look

$V = \{v_i = (3)^i \mid i \in \mathbb{N}\}$ together with $A = \{0, 1, 2\}$

Division algorithm $17 \in \mathbb{N}$

$$N'_0 = 17$$

$$17 = N'_0 = 3 \cdot 5 + 2 \qquad a_0 = 2 \in A$$

$$5 = N'_1 = 3 \cdot 1 + 2 \qquad a_1 = 2 \in A$$

$$1 = N'_2 = 3 \cdot 0 + 1 \qquad a_2 = 1 \in A$$

$$17 = ((1) \cdot 3 + 2) \cdot 3 + 2 \qquad \langle 17 \rangle_3 = 1\,2\,2$$

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The base $\frac{3}{2}$ number system

$$U = \left\{ u_i = \frac{1}{2} \left(\frac{3}{2}\right)^i \;\middle|\; i \in \mathbb{N} \right\} \quad \text{together with} \quad A = \{0, 1, 2\}$$

Modified division algorithm $\quad N \in \mathbb{N}$

$$\begin{aligned} N_0 &= N \\ 2\,N_0 &= 3\,N_1 + a_0 & \qquad a_0 \in A \\ 2\,N_1 &= 3\,N_2 + a_1 & \qquad a_1 \in A \\ &\ldots \end{aligned}$$

$$N = \sum_0^k a_i \, \frac{1}{2} \left(\frac{3}{2}\right)^i \qquad\qquad \langle N \rangle_{\frac{3}{2}} = \; a_k\, a_{k-1} \ldots a_1\, a_0$$

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Modified division algorithm $\qquad 5 \in \mathbb{N}$

$$N_0 = 5$$

$$2\,N_0 = 2 \cdot 5 = 3 \cdot 3 + 1 \qquad 1 \in A$$

$$2\,N_1 = 2 \cdot 3 = 3 \cdot 2 + 0 \qquad 0 \in A$$

$$2\,N_2 = 2 \cdot 2 = 3 \cdot 1 + 1 \qquad 1 \in A$$

$$2\,N_3 = 2 \cdot 1 = 3 \cdot 0 + 2 \qquad 2 \in A$$

$$5 = \frac{1}{2}\left[\left(\left((2) \cdot \frac{3}{2} + 1\right) \cdot \frac{3}{2} + 0\right) \cdot \frac{3}{2} + 1\right] \qquad \langle 5 \rangle_{\frac{3}{2}} = 2101$$

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Theorem

Every N *in* $\mathbb{N}$ *has an integer representation in the* $\frac{3}{2}$*-system.*

It is the unique finite $\frac{3}{2}$*-representation of* N *.*

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We call this representation the $\frac{3}{2}$-expansion of N

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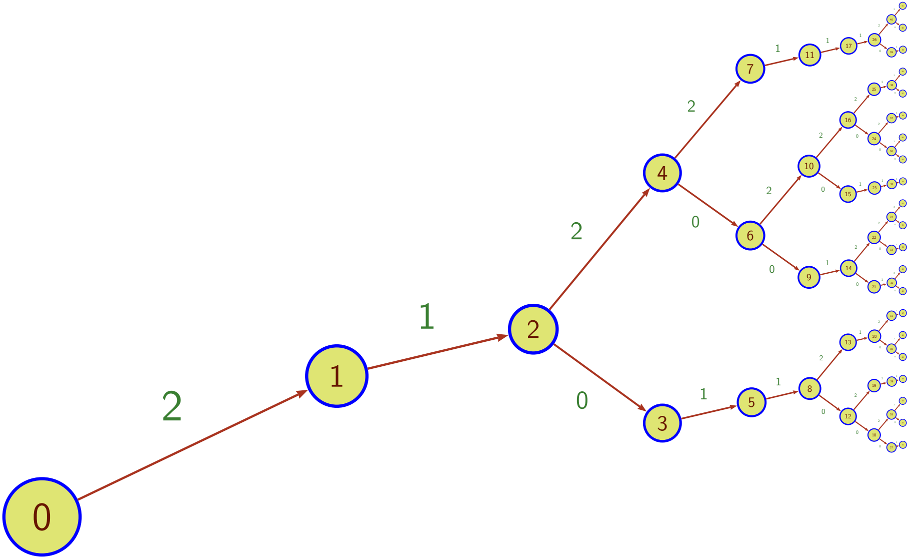
$$L_{\frac{3}{2}} \;=\; \{\langle N \rangle_{\frac{3}{2}} \;\big|\; N \in \mathbb{N}\} \;=\; ????$$



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The tree $T_{\frac{3}{2}}$ of the $\frac{3}{2}$-expansions

$L_{\frac{3}{2}}$ prefix-closed $\Longrightarrow$ $L_{\frac{3}{2}}$ spans the edges of a subtree $T_{\frac{3}{2}}$ of the full 3-ary tree.

The nodes of $T_{\frac{3}{2}}$ are labeled by the integers.

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Any two distinct subtrees of $T_{\frac{3}{2}}$ are not isomorphic.

The FLIP property

$$T \subseteq A^*$$

Definition

T has the Finite Left Iteration Property (FLIP) if

$$\forall u, v \in A^* \qquad \left\{ i \in \mathbb{N} \,\middle|\, u\,v^i \in \mathrm{Pre}(T) \right\} \qquad \text{is finite.}$$

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Proposition

$T_{\frac{p}{q}}$ *is a FLIP language.*

Digit conversion

D finite digit alphabet, that contains A .

$$\chi_D\colon D^* \to A^* \qquad \forall w \in D^* \qquad \pi(\chi_D(w)) = \pi(w) \ .$$

Proposition

For every D , χ_D *is realised by a letter-to letter sequential right transducer.*

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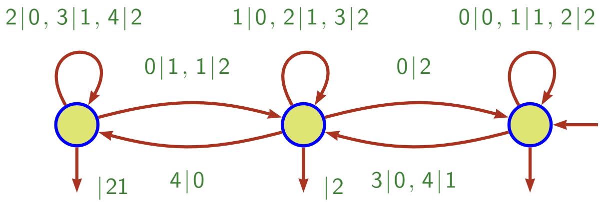
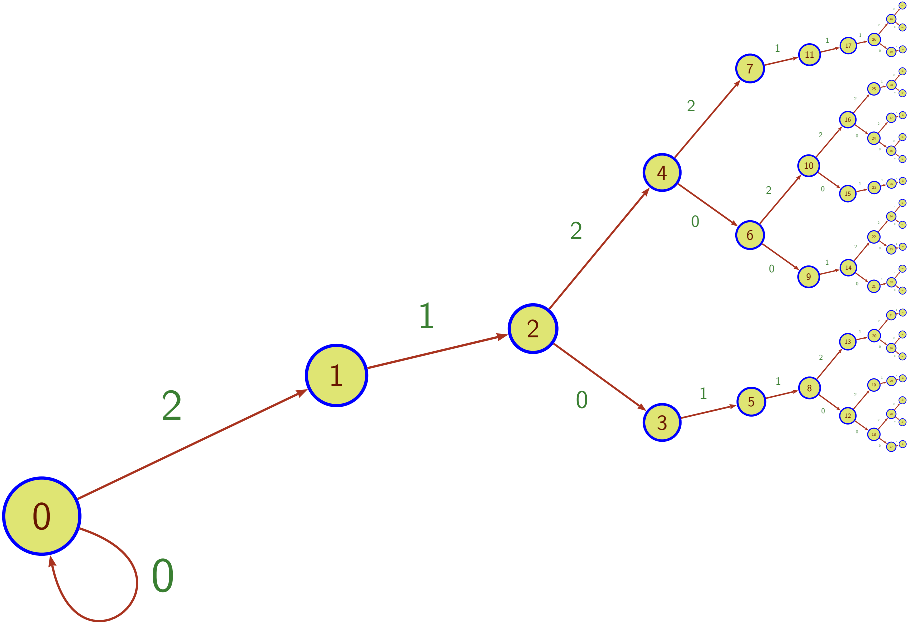


Some insight into the Mahler problem

Powers of rationals modulo 1 and rational base number systems

Israel J. Math., **168** (2008) 53–91.

Shigeki Akiyama, Christiane Frougny & Jacques Sakarovitch



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The $T_{\frac{p}{q}}$ are characterised by their *periodic signature*.

Part II

The signature of a tree

Signature of a tree

Definition

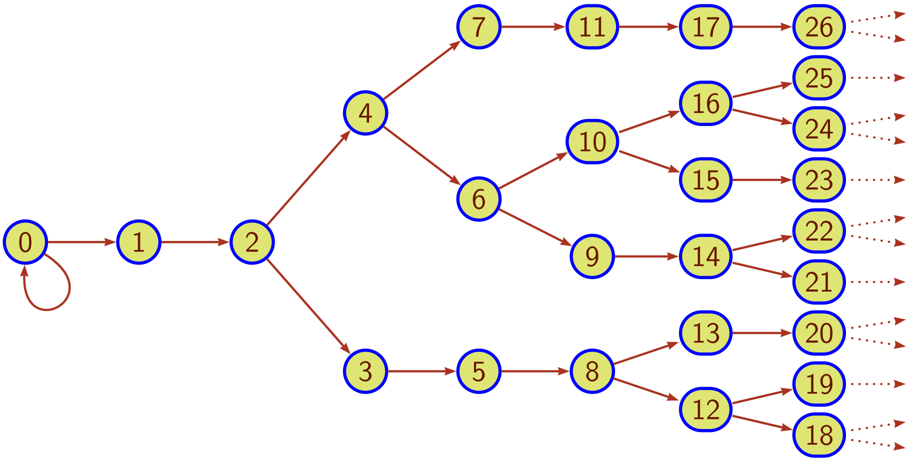
Signature of an ordered tree $\mathcal{T}$ =

