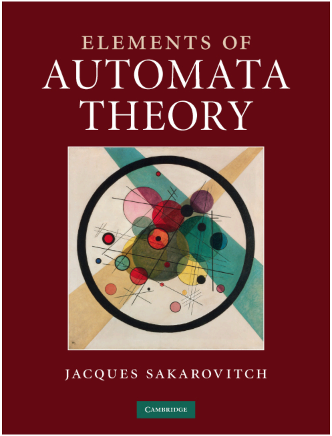
Introduction to weighted automata theory

Lectures given at
the 19th Estonian Winter School in Computer Science

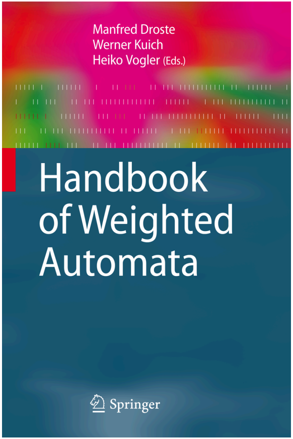
Jacques Sakarovitch

CNRS / Telecom ParisTech

Based on



Chapter III



Chapter 4

The presentation is very much inspired by a joint work with

Marie-Pierre Béal (Univ. Paris-Est)

and

Sylvain Lombardy (Univ. Bordeaux)

entitled

On the equivalence and conjugacy of weighted automata,

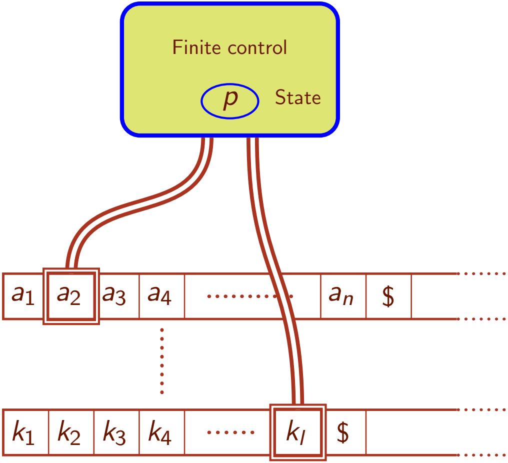
a first version of which has been published in *Proc. of CSR 2006*

and whose final complete version is still in preparation.

Lecture I

The model of (finite) weighted automata

A touch of general system theory

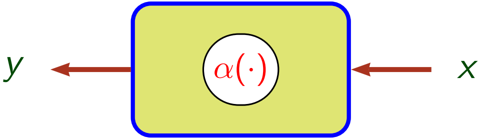


Paradigm of a machine for the computer scientists

A touch of general system theory

Paradigm of a machine for the rest of the world

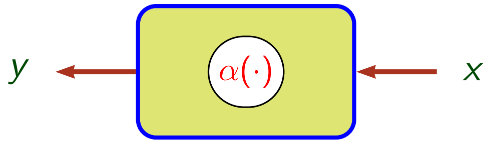
A touch of general system theory



$$y = \alpha(x)$$

Paradigm of a machine for the rest of the world

A touch of general system theory

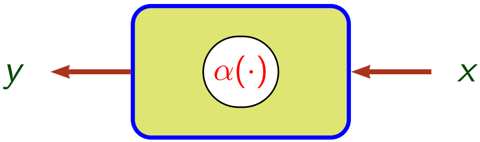


$$y = \alpha(x)$$

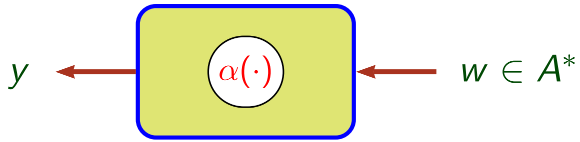
$$x \in \mathbb{R}^n\,, \quad y \in \mathbb{R}^m$$

Paradigm of a machine for the rest of the world

Getting back to computer science

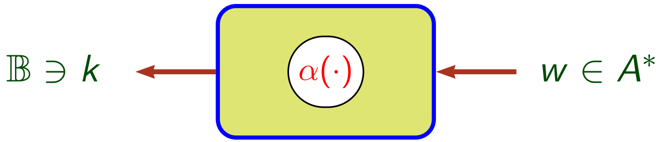


Getting back to computer science



The input belongs to a *free monoid* A^*

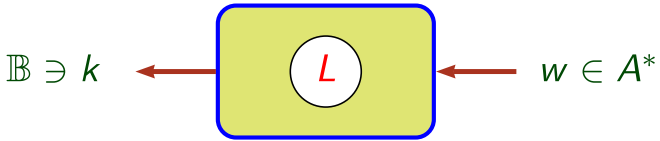
Getting back to computer science



The input belongs to a *free monoid* A^*

The output belongs to the *Boolean semiring* $\mathbb{B}$

Getting back to computer science



$$L \subseteq A^*$$

The input belongs to a *free monoid* A^*

The output belongs to the *Boolean semiring* $\mathbb{B}$

The function realised is *a language*

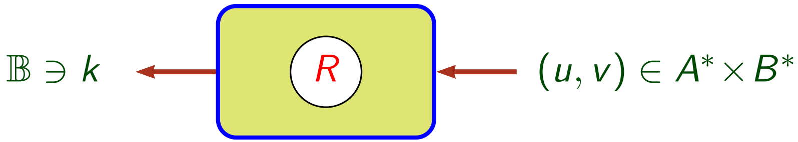
Getting back to computer science

$$\mathbb{B} \ni k \longleftarrow \boxed{\alpha(\cdot)} \longleftarrow (u, v) \in A^* \times B^*$$

The input belongs to a *direct product of free monoids* $A^* \times B^*$

The output belongs to *the Boolean semiring* $\mathbb{B}$

Getting back to computer science



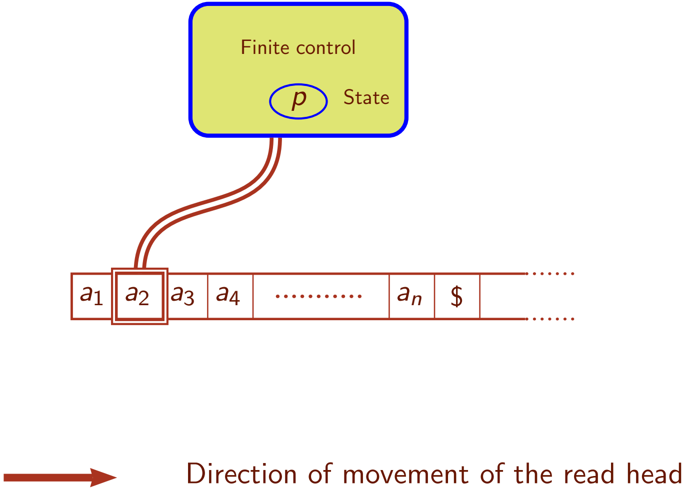
$$R \subseteq A^* \times B^*$$

The input belongs to a *direct product of free monoids* $A^* \times B^*$

The output belongs to *the Boolean semiring* $\mathbb{B}$

The function realised is *a relation between words*

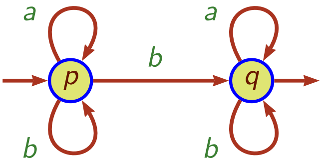
The simplest Turing Machine



The 1 way 1 tape Turing Machine (1W1TM)

The simplest Turing Machine is equivalent to finite automata

$\mathcal{B}_1$



The simplest Turing Machine is equivalent to finite automata

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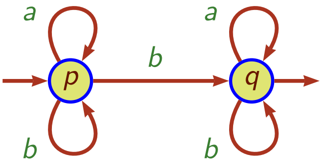
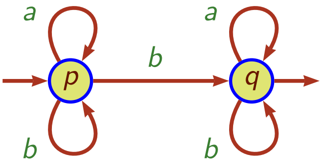


$b\,a\,b \quad \in A^*$

The simplest Turing Machine is equivalent to finite automata

$\mathcal{B}_1$



$b\,a\,b \quad \in A^*$

$$\longrightarrow p \xrightarrow{b} p \xrightarrow{a} p \xrightarrow{b} q \longrightarrow$$

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The simplest Turing Machine is equivalent to finite automata

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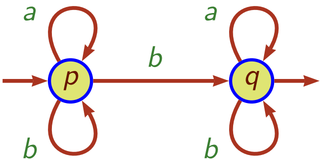


$L(\mathcal{B}_1) \subseteq A^*$

$b\,a\,b \quad \in A^*$

$$\longrightarrow p \xrightarrow{b} p \xrightarrow{a} p \xrightarrow{b} q \longrightarrow$$

$$\longrightarrow p \xrightarrow{b} q \xrightarrow{a} q \xrightarrow{b} q \longrightarrow$$

$$L(\mathcal{B}_1) = \{w \in A^* \mid w \in A^* b A^*\} = \{w \in A^* \mid |w|_b \geqslant 1\}$$

Rational (or regular) languages

Languages accepted (or recognized) by finite automata

=

Languages described by rational (or regular) expressions

=

Languages defined by MSO formulae

Remarkable features of the finite automaton model

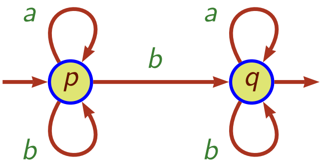
Decidable equivalence (decidable inclusion)

Closure under complement

Canonical automaton (minimal deterministic automaton)

Automata versus languages

$\mathcal{B}_1$



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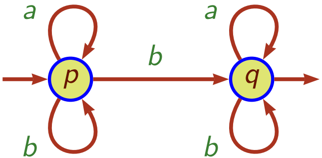
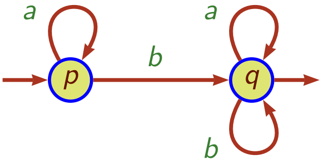


$L(\mathcal{B}_1) \subseteq A^*$

$\mathcal{B}_1'$



$L(\mathcal{B}_1') \subseteq A^*$

Automata versus languages

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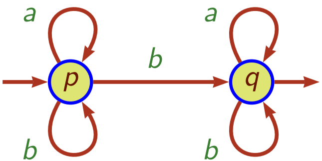

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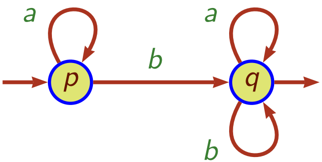

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$$L(\mathcal{B}_1) = L(\mathcal{B}_1') = \left\{ w \in A^* \,\middle|\, |w|_b \geqslant 1 \right\}$$

Automata versus languages

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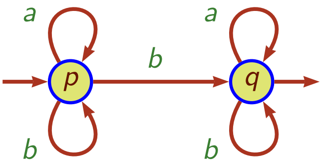
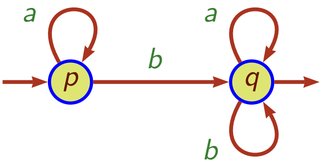


$L(\mathcal{B}_1) \subseteq A^*$

$\mathcal{B}_1'$



$L(\mathcal{B}_1') \subseteq A^*$

$$L(\mathcal{B}_1) = L(\mathcal{B}_1') = \left\{ w \in A^* \,\middle|\, |w|_b \geqslant 1 \right\} = A^* b A^*$$

Ambiguity: a preliminary to multiplicity

Here, *automaton* stands for *classical (Boolean) automaton*.

Definition

A (trim) automaton $\mathcal{A}$ is *unambiguous*
if *no* word
is the label of more than one successful computation of $\mathcal{A}$.

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It is decidable whether an automaton is ambiguous or not.

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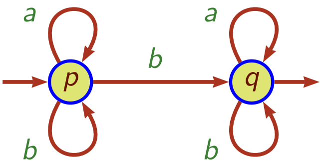
Proof ?

Ambiguity: a preliminary to multiplicity

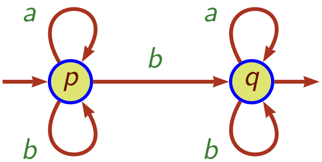
$\mathcal{B}_1$



$$L(\mathcal{B}_1) = A^* b\, A^*$$

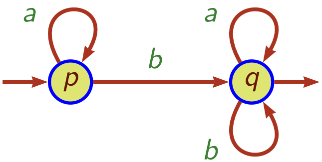
Ambiguity: a preliminary to multiplicity

$\mathcal{B}_1$



$L(\mathcal{B}_1) = A^* b\, A^*$

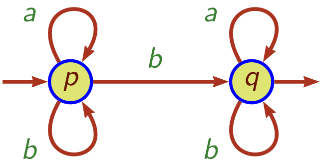
$\mathcal{B}_1'$



$L(\mathcal{B}_1') = A^* b\, A^*$

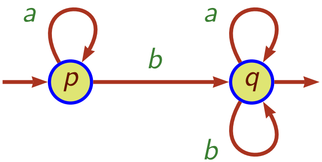
Ambiguity: a preliminary to multiplicity

$\mathcal{B}_1$



$L(\mathcal{B}_1) = A^* b A^*$

$\mathcal{B}_1'$



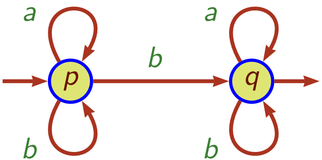
$L(\mathcal{B}_1') = A^* b A^*$

Counting the number of successful computations

$|\mathcal{B}_1| : \; b\,a\,b \quad \longmapsto \quad 2$ $\qquad$ $|\mathcal{B}_1'| : \; b\,a\,b \quad \longmapsto \quad 1$

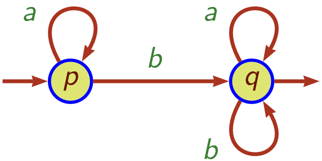
Ambiguity: a preliminary to multiplicity

$\mathcal{B}_1$



$L(\mathcal{B}_1) = A^* b A^*$

$\mathcal{B}'_1$

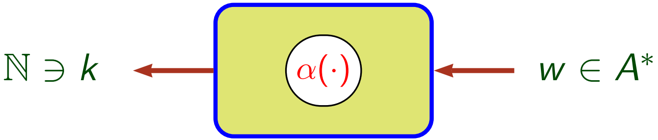


$L(\mathcal{B}'_1) = A^* b A^*$

Counting the number of successful computations

$|\mathcal{B}_1| : \; w \longmapsto |w|_b \qquad |\mathcal{B}'_1| : \; w \longmapsto 1$