sequence of the degrees of the nodes

in the breadth-first traversal of $\mathcal{T}$

Signature of a tree

Signature = sequence of the degrees



$\boldsymbol{s} =$

Signature of a tree

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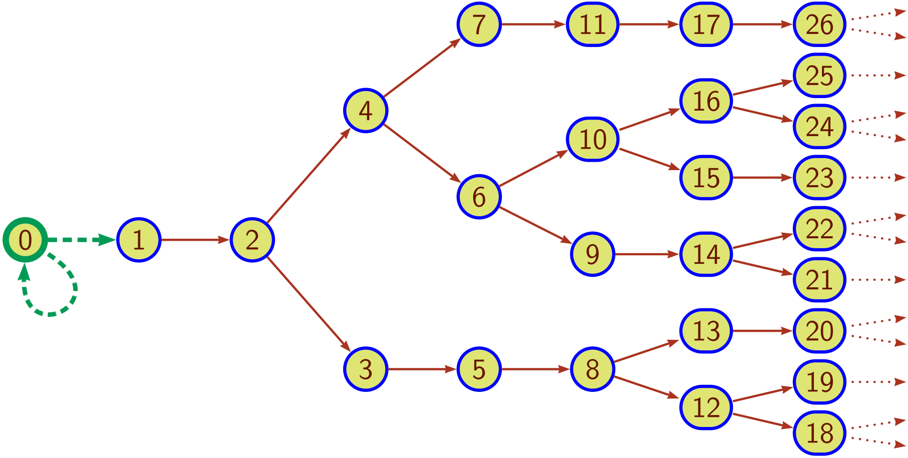
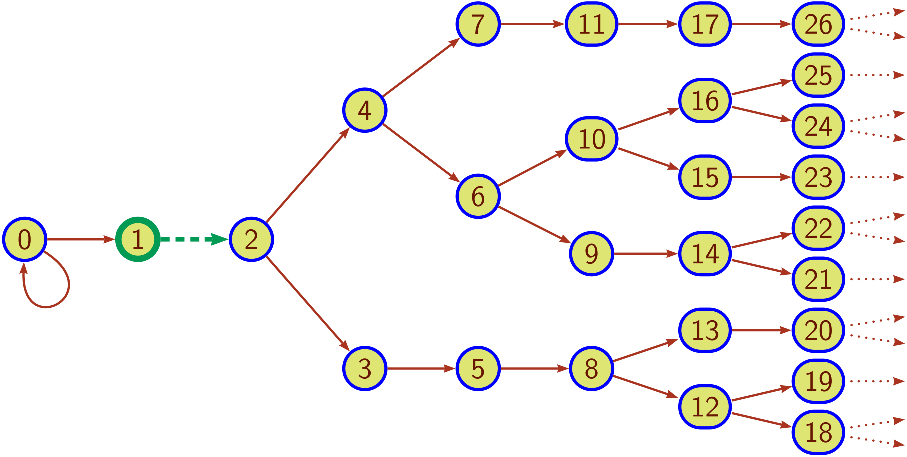


$s = 2$

Signature of a tree

Signature = sequence of the degrees



$s = 2\ 1$

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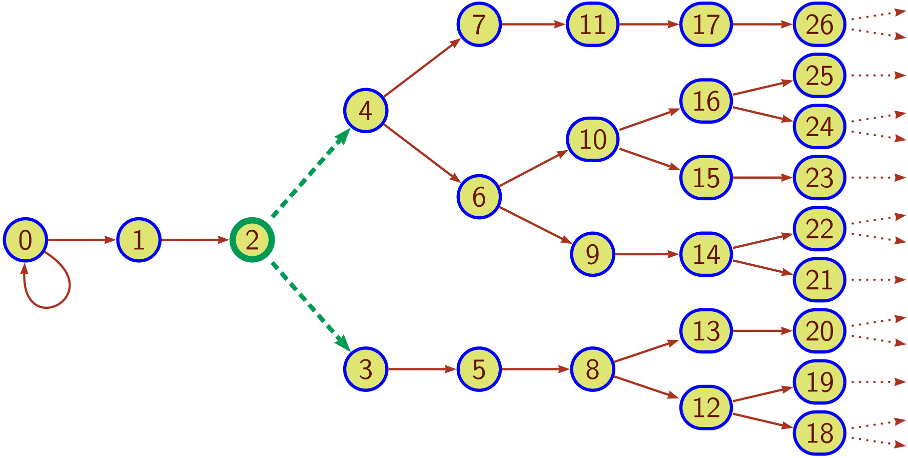

$s = 2\ 1\ 2$

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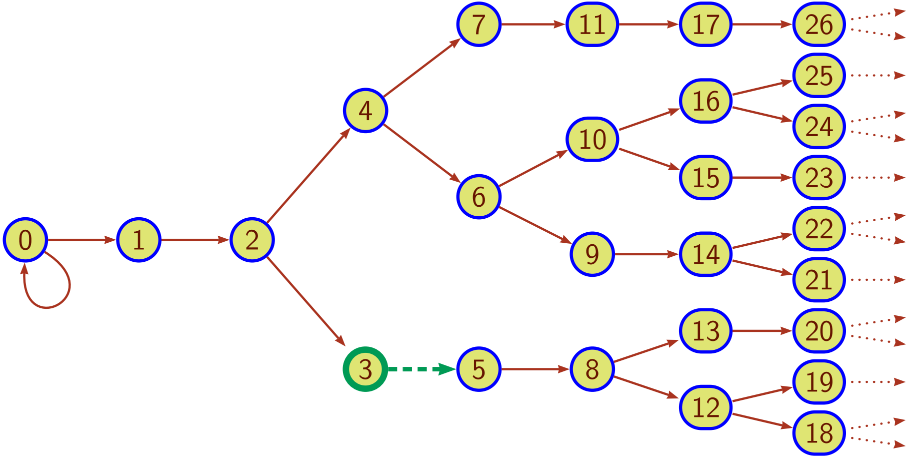

$s = 2\ 1\ 2\ 1$

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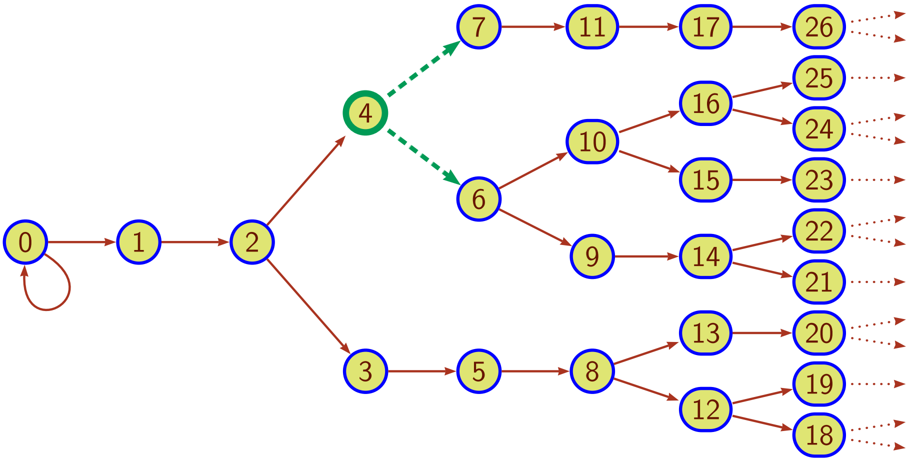
Signature = sequence of the degrees



$\boldsymbol{s} = \ 2 \ 1 \ 2 \ 1 \ 2$

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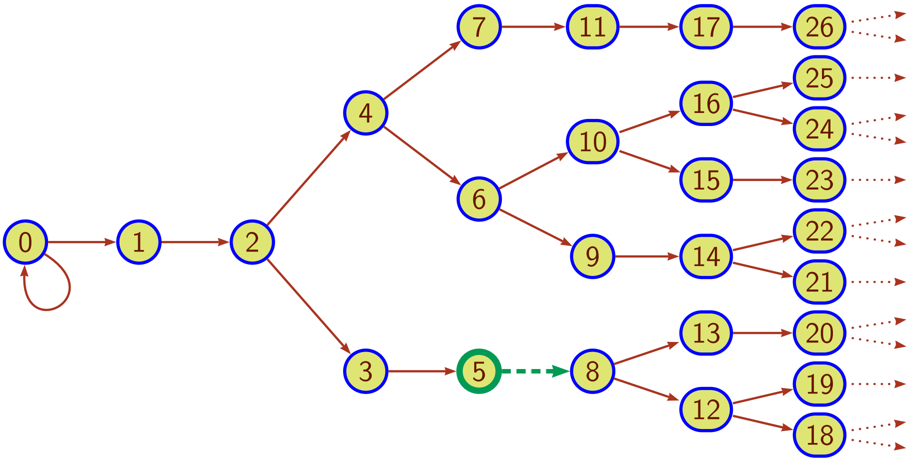

$$s = 2\ 1\ 2\ 1\ 2\ 1$$

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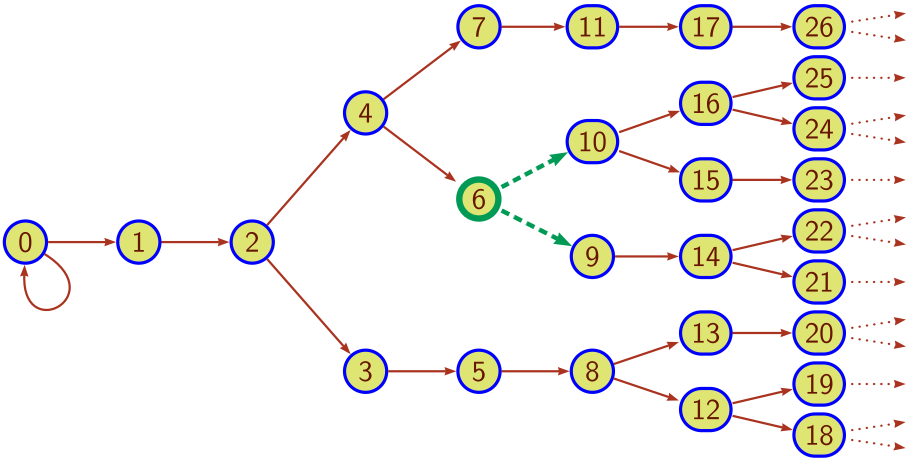

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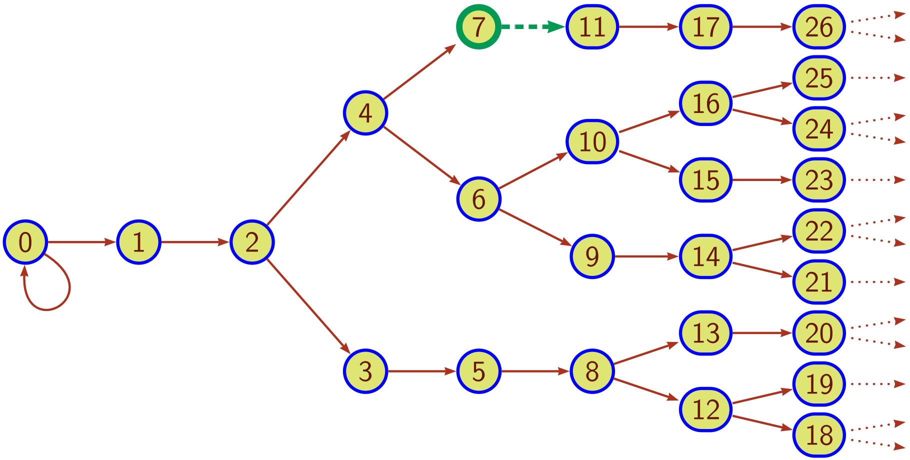