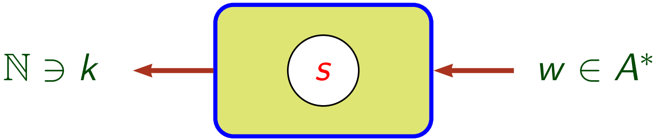
A new automaton model



The input belongs to a *free monoid* A^*

The output belongs to the *integer semiring* $\mathbb{N}$

A new automaton model



$s \colon A^* \to \mathbb{N}$

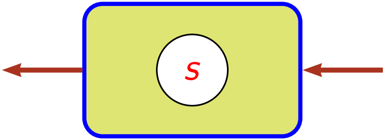
The input belongs to a *free monoid* A^*

The output belongs to the *integer semiring* $\mathbb{N}$

The function realised is *a function from* A^* to $\mathbb{N}$

A new automaton model

$\mathbb{N} \ni k$



$w \in A^*$

$s \colon A^* \to \mathbb{N}$ $\qquad$ $s \in \mathbb{N}\langle\langle A^* \rangle\rangle$

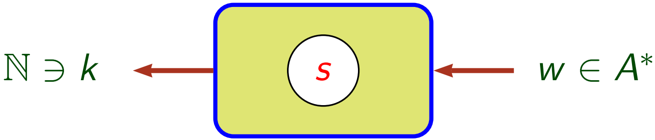
The input belongs to a *free monoid* A^*

The output belongs to the *integer semiring* $\mathbb{N}$

The function realised is *a function from* A^* to $\mathbb{N}$

we call it *a series*

A new automaton model



$$s \colon A^* \to \mathbb{N} \qquad\qquad s \in \mathbb{N}\langle\langle A^* \rangle\rangle$$

$$s_1 = b + a\,b + b\,a + 2\,b\,b + a\,a\,b + \cdots + 2\,b\,b\,a + 3\,b\,b\,b + \cdots$$

The input belongs to a *free monoid* A^*

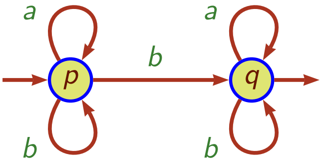
The output belongs to the *integer semiring* $\mathbb{N}$

The function realised is *a function from* A^* to $\mathbb{N}$

we call it *a series*

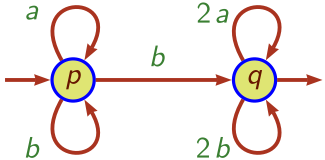
The weighted automaton model

$\mathcal{B}_1$



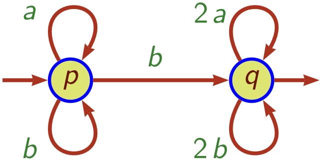
The weighted automaton model

$\mathcal{C}_1$



The weighted automaton model

$\mathcal{C}_1$

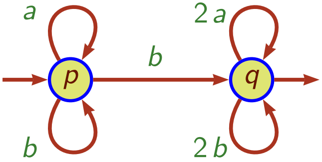


$$\xrightarrow{1} p \xrightarrow{b} p \xrightarrow{a} p \xrightarrow{b} q \xrightarrow{1}$$

$$\xrightarrow{1} p \xrightarrow{b} q \xrightarrow{2\,a} q \xrightarrow{2\,b} q \xrightarrow{1}$$

The weighted automaton model

$\mathcal{C}_1$



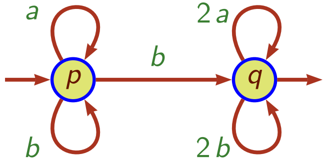
$$\xrightarrow{1} p \xrightarrow{b} p \xrightarrow{a} p \xrightarrow{b} q \xrightarrow{1}$$

$$\xrightarrow{1} p \xrightarrow{b} q \xrightarrow{2\,a} q \xrightarrow{2\,b} q \xrightarrow{1}$$

- Weight of a path c: *product* of the weights of transitions in c
- Weight of a word w: *sum* of the weights of paths with label w

The weighted automaton model

$\mathcal{C}_1$



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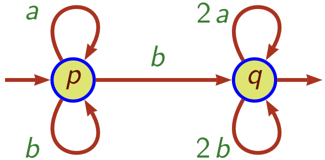
- Weight of a path c: *product* of the weights of transitions in c
- Weight of a word w: *sum* of the weights of paths with label w

$b\,a\,b \quad \longmapsto \quad 1 + 4 = 5$

The weighted automaton model

$\mathcal{C}_1$



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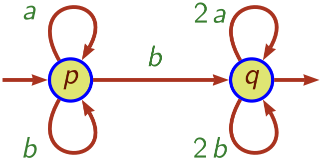
- Weight of a path c: *product* of the weights of transitions in c
- Weight of a word w: *sum* of the weights of paths with label w

$$b\,a\,b \quad \longmapsto \quad 1 + 4 = 5 \quad = \langle 101 \rangle_2$$

The weighted automaton model

$\mathcal{C}_1$



$|\mathcal{C}_1| \in \mathbb{N}\langle\langle A^* \rangle\rangle$

$$\xrightarrow{1} p \xrightarrow{b} p \xrightarrow{a} p \xrightarrow{b} q \xrightarrow{1}$$

$$\xrightarrow{1} p \xrightarrow{b} q \xrightarrow{2\,a} q \xrightarrow{2\,b} q \xrightarrow{1}$$

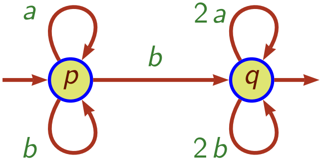
- Weight of a path c: *product* of the weights of transitions in c
- Weight of a word w: *sum* of the weights of paths with label w

$b\,a\,b \quad \longmapsto \quad 1 + 4 = 5$

$|\mathcal{C}_1|\colon A^* \longrightarrow \mathbb{N}$

The weighted automaton model

$\mathcal{C}_1$



$\left|\mathcal{C}_1\right| \in \mathbb{N}\langle\langle A^* \rangle\rangle$

$$\xrightarrow{1} p \xrightarrow{b} p \xrightarrow{a} p \xrightarrow{b} q \xrightarrow{1}$$

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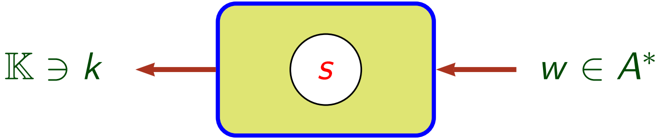
- Weight of a path c: *product* of the weights of transitions in c
- Weight of a word w: *sum* of the weights of paths with label w

$$\left|\mathcal{C}_1\right| = b + a\,b + 2\,b\,a + 3\,b\,b + a\,a\,b + 2\,a\,b\,a + \cdots + 5\,b\,a\,b + \cdots$$

The weighted automaton model



$$s \colon A^* \to \mathbb{K} \qquad\qquad s \in \mathbb{K}\langle\langle A^* \rangle\rangle$$

The input belongs to a *free monoid* A^*

The output belongs to a *semiring* $\mathbb{K}$

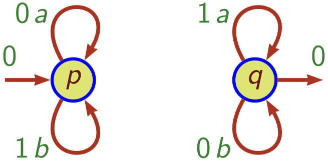
The function realised is *a function from* A^* to $\mathbb{K}$: *a series* in $\mathbb{K}\langle\langle A^* \rangle\rangle$

Richness of the model of weighted automata

- $\mathbb{B}$ 'classic' automata
- $\mathbb{N}$ 'usual' counting
- $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ numerical multiplicity
- $\langle\, \mathbb{Z} \cup +\infty, \min, + \,\rangle$ Min-plus automata
- $\langle\, \mathbb{Z}, \min, \max \,\rangle$ fuzzy automata
- $\mathfrak{P}\,(B^*) = \mathbb{B}\langle\!\langle B^* \rangle\!\rangle$ transducers
- $\mathbb{N}\langle\!\langle B^* \rangle\!\rangle$ weighted transducers
- $\mathfrak{P}\,(F(B))$ pushdown automata

Another example

$\mathcal{L}_1$



$|\mathcal{L}_1| \in \mathbb{Z}\text{min}\langle\langle A^* \rangle\rangle$

Another example

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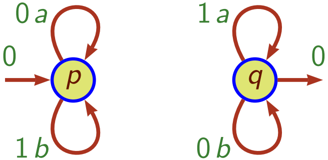


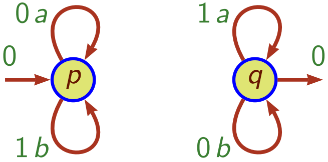
$|\mathcal{L}_1| \in \mathbb{Z}\mathrm{min}\langle\langle A^* \rangle\rangle$

$$\xrightarrow{0} p \xrightarrow{1\,b} p \xrightarrow{0\,a} p \xrightarrow{1\,b} p \xrightarrow{0}$$

$$\xrightarrow{0} q \xrightarrow{0\,b} q \xrightarrow{1\,a} q \xrightarrow{0\,b} q \xrightarrow{0}$$

Another example

$\mathcal{L}_1$



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- Weight of a path c:
  product, that is, the *sum*, of the weights of transitions in c
- Weight of a word w:
  sum, that is, the *min* of the weights of paths with label w.

Another example

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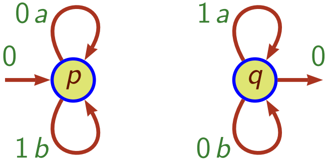

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- Weight of a path c:

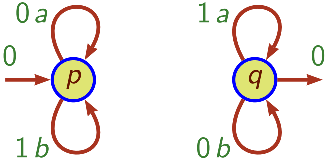
  product, that is, the *sum*, of the weights of transitions in c
- Weight of a word w:

  sum, that is, the *min* of the weights of paths with label w.

$b\,a\,b \longmapsto \min(1+0+1, 0+1+0) = 1$ $\qquad$ $|\mathcal{L}_1|\colon A^* \longrightarrow \mathbb{Z}\mathrm{min}$

Another example

$\mathcal{L}_1$



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- Weight of a path c:
  product, that is, the *sum*, of the weights of transitions in c
- Weight of a word w:
  sum, that is, the *min* of the weights of paths with label w.

$$|\mathcal{C}_1| = 0\,1_{A^*} + 0\,a + 0\,b + 1\,a\,b + 1\,b\,a + 0\,b\,b + \cdots + 1\,b\,a\,b + \cdots$$

Series play the role of languages

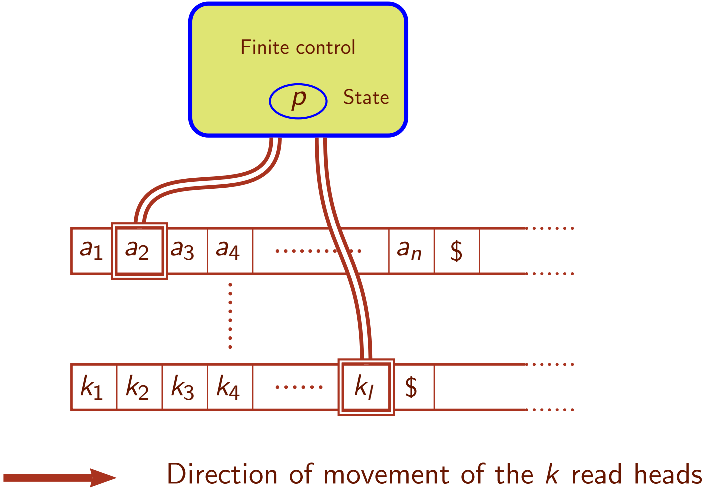
$\mathbb{K}\langle\langle A^* \rangle\rangle$ **plays the role of** $\mathfrak{P}(A^*)$

Weighted automata theory

is linear algebra

of computer science

The Turing Machine equivalent to finite transducers



The 1 way *k* tape Turing Machine (1WkTM)

Outline of the lectures

1. Rationality
2. Recognisability
3. Reduction and equivalence
4. Morphisms of automata

Lecture II

Rationality

Outline of Lecture II

- The set of series $\mathbb{K}\langle\langle A^* \rangle\rangle$ is a $\mathbb{K}$-algebra.
- Automata are (essentially) matrices: $\mathcal{A} = \langle\, I, E, T \,\rangle$
- Computing the behaviour of an automaton boils down to solving a linear system $X = E \cdot X + T$ (s)
- Solving the linear system (s) amounts to invert the matrix $(Id - E)$ (hence the name rational)
- The inversion of $Id - E$ is realised by an infinite sum $Id + E + E^2 + E^3 + \cdots$: the star of E
- What can be computed by a finite automaton is exactly what can be computed by the star operation (together with the algebra operations)

The semiring $\mathbb{K}\langle\langle A^* \rangle\rangle$

$\mathbb{K}$ semiring $\qquad\qquad$ A^* free monoid

$$s \in \mathbb{K}\langle\langle A^* \rangle\rangle \qquad s\colon A^* \to \mathbb{K} \qquad s\colon w \longmapsto \langle s, w \rangle$$

$$s = \sum_{w \in A^*} \langle s, w \rangle \, w$$

Point-wise addition $\qquad$ $\langle s + t, w \rangle = \langle s, w \rangle + \langle t, w \rangle$

Cauchy product $\qquad$ $\langle s\, t, w \rangle = \sum_{u\, v = w} \langle s, u \rangle \langle t, v \rangle$

$\{(u, v) \mid u\, v = w\}$ finite $\quad \Longrightarrow \quad$ Cauchy product well-defined

$\mathbb{K}\langle\langle A^* \rangle\rangle$ is a semiring

The semiring $\mathbb{K}\langle\langle M \rangle\rangle$

$\mathbb{K}$ semiring $\qquad$ M monoid

$$s \in \mathbb{K}\langle\langle M \rangle\rangle \qquad s \colon M \to \mathbb{K} \qquad s \colon m \longmapsto \langle s, m \rangle$$

$$s = \sum_{m \in M} \langle s, w \rangle \, w$$

Point-wise addition $\qquad \langle s + t, m \rangle = \langle s, m \rangle + \langle t, m \rangle$

Cauchy product $\qquad \langle s\, t, m \rangle = \sum_{x\, y = m} \langle s, x \rangle \langle t, y \rangle$

$\forall m \ \{(x, y) \mid x\, y = m\}$ finite $\quad \Longrightarrow \quad$ Cauchy product well-defined

The semiring $\mathbb{K}\langle\langle M\rangle\rangle$

Conditions for $\{(x,y) \mid x\,y = m\}$ finite for all m

Definition

M is *graded* if M equipped with a length function φ

$\varphi\colon M \to \mathbb{N}$ $\qquad\qquad \varphi(m\,m') = \varphi(m) + \varphi(m')$