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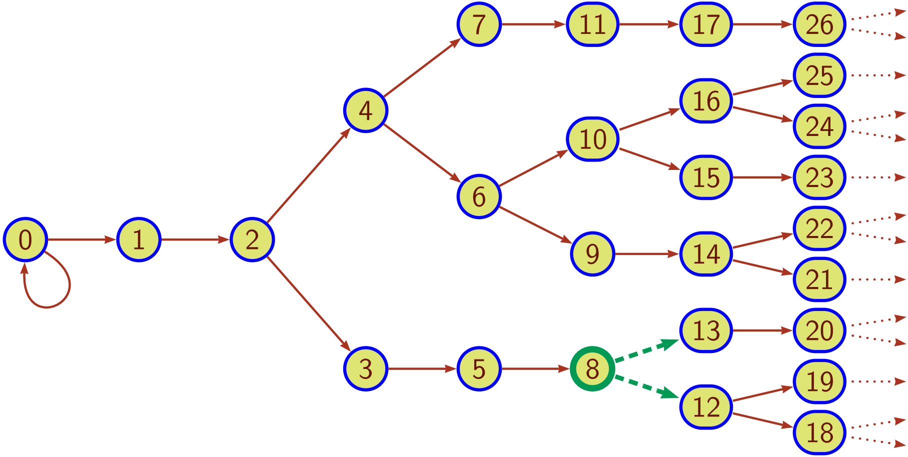

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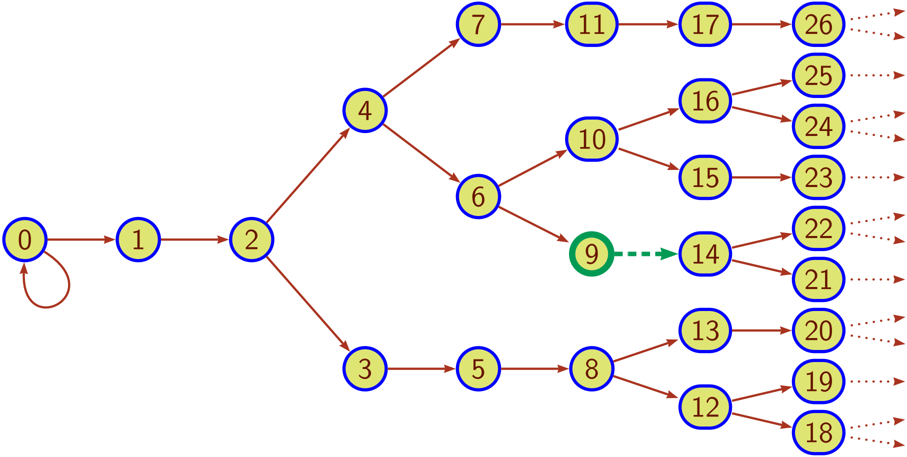

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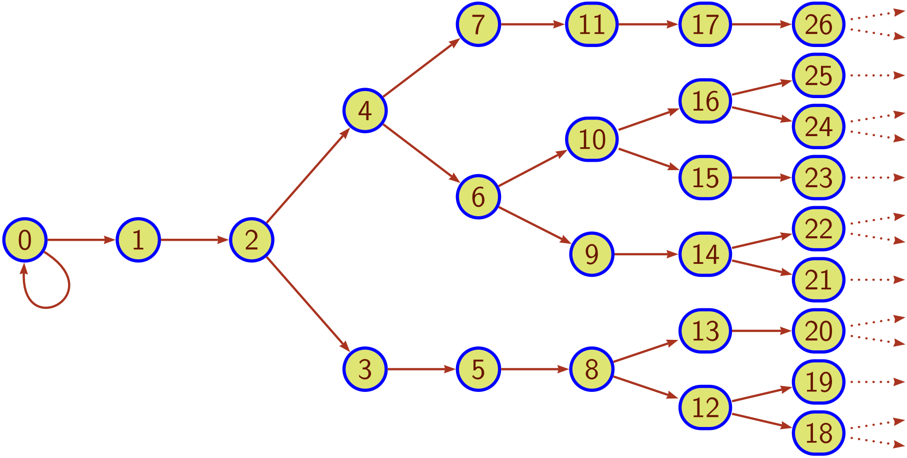


$$s = 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ \cdots$$

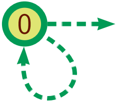
Tree from a signature

Signature = sequence of the degrees

$$\boldsymbol{s} = 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ \cdots$$

Tree from a signature

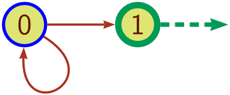
Signature = sequence of the degrees



$\boldsymbol{s} = {\color{red}2}\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ \cdots$

Tree from a signature

Signature = sequence of the degrees



$$\boldsymbol{s} = 2\ \color{red}{1}\ \color{black}2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ \cdots$$

Tree from a signature

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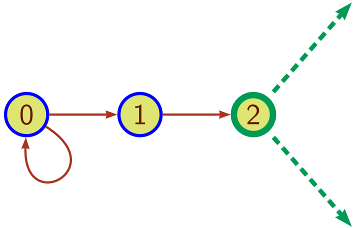

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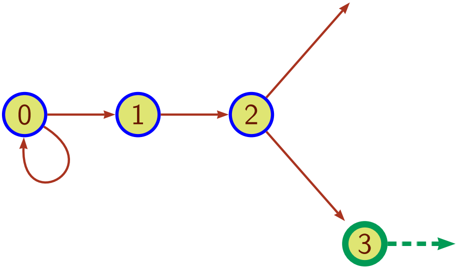

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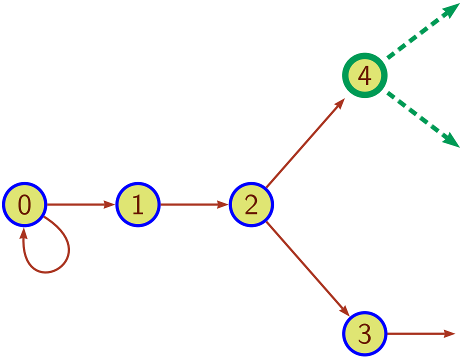

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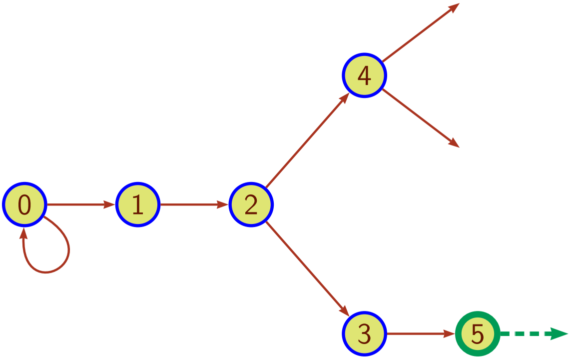

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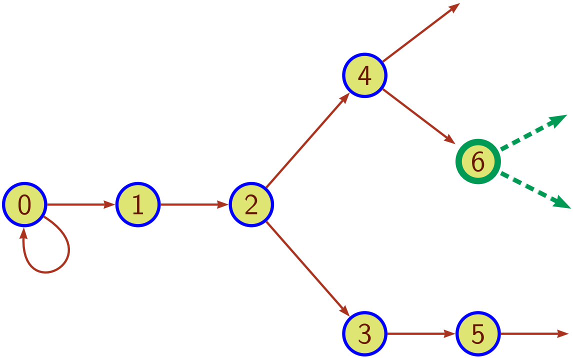
Signature = sequence of the degrees



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Tree from a signature

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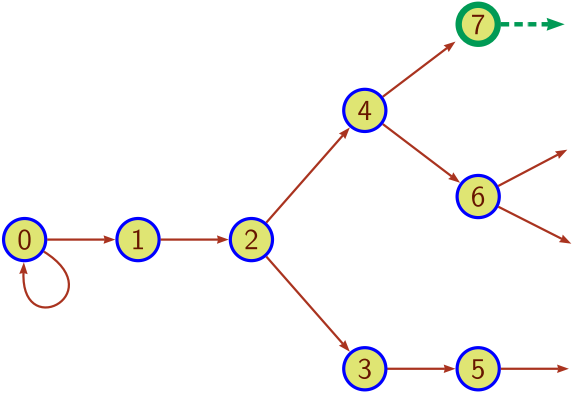

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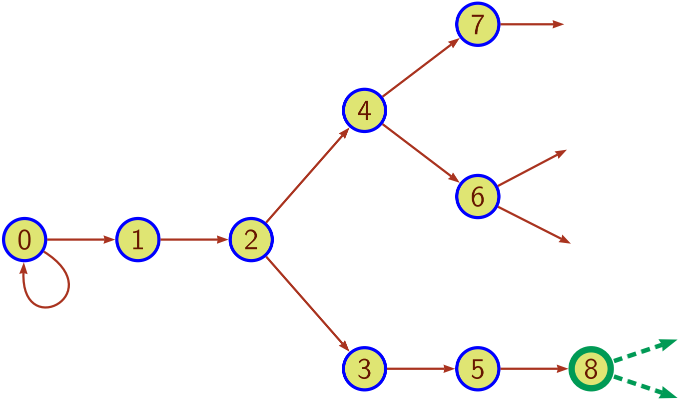

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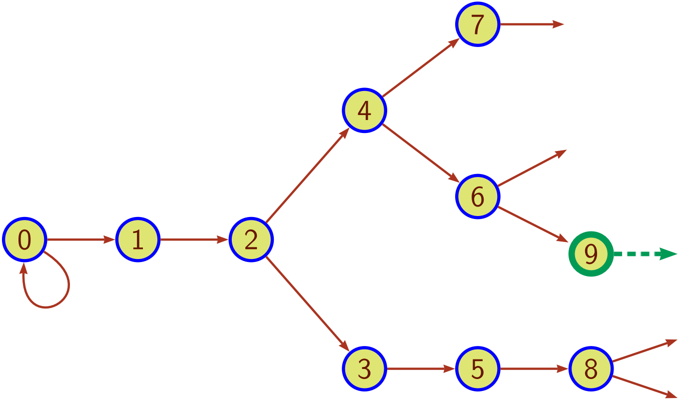

$\boldsymbol{s} =$ 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 $\cdots$