M f.g. and graded $\implies$ $\mathbb{K}\langle\langle M\rangle\rangle$ is a semiring

Examples

$\mathbb{M}$ trace monoid, then $\mathbb{K}\langle\langle M\rangle\rangle$ is a semiring

$\mathbb{K}\langle\langle A^* \times B^*\rangle\rangle$ is a semiring

$F(A)$, the free group on A , is not graded

The algebra $\mathbb{K}\langle\langle M \rangle\rangle$

$\mathbb{K}$ semiring $\qquad$ M f.g. graded monoid

$$s \in \mathbb{K}\langle\langle A^* \rangle\rangle \qquad s\colon A^* \to \mathbb{K} \qquad s\colon w \longmapsto \langle s, w \rangle$$

$$s = \sum_{w \in A^*} \langle s, w \rangle \, w$$

Point-wise addition $\qquad$ $\langle s + t, m \rangle = \langle s, m \rangle + \langle t, m \rangle$

Cauchy product $\qquad$ $\langle s\, t, m \rangle = \sum_{x\, y = m} \langle s, x \rangle \langle t, y \rangle$

External multiplication $\qquad$ $\langle k\, s, m \rangle = k \, \langle s, m \rangle$

$\mathbb{K}\langle\langle M \rangle\rangle$ is an algebra

The star operation

$$t \in \mathbb{K} \qquad\qquad t^* = \sum_{n\in\mathbb{N}} t^n$$

How to define infinite sums ?

One possible solution

Topology on $\mathbb{K}$

Definition of summable families and of their sum

t^* defined if $\{t^n\}_{n\in\mathbb{N}}$ summable

Other possible solutions

axiomatic definition of star, equational definition of star

The star operation

$t \in \mathbb{K}$

$$t^* = \sum_{n \in \mathbb{N}} t^n$$

The star operation

$$t \in \mathbb{K} \qquad\qquad t^* = \sum_{n \in \mathbb{N}} t^n$$

- $\forall \mathbb{K} \qquad (0_{\mathbb{K}})^* = 1_{\mathbb{K}}$
- $\mathbb{K} = \mathbb{N} \qquad \forall x \neq 0 \qquad x^*$ not defined.
- $\mathbb{K} = \mathcal{N} = \mathbb{N} \cup \{+\infty\} \qquad \forall x \neq 0 \qquad x^* = \infty$.
- $\mathbb{K} = \mathbb{Q} \qquad (\frac{1}{2})^* = 2$ with the natural topology, $(\frac{1}{2})^*$ is undefined with the discrete topology.

The star operation

$$t \in \mathbb{K} \qquad\qquad t^* = \sum_{n \in \mathbb{N}} t^n$$

In any case

$$t^* = 1_{\mathbb{K}} + t\, t^*$$

Star has the same flavor as the inverse

If $\mathbb{K}$ is a ring

$$t^* \, (1_{\mathbb{K}} - t) = 1_{\mathbb{K}}$$

$$\frac{1_{\mathbb{K}}}{1_{\mathbb{K}} - t} = 1_{\mathbb{K}} + t + t^2 + \cdots + t^n + \cdots$$

Star of series

$s \in \mathbb{K}\langle\langle A^* \rangle\rangle$ When is $s^* = \sum_{n \in \mathbb{N}} s^n$ defined ?

Topology on $\mathbb{K}$ yields topology on $\mathbb{K}\langle\langle A^* \rangle\rangle$

s proper $s_0 = \langle s, 1_{A^*} \rangle = 0_{\mathbb{K}}$

s proper $\Longrightarrow$ s^* defined

Rational series

$\mathbb{K}\langle A^* \rangle \subseteq \mathbb{K}\langle\langle A^* \rangle\rangle$ subalgebra of polynomials

$\mathbb{K}\mathrm{Rat}\, A^*$ closure of $\mathbb{K}\langle A^* \rangle$ under

- sum
- product
- exterior multiplication
- and star

$\mathbb{K}\mathrm{Rat}\, A^* \subseteq \mathbb{K}\langle\langle A^* \rangle\rangle$ subalgebra of rational series

Fundamental theorem of finite automata

Theorem

$$s \in \mathbb{K}\mathrm{Rat}\, A^* \qquad \Longleftrightarrow \qquad \exists \mathcal{A} \in \mathsf{WA}\,(A^*) \quad s = |\mathcal{A}|$$

Fundamental theorem of finite automata

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Kleene theorem ?

Fundamental theorem of finite automata

Theorem

$$s \in \mathbb{K}\mathrm{Rat}\, A^* \qquad \Longleftrightarrow \qquad \exists \mathcal{A} \in \mathsf{WA}\,(A^*) \quad s = |\mathcal{A}|$$

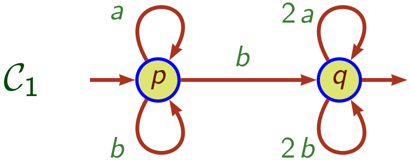
Kleene theorem ?

Theorem

M *finitely generated graded monoid*

$$s \in \mathbb{K}\mathrm{Rat}\, M \qquad \Longleftrightarrow \qquad \exists \mathcal{A} \in \mathsf{WA}\,(M) \quad s = |\mathcal{A}|$$

Automata are matrices



$$\mathcal{C}_1 = \langle\, I_1, E_1, T_1 \,\rangle = \left\langle \begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} a+b & b \\ 0 & 2\,a + 2\,b \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle \quad .$$

Automata are matrices

$\mathcal{A} = \langle\, I, E, T \,\rangle$ $\qquad$ $E =$ incidence matrix

Automata are matrices

$$\mathcal{A} = \langle\, I, E, T \,\rangle \qquad\qquad E = \text{incidence matrix}$$

Notation

$\mathbf{wl}(x)$ = weighted label of x

In our model, e transition $\Rightarrow$ $\mathbf{wl}(e) = k\,a$

Automata are matrices

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$$E_{p,q} = \sum \{\mathbf{wl}(e) \mid e \quad \text{transition from } p \text{ to } q\}$$

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Lemma

$$E^{n}_{p,q} = \sum \{\mathbf{wl}(c) \mid c \quad \textit{computation from } p \textit{ to } q \textit{ of length } n\}$$

Automata are matrices

$$\mathcal{A} = \langle\, I, E, T \,\rangle \qquad\qquad E = \text{incidence matrix}$$

$$E_{p,q} = \sum \{\mathbf{wl}(e) \mid e \quad \text{transition from } p \text{ to } q\}$$

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$$E^* = \sum_{n \in \mathbb{N}} E^n$$

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$$E^*_{p,q} = \sum \{\mathbf{wl}(c) \mid c \quad \text{computation from } p \text{ to } q\}$$

$$\left|\mathcal{A}\right| = I \cdot E^* \cdot T$$

Automata are matrices

$\mathbb{K}$ semiring $\qquad$ M graded monoid

$$\mathbb{K}\langle\langle M \rangle\rangle^{Q\times Q} \quad \text{is isomorphic to} \quad \mathbb{K}^{Q\times Q}\langle\langle M \rangle\rangle$$

$E \in \mathbb{K}\langle\langle M \rangle\rangle^{Q\times Q}$ $\qquad$ E proper $\implies$ E^* defined

Automata are matrices

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Theorem

The entries of E^* *are*

in the rational closure of the entries of E

Fundamental theorem of finite automata

$\mathbb{K}$ semiring $\qquad M$ graded monoid

$\mathbb{K}\langle\langle M\rangle\rangle^{Q\times Q}$ is isomorphic to $\mathbb{K}^{Q\times Q}\langle\langle M\rangle\rangle$

$E \in \mathbb{K}\langle\langle M\rangle\rangle^{Q\times Q}$ $\qquad E$ proper $\Longrightarrow$ E^* defined

Theorem
The entries of E^* *are*

in the rational closure of the entries of E

Theorem
The family of behaviours of weighted automata over M

with coefficients in $\mathbb{K}$ *is rationally closed.*

The collect theorem

$$\mathbb{K}\langle\langle A^* \times B^* \rangle\rangle \text{ is isomorphic to } [\mathbb{K}\langle\langle B^* \rangle\rangle]\,\langle\langle A^* \rangle\rangle$$

Theorem

Under the above isomorphism,

$$\mathbb{K}\mathrm{Rat}\, A^* \times B^* \text{ corresponds to } [\mathbb{K}\mathrm{Rat}\, B^*]\,\mathrm{Rat}\, A^*$$

Lecture III

Recognisability

Outline of Lecture III

- Representation and recognisable series.
- Automata over free monoids are representations
- The notion of action and deterministic automata
- The reachability space and the control morphism
- The notion of quotient and the minimal automaton
- The observation morphism
- The representation theorem

Recognisable series

$\mathbb{K}$ semiring $\quad\quad$ A^* free monoid

Recognisable series

$\mathbb{K}$ semiring $\qquad$ A^* free monoid

$\mathbb{K}$-representation

Q finite $\qquad$ $\mu\colon A^* \to \mathbb{K}^{Q\times Q}$ morphism

$(\,I, \mu, T\,)$ $\qquad$ $I \in \mathbb{K}^{1\times Q}$ $\qquad$ $\mu\colon A^* \to \mathbb{K}^{Q\times Q}$ $\qquad$ $T \in \mathbb{K}^{Q\times 1}$

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$s \in \mathbb{K}\langle\langle A^* \rangle\rangle$ recognisable if s realised by a $\mathbb{K}$-representation

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$s \in \mathbb{K}\langle\!\langle A^* \rangle\!\rangle$ recognisable if s realised by a $\mathbb{K}$-representation

$\mathbb{K}\mathrm{Rec}\, A^* \subseteq \mathbb{K}\langle\!\langle A^* \rangle\!\rangle$ $\qquad$ submodule of recognisable series

Recognisable series

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Example

$$I = \begin{pmatrix} 1 & 0 \end{pmatrix}\ , \quad \mu(a) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\ , \quad \mu(b) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\ , \quad T = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$(\, I, \mu, T\,)$ realises $\displaystyle\sum_{w\in A^*} |w|_b\, w \qquad \in \mathbb{K}\mathrm{Rec}\, A^*$

Recognisable series

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$s \in \mathbb{K}\langle\langle M\rangle\rangle$ recognisable if s realised by a $\mathbb{K}$-representation

$\mathbb{K}\mathrm{Rec}\, M \subseteq \mathbb{K}\langle\langle M\rangle\rangle$ submodule of recognisable series

The key lemma

$\mathbb{K}$ semiring $\qquad$ A^* free monoid

The key lemma

$\mathbb{K}$ semiring $\qquad$ A^* free monoid

$\mu \colon A^* \to \mathbb{K}^{Q\times Q}$ $\qquad$ defined by $\qquad$ $\{\mu(a)\}_{a\in A}$

The key lemma

$\mathbb{K}$ semiring M monoid

$\mu\colon A^* \to \mathbb{K}^{Q \times Q}$ defined by $\{\mu(a)\}_{a \in A}$

The key lemma

$\mathbb{K}$ semiring $\qquad$ M monoid

$\mu\colon M \to \mathbb{K}^{Q \times Q}$ $\qquad$ defined by $\qquad$?

The key lemma

$\mathbb{K}$ semiring $\qquad$ A^* free monoid

$\mu\colon A^* \to \mathbb{K}^{Q\times Q}$ $\qquad$ defined by $\qquad$ $\{\mu(a)\}_{a\in A}$

The key lemma

$\mathbb{K}$ semiring $\qquad A^*$ free monoid

$\mu\colon A^* \to \mathbb{K}^{Q\times Q}$ defined by $\{\mu(a)\}_{a\in A}$

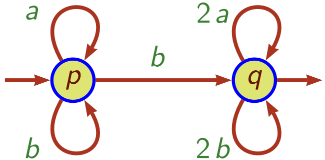
Lemma

$$\mu\colon A^* \to \mathbb{K}^{Q\times Q} \qquad X = \sum_{a\in A} \mu(a)\, a$$

$$\forall w \in A^* \qquad \langle X^*, w\rangle = \mu(w)$$

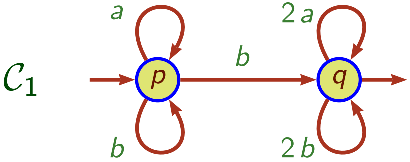
Automata are matrices

$\mathcal{C}_1$



$$\mathcal{C}_1 = \langle\, I_1, E_1, T_1 \,\rangle = \left\langle \begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} a+b & b \\ 0 & 2\,a + 2\,b \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle \quad .$$

Automata over free monoids are representations



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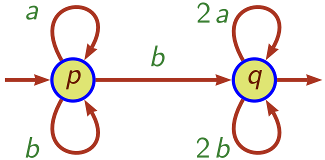
$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} a + \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} b$$

Automata over free monoids are representations

$\mathcal{C}_1$ 

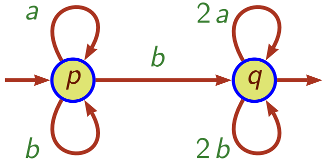
$$\mathcal{C}_1 = \langle\, I_1, E_1, T_1 \,\rangle = \left\langle \begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} a+b & b \\ 0 & 2a+2b \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle \ .$$

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} a + \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} b$$

$$\mathcal{C}_1 = (\, I_1, \mu_1, T_1 \,) \qquad \mu_1(a) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} , \quad \mu_1(b) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

Automata over free monoids are representations

$\mathcal{C}_1$



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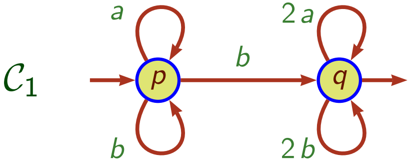
$$|\mathcal{C}_1| = I_1 \cdot {E_1}^* \cdot T_1 = \sum_{w \in A^*} (I_1 \cdot \mu_1(w) \cdot T_1)\, w$$

Automata over free monoids are representations



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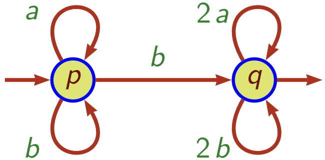
$$|\mathcal{C}_1| = I_1 \cdot {E_1}^* \cdot T_1 = \sum_{w \in A^*} (I_1 \cdot \mu_1(w) \cdot T_1)\, w \qquad\qquad |\mathcal{C}_1| \in \mathbb{K}\mathrm{Rec}\, A^*$$