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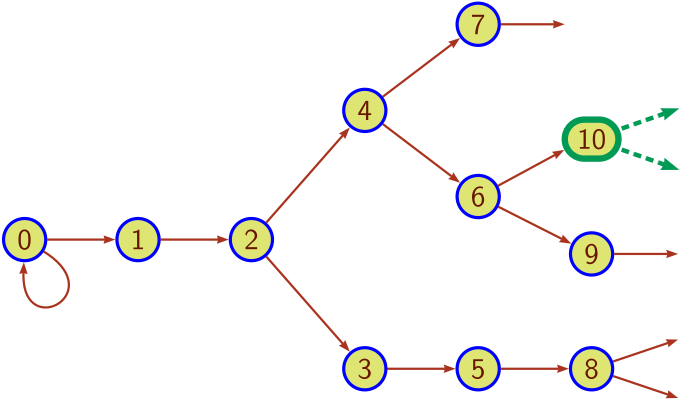

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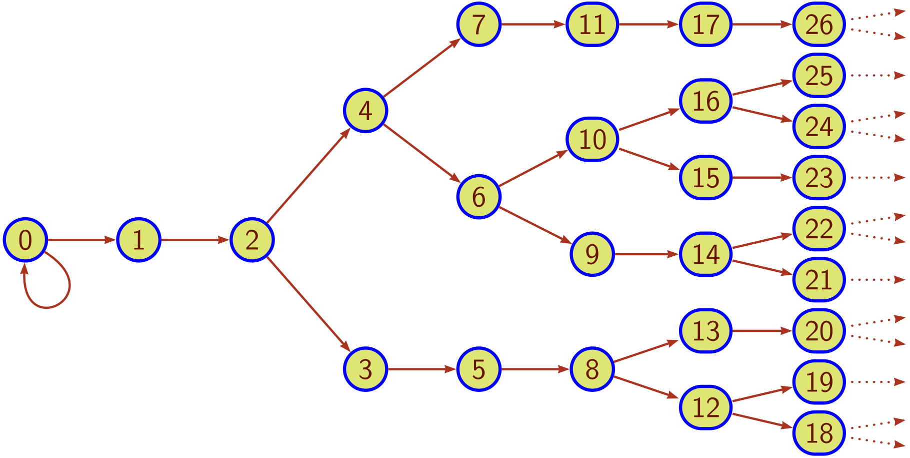


$$\boldsymbol{s} = 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ \cdots$$

Labelled signature of a labelled tree

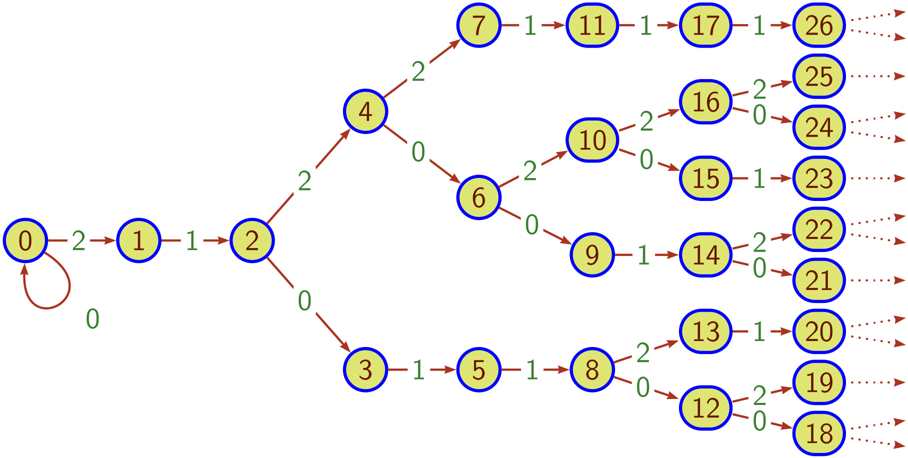
Arcs of $\mathcal{T}$ labelled in an ordered alphabet A

Definition

Labelled signature of an ordered tree $\mathcal{T}$ =
signature of $\mathcal{T}$ +
sequence of the labels of the arcs
in the breadth-first traversal of $\mathcal{T}$

labelled signature $(\boldsymbol{s}, \boldsymbol{\lambda})$

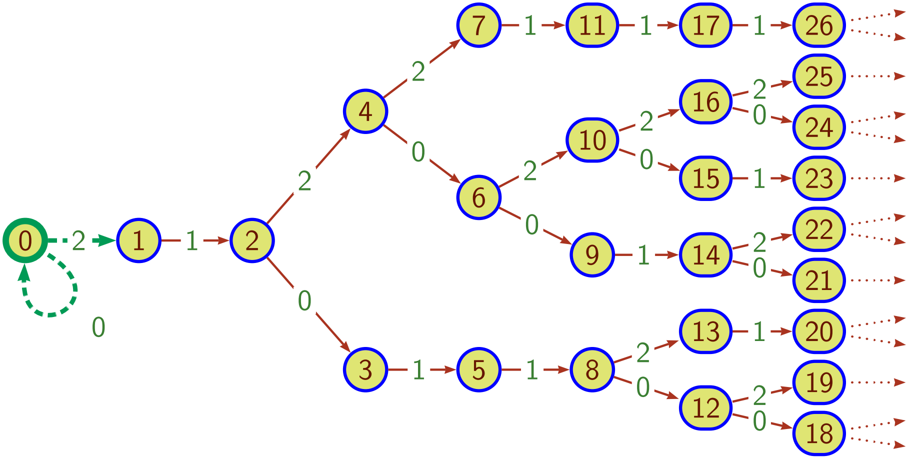
Labelled signature of a labelled tree



$s =$

$\lambda =$

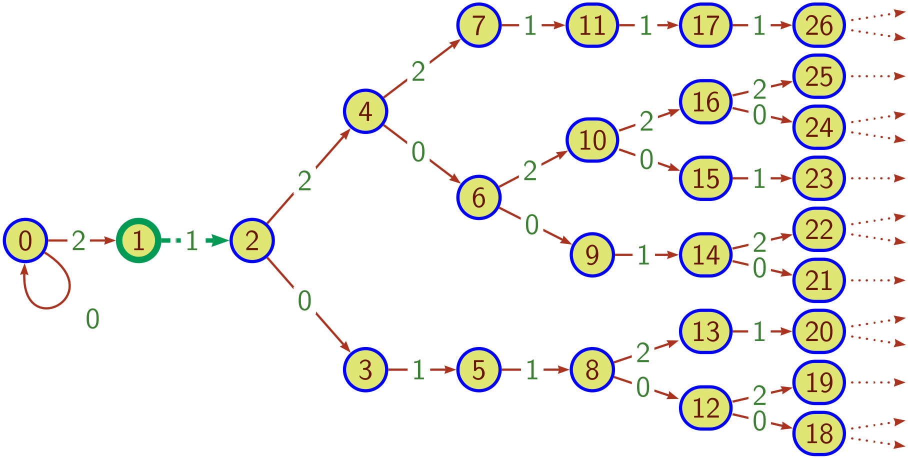
Labelled signature of a labelled tree



$\boldsymbol{s} = 2$

$\boldsymbol{\lambda} = 0\,2$

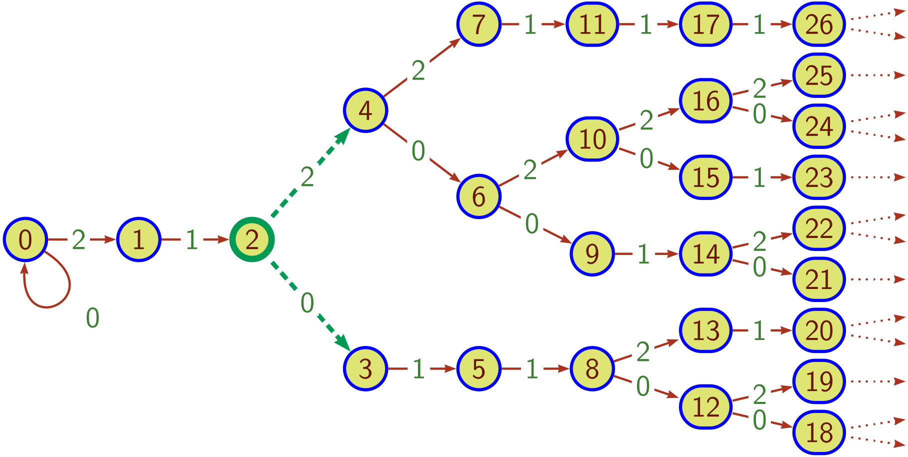
Labelled signature of a labelled tree



$\boldsymbol{s} = \ 2\ 1$

$\boldsymbol{\lambda} = 0\,2\,1$

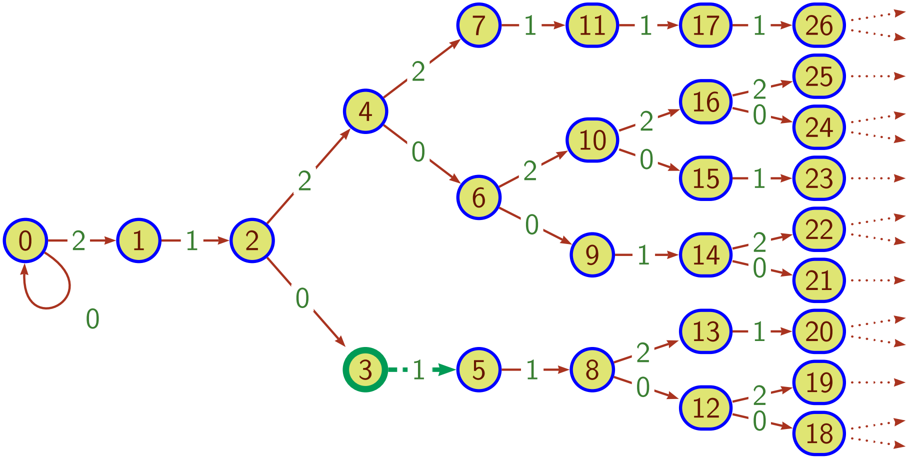
Labelled signature of a labelled tree



$\boldsymbol{s} = 2\ 1\ 2$

$\boldsymbol{\lambda} = 0\,2\,1\,0\,2$

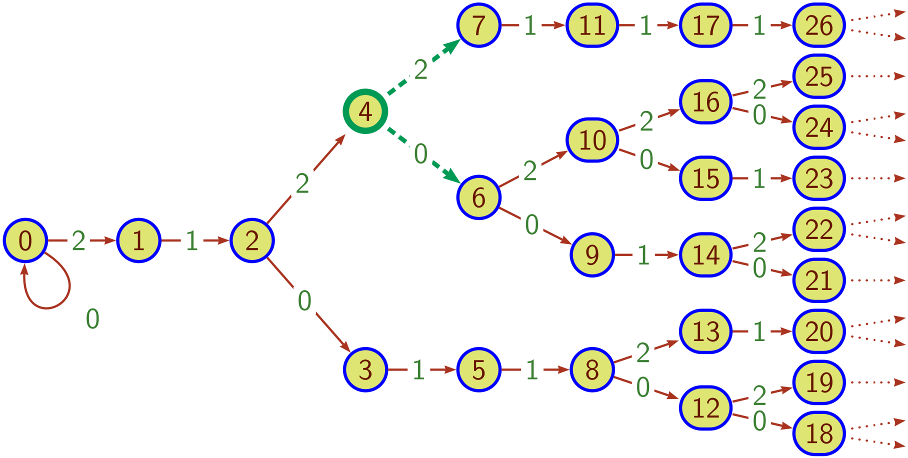
Labelled signature of a labelled tree



$\boldsymbol{s} = \ 2\ 1\ 2\ 1$

$\boldsymbol{\lambda} = 0\,2\,1\,0\,2\,1$

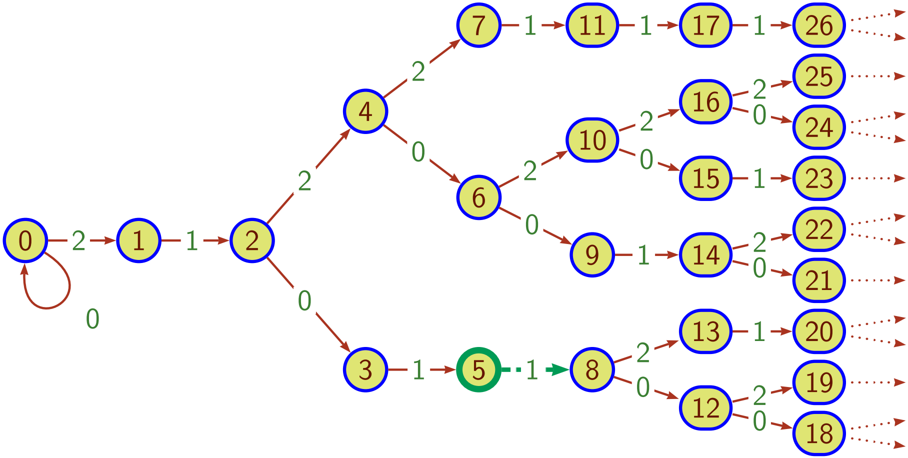
Labelled signature of a labelled tree



$$\boldsymbol{s} = \; 2 \;\; 1 \;\; 2 \;\; 1 \;\; 2$$

$$\boldsymbol{\lambda} = 0\,2\,1\,0\,2\,1\,0\,2$$

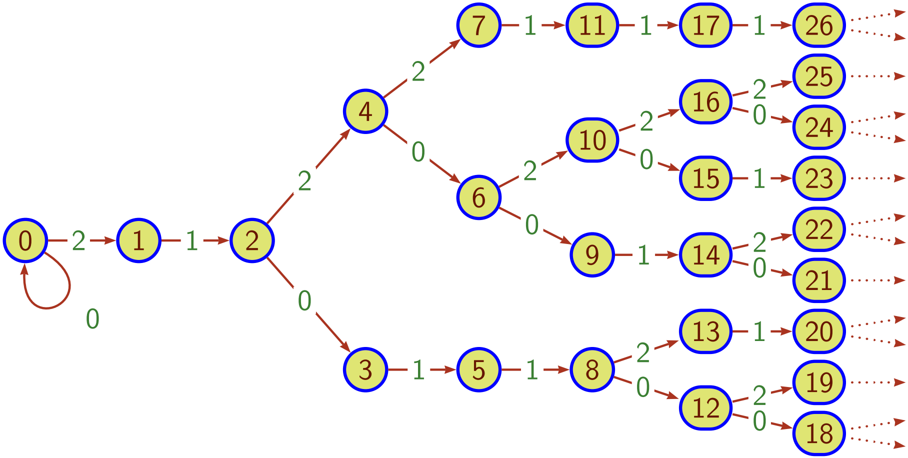
Labelled signature of a labelled tree



$\boldsymbol{s} = \ 2 \ 1 \ 2 \ 1 \ 2 \ 1$

$\boldsymbol{\lambda} = 0\,2\,1\,0\,2\,1\,0\,2\,1$

Labelled signature of a labelled tree



$$\boldsymbol{s} = 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ \cdots$$

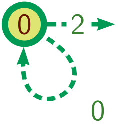
$$\boldsymbol{\lambda} = 0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\ \cdots$$

Labelled tree from a labelled signature

$$\boldsymbol{s} = \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; 2 \; 1 \; \cdots$$

$$\boldsymbol{\lambda} = 0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1 \; \cdots$$

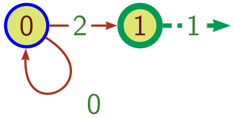
Labelled tree from a labelled signature



$\boldsymbol{s} = 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ \cdots$

$\boldsymbol{\lambda} = 0\,2\ 1\ 0\,2\ 1\ 0\,2\ 1\ 0\,2\ 1\ 0\,2\ 1\ 0\,2\ 1\ 0\,2\ 1\ 0\,2\ 1\ 0\,2\ 1\ \cdots$

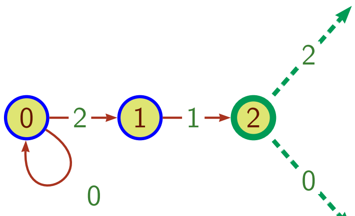
Labelled tree from a labelled signature



$s =$ 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 ⋯

$\lambda =$ 0 2 1 0 2 1 0 2 1 0 2 1 0 2 1 0 2 1 0 2 1 0 2 1 0 2 1 ⋯

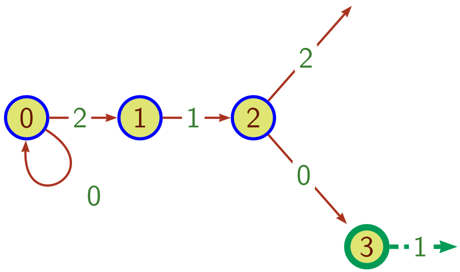
Labelled tree from a labelled signature



$$\boldsymbol{s} = 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ \cdots$$

$$\boldsymbol{\lambda} = 0\,2\ 1\ 0\,2\ 1\ 0\,2\ 1\ 0\,2\ 1\ 0\,2\ 1\ 0\,2\ 1\ 0\,2\ 1\ 0\,2\ 1\ 0\,2\ 1\ \cdots$$

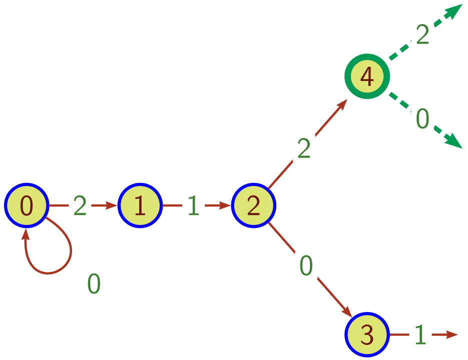
Labelled tree from a labelled signature



$\boldsymbol{s} = 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ \cdots$

$\boldsymbol{\lambda} = 0\,2\ 1\ 0\,2\ 1\ 0\,2\ 1\ 0\,2\ 1\ 0\,2\ 1\ 0\,2\ 1\ 0\,2\ 1\ 0\,2\ 1\ 0\,2\ 1\ \cdots$

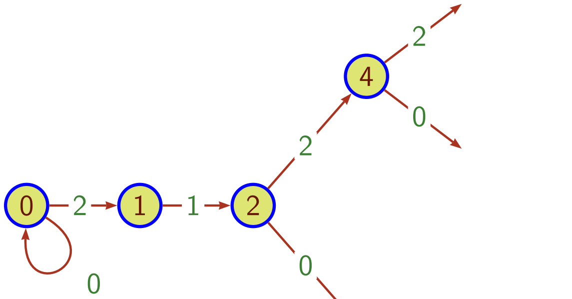
Labelled tree from a labelled signature



$\boldsymbol{s} =$ 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 ⋯

$\boldsymbol{\lambda} =$ 0 2 1 0 2 1 0 2 1 0 2 1 0 2 1 0 2 1 0 2 1 0 2 1 0 2 1 ⋯

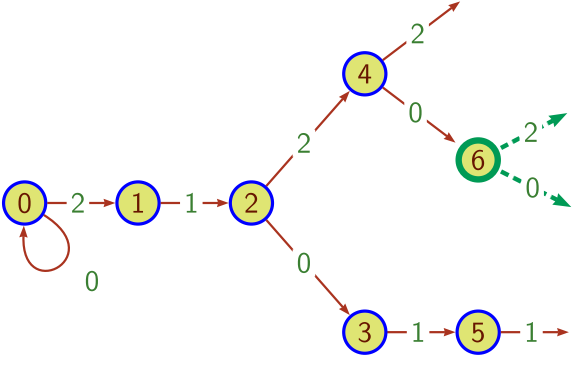
Labelled tree from a labelled signature



$$\boldsymbol{s} = \ 2\ 1\ 2\ 1\ 2\ \color{red}{1}\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ \cdots$$

$$\boldsymbol{\lambda} = 0\,2\,1\,0\,2\,1\,0\,2\,\color{red}{1}\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\,0\,2\,1\ \cdots$$

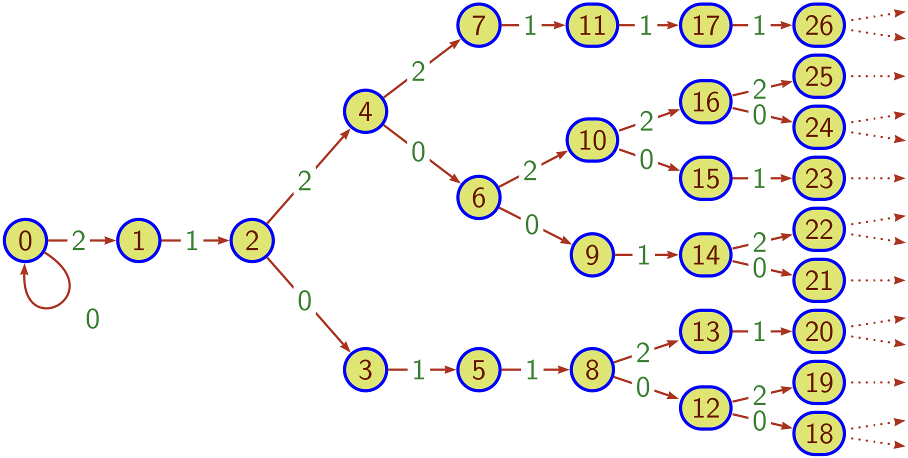
Labelled tree from a labelled signature



$$\boldsymbol{s} = \; 2\; 1\; 2\; 1\; 2\; 1\; 2\; 1\; 2\; 1\; 2\; 1\; 2\; 1\; 2\; 1\; 2\; 1\; \cdots$$

$$\boldsymbol{\lambda} = 0\,2\; 1\; 0\,2\; 1\; 0\,2\; 1\; 0\,2\; 1\; 0\,2\; 1\; 0\,2\; 1\; 0\,2\; 1\; 0\,2\; 1\; 0\,2\; 1\; \cdots$$