Automata over free monoids are representations

$\mathcal{C}_1$



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Conversely, representations are automata

The Kleene-Schützenberger Theorem

Fundamental Theorem of Finite Automata and Key Lemma

yield

Theorem

A *finite* $\Rightarrow$ $\mathbb{K}\mathrm{Rec}\,A^* = \mathbb{K}\mathrm{Rat}\,A^*$

The reachability set

$$\mathcal{A} = (\ I, \mu, T\ )$$

The reachability set

$$\mathcal{A} = (\, I, \mu, T \,)$$

Reachability set

$$\mathbf{R}_{\mathcal{A}} = \{ I \cdot \mu(w) \mid w \in A^* \}$$

$$\mathbf{R}_{\mathcal{A}} \subseteq \mathbb{K}^{Q}$$

Reachability space

$$\langle \mathbf{R}_{\mathcal{A}} \rangle$$

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$$\langle \mathbf{R}_{\mathcal{A}} \rangle$$

A^* acts on $\mathbf{R}_{\mathcal{A}}$:

$$(I \cdot \mu(w)) \bullet a = (I \cdot \mu(w)) \cdot \mu(a) = I \cdot \mu(w\, a)$$

The reachability set

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Reachability set

Reachability space

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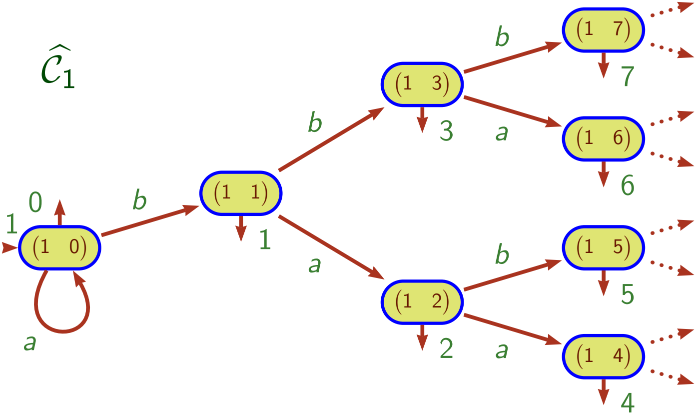
This action turns

$\mathbf{R}_{\mathcal{A}}$ into a deterministic automaton $\widehat{\mathcal{A}}$

(possibly infinite)

The reachability set

$$\mathcal{C}_1 = (\, I_1, \mu_1, T_1 \,)$$



The reachability set

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Reachability set

Reachability space

$$\mathbf{R}_{\mathcal{A}} = \{ I \cdot \mu(w) \mid w \in A^* \} \qquad \mathbf{R}_{\mathcal{A}} \subseteq \mathbb{K}^{Q} \qquad \langle \mathbf{R}_{\mathcal{A}} \rangle$$

$\mathbf{R}_{\mathcal{A}}$ is turned into a deterministic automaton $\widehat{\mathcal{A}}$

The reachability set

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Reachability set

Reachability space

$\mathbf{R}_{\mathcal{A}} = \{ I \cdot \mu(w) \,|\, w \in A^* \}$ $\qquad$ $\mathbf{R}_{\mathcal{A}} \subseteq \mathbb{K}^Q$ $\qquad$ $\langle \mathbf{R}_{\mathcal{A}} \rangle$

$\mathbf{R}_{\mathcal{A}}$ is turned into a deterministic automaton $\widehat{\mathcal{A}}$

If $\mathbb{K} = \mathbb{B}$, $\widehat{\mathcal{A}}$ is the (classical) determinisation of $\mathcal{A}$

The reachability set

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Reachability set

Reachability space

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If $\mathbb{K}$ is *locally finite*, $\mathbf{R}_{\mathcal{A}}$ and $\widehat{\mathcal{A}}$ are finite.

The reachability set

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Reachability set

$$\mathbf{R}_{\mathcal{A}} = \{ I \cdot \mu(w) \,|\, w \in A^* \}$$

$\mathbf{R}_{\mathcal{A}} \subseteq \mathbb{K}^Q$

Reachability space

$\langle \mathbf{R}_{\mathcal{A}} \rangle$

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If $\mathbb{K}$ is *locally finite*, $\mathbf{R}_{\mathcal{A}}$ and $\widehat{\mathcal{A}}$ are finite.

Counting in a locally finite semiring is not really counting

The control morphism

$$\mathcal{A} = (\, I, \mu, T \,)$$

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Reachability space

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Reachability space

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$$\Psi_{\mathcal{A}} \colon \mathbb{K}\langle A^* \rangle \longrightarrow \mathbb{K}^Q \qquad \forall w \in A^* \quad \Psi_{\mathcal{A}}(w) = I \cdot \mu(w)$$

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$$\begin{array}{c} \mathbb{K}\langle A^* \rangle \\ \Psi_{\mathcal{A}} \Big\downarrow \\ \mathbb{K}^Q \end{array}$$

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$$\begin{array}{ccc} \mathbb{K}\langle A^* \rangle & \qquad & w \\ \Big\downarrow \Psi_{\mathcal{A}} & & \Big\downarrow \Psi_{\mathcal{A}} \\ \mathbb{K}^Q & & x \end{array}$$

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$$\begin{tikzcd}
\mathbb{K}\langle A^* \rangle \arrow[r, Rightarrow, "A^*"] \arrow[d, "\Psi_{\mathcal{A}}"'] & \mathbb{K}\langle A^* \rangle \\
\mathbb{K}^Q &
\end{tikzcd}
\qquad
\begin{tikzcd}
w \arrow[r, mapsto] \arrow[d, "\Psi_{\mathcal{A}}"'] & w\,a \\
x &
\end{tikzcd}$$

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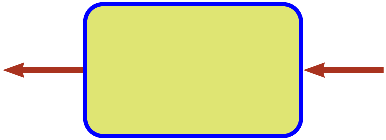
\mathbb{K}^Q \arrow[r, Rightarrow, "A^*"] & \mathbb{K}^Q
\end{tikzcd}
\qquad
\begin{tikzcd}
w \arrow[r, mapsto] \arrow[d, "\Psi_{\mathcal{A}}"'] & w\,a \arrow[d, "\Psi_{\mathcal{A}}"] \\
x \arrow[r, mapsto] & x \cdot \mu(a)
\end{tikzcd}$$

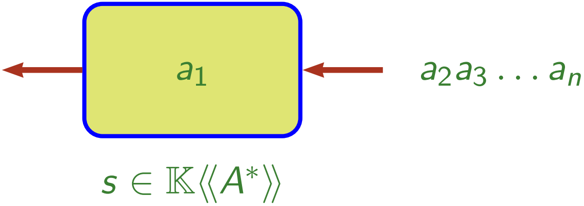
The control morphism is a morphism of actions

A basic construct: the quotient of series



The input belongs to a free monoid A^*

A basic construct: the quotient of series



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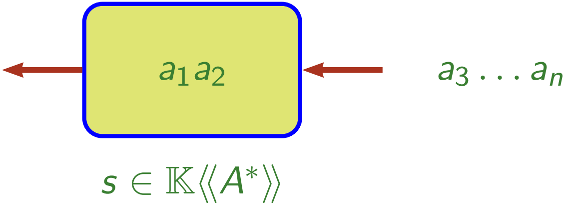