Labelled tree from a labelled signature



$$\boldsymbol{s} = \; 2\;1\;2\;1\;2\;1\;2\;1\;2\;1\;2\;1\;2\;1\;2\;1\;2\;1\;\cdots$$

$$\boldsymbol{\lambda} = 02\,1\,02\,1\,02\,1\,02\,1\,02\,1\,02\,1\,02\,1\,02\,1\,02\,1\;\cdots$$

Another example: the s-morphic signatures

$\sigma : A^* \to A^*$ morphism

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$$\sigma(a) = a\,b \qquad \sigma(b) = a$$

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$$\sigma : A^* \to A^* \text{ morphism}$$

$$\sigma(a) = a\,b \qquad \sigma(b) = a$$

$$\sigma^2(a) \;=\; a \;\; b \;\; a$$

Another example: the s-morphic signatures

$\sigma : A^* \to A^*$ morphism

$$\sigma(a) = a\,b \qquad \sigma(b) = a$$

$\sigma^3(a) \;=\; a \;\; b \;\; a \;\; a \;\; b$

Another example: the s-morphic signatures

$\sigma : A^* \to A^*$ morphism

$$\sigma(a) = a\,b \qquad \sigma(b) = a$$

$$\sigma^4(a) \;=\; a \;\; b \;\; a \;\; a \;\; b \;\; a \;\; b \;\; a$$

Another example: the s-morphic signatures

$\sigma : A^* \to A^*$ morphism

$$\sigma(a) = a\,b \qquad \sigma(b) = a$$

$$\sigma^5(a) \;=\; a \;\; b \;\; a \;\; a \;\; b \;\; a \;\; b \;\; a \;\; a \;\; b \;\; a \;\; a \;\; b$$

Another example: the s-morphic signatures

$\sigma : A^* \to A^*$ morphism

$$\sigma(a) = a\,b \qquad \sigma(b) = a$$

$$\sigma^6(a) = a\ b\ a\ a\ b\ a\ b\ a\ a\ b\ a\ a\ b\ a\ b\ a\ a\ b\ a\ b\ a$$

Another example: the s-morphic signatures

$\sigma : A^* \to A^*$ morphism

$$\sigma(a) = ab \qquad \sigma(b) = a$$

$$\sigma^\omega(a) \;=\; a\ b\ a\ a\ b\ a\ b\ a\ a\ b\ a\ a\ b\ a\ b\ a\ a\ b\ a\ b\ a \cdots$$

Another example: the s-morphic signatures

$$\sigma\colon A^* \to A^* \ \text{ morphism}$$

$$f_\sigma\colon A^* \to D^* \ \text{ morphism}$$

$$\sigma(a) = a\,b \qquad \sigma(b) = a$$

$$f_\sigma(a) = |\sigma(a)| = 2 \qquad f_\sigma(b) = |\sigma(b)| = 1$$

$$\sigma^\omega(a) \ = \ a \ b \ a \ a \ b \ a \ b \ a \ a \ b \ a \ a \ b \ a \ b \ a \ a \ b \ a \ b \ a \cdots$$

$$f_\sigma(\sigma^\omega(a)) = \ 2 \ 1 \ 2 \ 2 \ 1 \ 2 \ 1 \ 2 \ 2 \ 1 \ 2 \ 2 \ 1 \ 2 \ 1 \ 2 \ 2 \ 1 \ 2 \ 1 \ 2 \cdots$$

Another example: the s-morphic signatures

$$\sigma\colon A^* \to A^* \ \text{ morphism}$$

$$f_\sigma\colon A^* \to D^* \ \text{ morphism} \qquad g\colon A^* \to B^* \ \text{ morphism}$$

$$\sigma(a) = a\,b \qquad \sigma(b) = a$$

$$f_\sigma(a) = |\sigma(a)| = 2 \qquad f_\sigma(b) = |\sigma(b)| = 1$$

$$g(a) = 0\,1 \qquad g(b) = 0$$

$$\sigma^\omega(a) \;=\; a\ b\ a\ a\ b\ a\ b\ a\ a\ b\ a\ a\ b\ a\ b\ a\ a\ b\ a\ b\ a \cdots$$

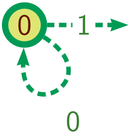
$$f_\sigma(\sigma^\omega(a)) = \;2\ 1\ 2\ 2\ 1\ 2\ 1\ 2\ 2\ 1\ 2\ 2\ 1\ 2\ 1\ 2\ 2\ 1\ 2\ 1\ 2 \cdots$$

$$g(\sigma^\omega(a)) = 0\,1\ 0\ 0\,1\,0\,1\ 0\ 0\,1\ 0\ 0\,1\,0\,1\ 0\ 0\,1\,0\,1\ 0\ 0\,1\ 0\ 0\,1\,0\,1\ 0\ 0\,1\ 0\ 0\,1\ \cdots$$

Another example: the s-morphic signatures

$$\boldsymbol{s} = 2\ 1\ 2\ 2\ 1\ 2\ 1\ 2\ 2\ 1\ 2\ 2\ 1\ 2\ 1\ \cdots$$
$$\boldsymbol{\lambda} = 01\ 0\ 0101\ 0\ 01\ 0\ 0101\ 0\ 0101\ 0\ 01\ 0\ \cdots$$

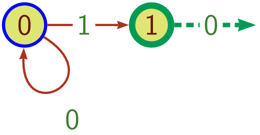
Another example: the s-morphic signatures



$$s = \ {\color{red}2} \ 1 \ 2 \ 2 \ 1 \ 2 \ 1 \ 2 \ 2 \ 1 \ 2 \ 2 \ 1 \ 2 \ 1 \cdots$$

$$\lambda = {\color{red}01} \ 0 \ 0101 \ 0 \ 01 \ 0 \ 0101 \ 0 \ 0101 \ 0 \ 01 \ 0 \ \cdots$$

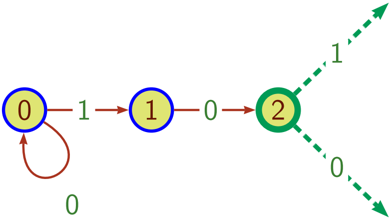
Another example: the s-morphic signatures



$$\boldsymbol{s} = \ 2 \ \ {\color{red}1} \ \ 2 \ \ 2 \ \ 1 \ \ 2 \ \ 1 \ \ 2 \ \ 2 \ \ 1 \ \ 2 \ \ 2 \ \ 1 \ \ 2 \ \ 1 \cdots$$

$$\boldsymbol{\lambda} = 01 \ {\color{red}0} \ 0101 \ 0 \ 01 \ 0 \ 0101 \ 0 \ 0101 \ 0 \ 01 \ 0 \ \cdots$$

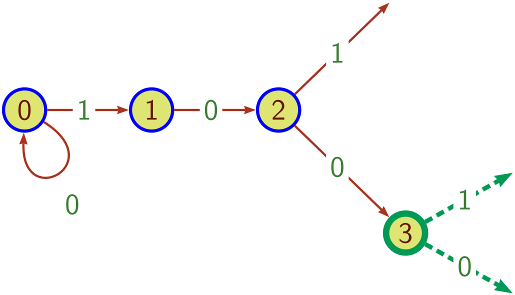
Another example: the s-morphic signatures



$$\boldsymbol{s} = \; 2 \;\; 1 \;\; \color{red}{2} \;\; 2 \;\; 1 \;\; 2 \;\; 1 \;\; 2 \;\; 2 \;\; 1 \;\; 2 \;\; 2 \;\; 1 \;\; 2 \;\; 1 \cdots$$

$$\boldsymbol{\lambda} = 01 \;\; 0 \;\; \color{red}{01} \, 01 \;\; 0 \;\; 01 \;\; 0 \;\; 0101 \;\; 0 \;\; 0101 \;\; 0 \;\; 01 \;\; 0 \;\; \cdots$$

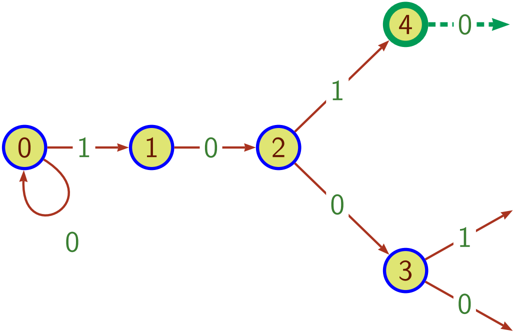
Another example: the s-morphic signatures



$\boldsymbol{s} = 2 \; 1 \; 2 \; {\color{red}2} \; 1 \; 2 \; 1 \; 2 \; 2 \; 1 \; 2 \; 2 \; 1 \; 2 \; 1 \cdots$

$\boldsymbol{\lambda} = 01 \; 0 \; 01{\color{red}01} \; 0 \; 01 \; 0 \; 0101 \; 0 \; 0101 \; 0 \; 01 \; 0 \cdots$

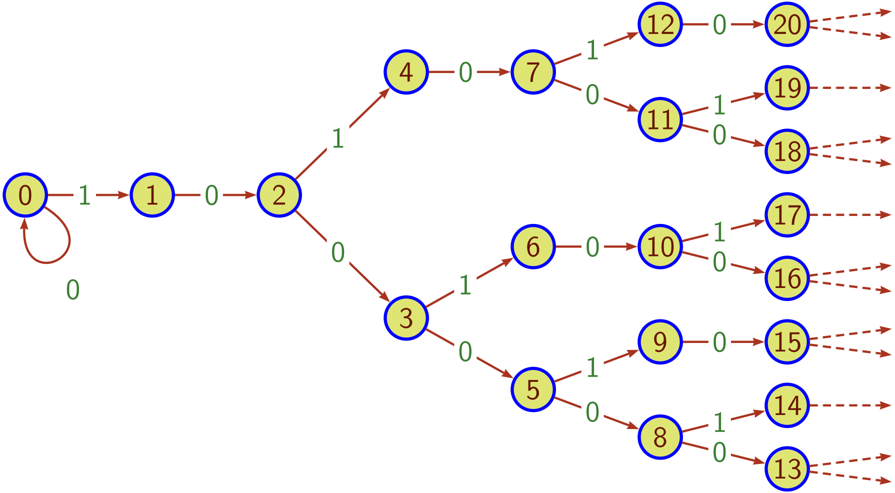
Another example: the s-morphic signatures



$$s = 2\ \ 1\ \ 2\ \ 2\ \ 1\ \ 2\ \ 1\ \ 2\ \ 2\ \ 1\ \ 2\ \ 2\ \ 1\ \ 2\ \ 1 \cdots$$

$$\lambda = 01\ \ 0\ \ 0101\ \ 0\ \ 01\ \ 0\ \ 0101\ \ 0\ \ 0101\ \ 0\ \ 01\ \ 0 \cdots$$

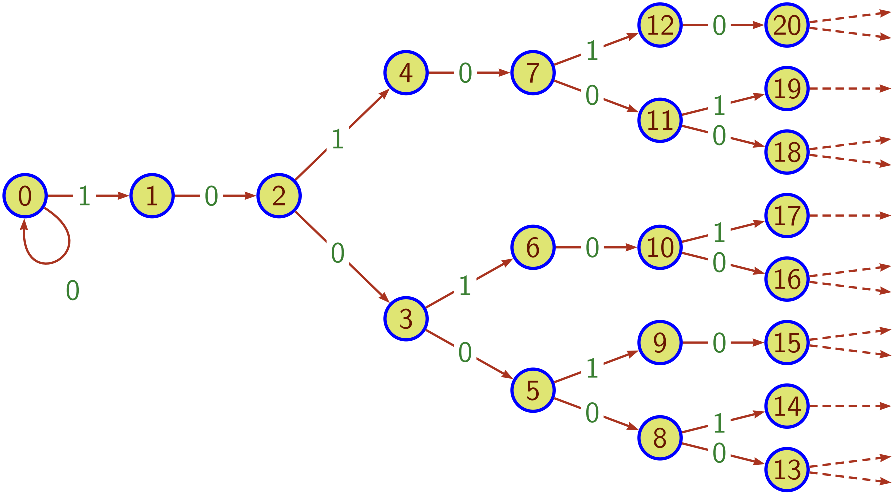
Another example: the s-morphic signatures



$$\boldsymbol{s} = \; 2 \;\; 1 \;\; 2 \;\; 2 \;\; 1 \;\; 2 \;\; 1 \;\; 2 \;\; 2 \;\; 1 \;\; 2 \;\; 2 \;\; 1 \;\; 2 \;\; 1 \cdots$$

$$\boldsymbol{\lambda} = 01 \;\; 0 \;\; 0101 \;\; 0 \;\; 01 \;\; 0 \;\; 0101 \;\; 0 \;\; 0101 \;\; 0 \;\; 01 \;\; 0 \cdots$$

Another example: the s-morphic signatures



$$T = \{0,1\}^* \setminus \{0,1\}^* 11 \{0,1\}^*$$

Another example: the s-morphic signatures

Theorem (Cobham 72, Rigo–Maes 02, M.–S. 14)

A prefix-closed language is regular iff its labelled signature is s-morphic.

Part III

The signature of $T_{\frac{p}{q}}$ *is periodic*

Signature of $T_{\frac{p}{q}}$

p, q coprime integers $p > q \geqslant 1$

Signature of $T_{\frac{p}{q}}$

p, q coprime integers $p > q \geqslant 1$

Theorem

The (labelled) signature of $T_{\frac{p}{q}}$ is purely periodic.

Rhythm

p, q coprime integers $p > q \geqslant 1$

A purely periodic signature

$$\boldsymbol{s} = \mathbf{r}^{\omega}$$

Rhythm

p, q coprime integers $p > q \geqslant 1$

A purely periodic signature

$$\boldsymbol{s} = \mathbf{r}^{\omega}$$

Definition

$\mathbf{r}$ rhythm of directing parameter (q, p)

$$\mathbf{r} = (r_0, r_1, \ldots, r_{q-1}) \qquad \sum_{i=0}^{q-1} r_i = p$$

Rhythm

p, q coprime integers $p > q \geqslant 1$

A purely periodic signature

$$\boldsymbol{s} \;=\; \mathbf{r}^{\omega}$$

Definition

$\mathbf{r}$ rhythm of directing parameter (q, p)

$$\mathbf{r} \;=\; (r_0, r_1, \ldots, r_{q-1}) \qquad\qquad \sum_{i=0}^{q-1} r_i \;=\; p$$

Example

Rhythms of dir. par. $(3, 5)$: $(3, 1, 1)$ $(2, 2, 1)$ $(1, 2, 2)$

Rhythm

p, q coprime integers $p > q \geqslant 1$

Geometric representation

$\mathbf{r} = (r_0, r_1, \ldots, r_{q-1})$

$$\texttt{path}(\mathbf{r}) = y^{r_0} x\, y^{r_1} x\, y^{r_2} \cdots x\, y^{r_{q-1}} x$$

Rhythm

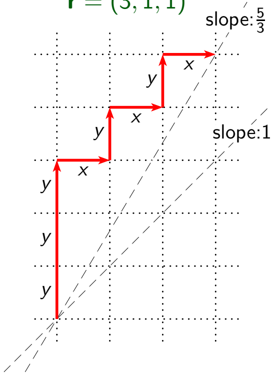
p, q coprime integers $p > q \geqslant 1$

Geometric representation

$\mathbf{r} = (r_0, r_1, \ldots, r_{q-1})$

$$\mathtt{path}(\mathbf{r}) = y^{r_0} x\, y^{r_1} x\, y^{r_2} \cdots x\, y^{r_{q-1}} x$$

$\mathbf{r} = (3, 1, 1)$



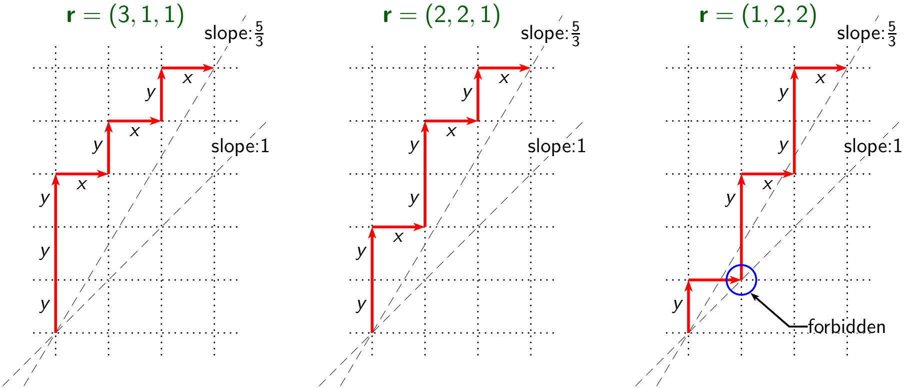
Rhythm

p, q coprime integers $p > q \geqslant 1$

Geometric representation

$\mathbf{r} = (r_0, r_1, \ldots, r_{q-1})$

$$\mathtt{path}(\mathbf{r}) = y^{r_0} x\, y^{r_1} x\, y^{r_2} \cdots x\, y^{r_{q-1}} x$$



Christoffel rhythm $\mathbf{r}_{\frac{p}{q}}$

p, q coprime integers $p > q \geqslant 1$

$\mathbf{r}$ Christoffel rhythm if $\texttt{path}(\mathbf{r})$ Christoffel word

Christoffel rhythm $\mathbf{r}_{\frac{p}{q}}$

p, q coprime integers $p > q \geqslant 1$

$\mathbf{r}$ Christoffel rhythm if $\texttt{path}(\mathbf{r})$ Christoffel word

$\texttt{path}(\mathbf{r})$ Christoffel word if no integer point between $\texttt{path}(\mathbf{r})$ and slope

Christoffel rhythm $\mathbf{r}_{\frac{p}{q}}$

p, q coprime integers $p > q \geqslant 1$

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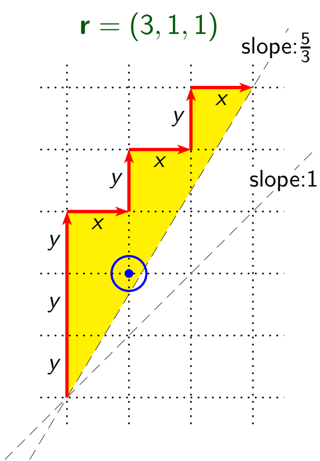
$\texttt{path}(\mathbf{r})$ Christoffel word if no integer point between $\texttt{path}(\mathbf{r})$ and slope



Signature of $T_{\frac{p}{q}}$

p, q coprime integers, $p > q \geqslant 1$

Theorem

The signature of $T_{\frac{p}{q}}$ is purely periodic of period $\mathbf{r}_{\frac{p}{q}}$.

Rhythm and labelling

p, q coprime integers $p > q \geqslant 1$ A *ordered* alphabet

Rhythm and labelling

p, q coprime integers $p > q \geqslant 1$ $\qquad$ A *ordered* alphabet

A purely periodic labelled signature

$$(\boldsymbol{s}, \boldsymbol{\lambda}) \;=\; (\mathbf{r}^{\omega}, \boldsymbol{\gamma}^{\omega})$$

Rhythm and labelling

p, q coprime integers $p > q \geqslant 1$ $\quad$ A *ordered* alphabet

A purely periodic labelled signature

$$(\boldsymbol{s}, \boldsymbol{\lambda}) = (\mathbf{r}^{\omega}, \boldsymbol{\gamma}^{\omega})$$

r rhythm of dir. par. (q, p) $\quad$ $\boldsymbol{\gamma} = (\gamma_0, \gamma_1, \ldots, \gamma_{p-1})$ $\quad \gamma_i \in A$

Rhythm and labelling

p, q coprime integers $p > q \geqslant 1$ A *ordered* alphabet

A purely periodic labelled signature

$$(\boldsymbol{s}, \boldsymbol{\lambda}) = (\mathbf{r}^{\omega}, \boldsymbol{\gamma}^{\omega})$$

$\mathbf{r}$ rhythm of dir. par. (q, p) $\boldsymbol{\gamma} = (\gamma_0, \gamma_1, \ldots, \gamma_{p-1})$ $\gamma_i \in A$

Definition

$$\mathbf{r} = (r_0, r_1, \ldots, r_{q-1})$$

$\boldsymbol{\gamma} = u_0\, u_1 \cdots u_{q-1}$ factorisation induced by $\mathbf{r}$ $|u_i| = r_i$

$\boldsymbol{\gamma}$ consistent with $\mathbf{r}$ every u_i increasing word

Rhythm and labelling

p, q coprime integers $p > q \geqslant 1$ $\qquad$ A *ordered* alphabet

A purely periodic labelled signature

$$(\boldsymbol{s}, \boldsymbol{\lambda}) \;=\; (\mathbf{r}^{\omega}, \boldsymbol{\gamma}^{\omega})$$

$\mathbf{r}$ rhythm of dir. par. (q, p) $\qquad$ $\boldsymbol{\gamma} = (\gamma_0, \gamma_1, \ldots, \gamma_{p-1})$ $\quad \gamma_i \in A$

Definition

$$\mathbf{r} = (r_0, r_1, \ldots, r_{q-1})$$

$\boldsymbol{\gamma} = u_0\, u_1 \cdots u_{q-1}$ factorisation induced by $\mathbf{r}$ $\qquad$ $|u_i| = r_i$

$\boldsymbol{\gamma}$ consistent with $\mathbf{r}$ $\qquad$ every u_i increasing word

Examples

$\mathbf{r} = (3, 1, 1)$ $\qquad$ $\boldsymbol{\gamma} = 0\,1\,2\,1\,0$ $\quad$ $\boldsymbol{\gamma} = 0\,3\,5\,6\,4$ $\quad$ consistent

$\mathbf{r} = (2, 2, 1)$ $\qquad$ $\boldsymbol{\gamma} = 0\,1\,2\,1\,0$ not consistent $\quad$ $\boldsymbol{\gamma} = 0\,3\,5\,6\,4$ consistent

Christoffel labelling

p, q coprime integers $p > q \geqslant 1$ alphabet: $\{0, 1, \ldots, p-1\}$

Christoffel labelling

p, q coprime integers $p > q \geqslant 1$ alphabet: $\{0, 1, \ldots, p-1\}$

Definition

$$\gamma_{\frac{p}{q}} = \big(0,\, (q \,\%\, p),\, (2\,q \,\%\, p), \ldots,\, ((p-1)\,q \,\%\, p)\big) \ .$$

Christoffel labelling

p, q coprime integers $p > q \geqslant 1$ alphabet: $\{0, 1, \ldots, p-1\}$

Definition

$$\gamma_{\frac{p}{q}} = \big(0, (q \,\%\, p), (2q \,\%\, p), \ldots, ((p-1)\, q \,\%\, p)\big) \quad .$$

Examples

$\mathbf{r}_{\frac{3}{2}} = (2, 1)$ $\gamma_{\frac{3}{2}} = 0\,2\,1$ $\mathbf{r}_{\frac{5}{3}} = (2, 2, 1)$ $\gamma_{\frac{5}{3}} = 0\,3\,1\,4\,2$

Christoffel labelling

p, q coprime integers $p > q \geqslant 1$ alphabet: $\{0, 1, \ldots, p-1\}$

Definition

$$\gamma_{\frac{p}{q}} \;=\; \big(\, 0,\; (q \,\%\, p),\; (2\,q \,\%\, p), \ldots,\; ((p-1)\,q \,\%\, p) \,\big) \;\; .$$

Examples

$\mathbf{r}_{\frac{3}{2}} = (2, 1)$ $\gamma_{\frac{3}{2}} = 0\,2\,1$ $\mathbf{r}_{\frac{5}{3}} = (2, 2, 1)$ $\gamma_{\frac{5}{3}} = 0\,3\,1\,4\,2$

Proposition

$\gamma_{\frac{p}{q}}$ *is consistent with* $\mathbf{r}_{\frac{p}{q}}$

Signature of $T_{\frac{p}{q}}$

p, q coprime integers, $p > q \geqslant 1$

Theorem

The labelled signature of $T_{\frac{p}{q}}$ *is purely periodic of period* $(\mathbf{r}_{\frac{p}{q}}, \boldsymbol{\gamma}_{\frac{p}{q}})$.

Part IV

Trees with periodic signature are essentially $T_{\frac{p}{q}}$

Special labelling

p, q coprime integers $p > q \geqslant 1$

Special labelling

p, q coprime integers $p > q \geqslant 1$

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Definition

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special labelling associated with $\mathbf{r}$

$$\gamma_i \in u_k\,,\ \gamma_{i+1} \in u_{k+j} \qquad \Longrightarrow \qquad \gamma_{i+1} = \gamma_i + q - j\,p$$

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$\mathbf{r} = (3, 1, 1)$ $\boldsymbol{\gamma}_{\mathbf{r}} = 0\,3\,6\,4\,2$

$\mathbf{r} = (4, 0, 1)$ $\boldsymbol{\gamma}_{\mathbf{r}} = 0\,3\,6\,9\,2$

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Observation

The special labelling associated with $\mathbf{r}$ is consistent with $\mathbf{r}$

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Proposition

$$\boldsymbol{\gamma}_{\mathbf{r}_{\frac{p}{q}}} = \boldsymbol{\gamma}_{\frac{p}{q}}$$

The tree $T_{\mathbf{r}}$

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$\mathbf{r}$ rhythm of directing parameter (q, p) $\boldsymbol{\gamma}_{\mathbf{r}}$ special labelling

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$T_{\mathbf{r}}$ labelled tree with labelled signature $(\mathbf{r}^{\omega}, \boldsymbol{\gamma}_{\mathbf{r}}^{\omega})$

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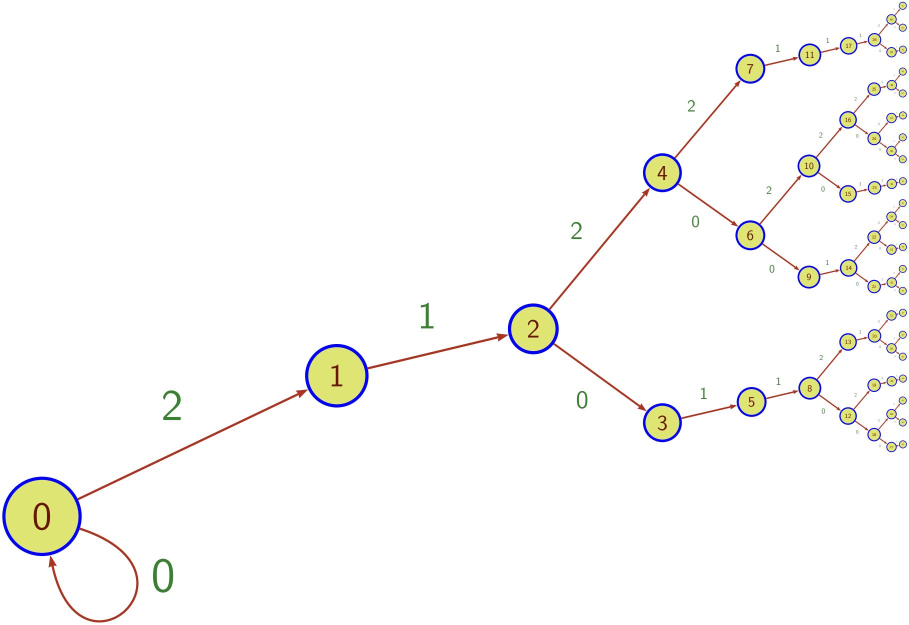
If $\dfrac{p}{q}$ is an integer, then $T_{\mathbf{r}}$ is regular, otherwise $T_{\mathbf{r}}$ is a FLIP language.



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Part V

Complements : the Mahler problem

The fractional part of the powers of rational numbers

Notation

$\theta \in \mathbb{R}$ $\{\theta\}$ fractional part of θ

Problem

$\theta \in \mathbb{R}\,,\ \theta > 1$ Distribution of $S(\theta) = \{\theta^n\}_{n\in\mathbb{N}}$ **?**

Theorem

For almost all θ, $S(\theta)$ is uniformly distributed.

The fractional part of the powers of rational numbers

Very few results are known for specific values of θ.

Proposition

θ *Pisot* $\implies$ 0 *is the only limit point of* $S(\theta)$ *(in* $\mathbb{R}/\mathbb{Z}$ *).*

Experimental results show that $S(\theta)$ looks :

- *uniformly distributed* for transcendental θ,
- *very chaotic* for rational θ.

Theorem (Pisot ?? — Vijayaraghavan 40)

θ *rational* $\implies$ $S(\theta)$ *has infinitely many limit points.*

Parametrization of the problem

Fix the rational $\frac{p}{q}$, $p > q \geqslant 2$ coprime integers.

New problem

$\xi \in \mathbb{R}$ Distribution of $M_{\frac{p}{q}}(\xi) = \left\{ \xi \left(\frac{p}{q} \right)^n \right\}_{n \in \mathbb{N}}$ **?**

Theorem

For almost all ξ, $M_{\frac{p}{q}}(\xi)$ is uniformly distributed.

The (generalized) Mahler approach

Notation

$I \subsetneq [0,1[$ $\quad$ I will be a finite union of semi-closed intervals.

$$\mathbf{Z}_{\frac{p}{q}}(I) = \{\xi \in \mathbb{R} \mid \quad M_{\frac{p}{q}}(\xi) \quad \text{is eventually contained in } I\ \} \ .$$

Two directions of research:

Look for I as large as possible such that $\mathbf{Z}_{\frac{p}{q}}(I)$ is empty.

Look for I as small as possible such that $\mathbf{Z}_{\frac{p}{q}}(I)$ is non empty.

Theorem (Mahler 68)

$$\mathbf{Z}_{\frac{3}{2}}\left([0, \tfrac{1}{2}[\right) \quad \textit{is at most countable.}$$

Open problem $\qquad$ Is $\mathbf{Z}_{\frac{3}{2}}\left([0, \frac{1}{2}[\right)$ non empty?

The search for big I with empty $\mathbf{Z}_{\frac{p}{q}}(I)$

Theorem (Flatto, Lagarias, Pollington 95)

The set of reals s

such that $\mathbf{Z}_{\frac{p}{q}}\left([s, s + \frac{1}{p}[\right)$ *is empty*

is dense in $[0, 1 - \frac{1}{p}]$.

Theorem (Bugeaud 04)

The same set is of Lebesgue measure $1 - \frac{1}{p}$.

The search for small I with non empty $\mathbf{Z}_{\frac{p}{q}}(I)$

Theorem (Pollington 81)

$\mathbf{Z}_{\frac{3}{2}}\left(\left[\frac{4}{65}, \frac{61}{65}\right[\right)$ *is non empty.*

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Theorem (A.-F.-S. 05)

Let $p \geqslant 2q-1$. *There exists* $Y_{\frac{p}{q}} \subset [0,1[$ *of measure* $\frac{q}{p}$

such that $\mathbf{Z}_{\frac{p}{q}}\left(Y_{\frac{p}{q}}\right)$ *is (countable) infinite.*

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Indeed $\mathbf{Z}_{\frac{p}{q}}\left(Y_{\frac{p}{q}}\right) = \{\xi \in \mathbb{R}_+ \mid \xi \text{ has two } \frac{p}{q}\text{-expansions } \}$.

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