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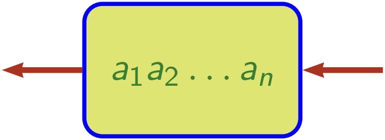

$s \in \mathbb{K}\langle\langle A^* \rangle\rangle$

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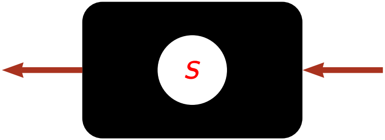
A basic construct: the quotient of series

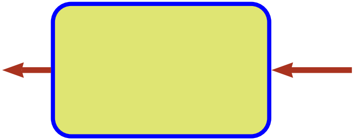


$$s \in \mathbb{K}\langle\langle A^* \rangle\rangle$$

The input belongs to a free monoid A^*

A basic construct: the quotient of series

$$\langle s, a_1 \ldots a_n \rangle = k$$



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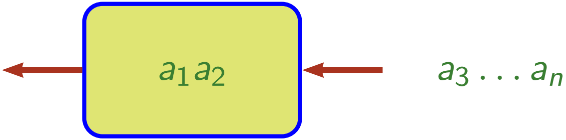

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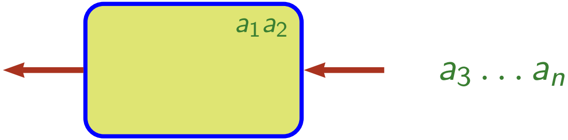

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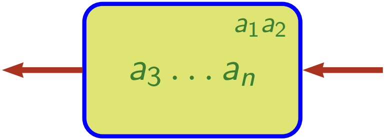

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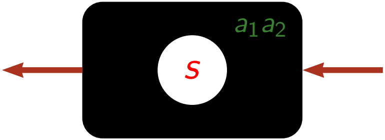
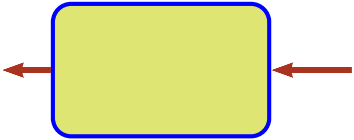


$$s \in \mathbb{K}\langle\langle A^* \rangle\rangle$$

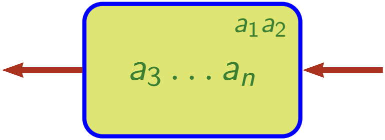
A basic construct: the quotient of series

$$\langle s, a_1 \ldots a_n \rangle = k$$



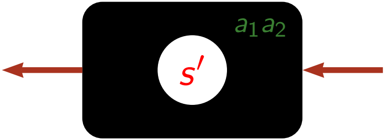
$$s \in \mathbb{K}\langle\langle A^* \rangle\rangle$$

A basic construct: the quotient of series



$s' \in \mathbb{K}\langle\langle A^* \rangle\rangle$

A basic construct: the quotient of series



$$s' \in \mathbb{K}\langle\langle A^* \rangle\rangle$$

A basic construct: the quotient of series

$$k = \langle s', a_3 \ldots a_n \rangle = \langle s, a_1 a_2 a_3 \ldots a_n \rangle$$

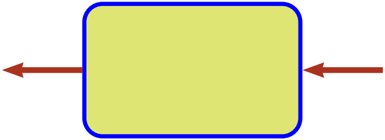
A basic construct: the quotient of series

$$k = \langle s', a_3 \ldots a_n \rangle = \langle s, a_1 a_2 a_3 \ldots a_n \rangle$$

$$s' = [a_1 a_2]^{-1} s$$

The series s' is *the quotient* of s by $a_1 a_2$

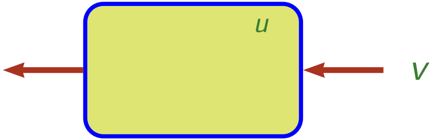
A basic construct: the quotient of series



$u\,v$

$s \in \mathbb{K}\langle\langle A^* \rangle\rangle$

A basic construct: the quotient of series



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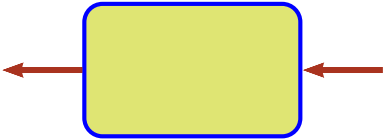
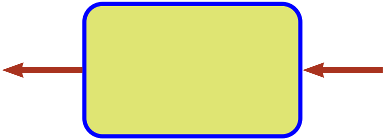


$$k = \langle s', v \rangle = \langle s, u\, v \rangle$$

A basic construct: the quotient of series



$$k = \langle s', v \rangle = \langle s, u\,v \rangle$$

$$s' = u^{-1} s$$

The series s' is *the quotient* of s by u

The quotient operation

$$s \in \mathbb{K}\langle\langle A^* \rangle\rangle$$

$$v \in A^* \qquad v^{-1}\, s = \sum_{w \in A^*} \langle s, v\, w \rangle\, w$$

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$v^{-1} \colon \mathbb{K}\langle\langle A^* \rangle\rangle \longrightarrow \mathbb{K}\langle\langle A^* \rangle\rangle$ endomorphism of $\mathbb{K}$-modules

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$$v^{-1}(s + t) = v^{-1}\, s + v^{-1}\, t \qquad v^{-1}(k\, s) = k\,(v^{-1}\, s)$$

The quotient operation

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$$\mathbb{K}\langle\langle A^* \rangle\rangle \overset{A^*}{\Longrightarrow} \mathbb{K}\langle\langle A^* \rangle\rangle \qquad s \longmapsto v^{-1}s$$

Quotient is a (right) action of A^* on $\mathbb{K}\langle\langle A^* \rangle\rangle$

The quotient operation

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Quotient is a (right) action of A^* on $\mathbb{K}\langle\langle A^* \rangle\rangle$

$$(u\, v)^{-1}\, s = v^{-1}\, (u^{-1}\, s)$$

The minimal automaton

$$s \in \mathbb{K}\langle\langle A^* \rangle\rangle$$

$$\mathbf{R}_s = \left\{ v^{-1}\, s \,\middle|\, v \in A^* \right\}$$

The minimal automaton

$$s \in \mathbb{K}\langle\langle A^* \rangle\rangle$$

$$\mathbf{R}_s = \left\{ v^{-1}\, s \,\middle|\, v \in A^* \right\}$$

Quotient turns

$\mathbf{R}_s$ into the minimal automaton $\mathcal{A}_s$ of s

(possibly infinite)

The observation morphism

$$\mathcal{A} = (\, I, \mu, T \,)$$

$$\Phi_{\mathcal{A}} \colon \mathbb{K}^{Q} \longrightarrow \mathbb{K}\langle\langle A^* \rangle\rangle \qquad\qquad \Phi_{\mathcal{A}}(x) = \left|(\, x, \mu, T \,)\right| = \sum_{w \in A^*} (x \cdot \mu(w) \cdot T)\, w$$

The observation morphism

$$\mathcal{A} = (\, I, \mu, T \,)$$

$$\Phi_{\mathcal{A}} \colon \mathbb{K}^{Q} \longrightarrow \mathbb{K}\langle\langle A^* \rangle\rangle \qquad \Phi_{\mathcal{A}}(x) = \left|(\, x, \mu, T \,)\right| = \sum_{w \in A^*} (x \cdot \mu(w) \cdot T)\, w$$

$$s = \left|(\, I, \mu, T \,)\right| = \Phi_{\mathcal{A}}(I) \qquad w^{-1}s = \left|(\, I \cdot \mu(w), \mu, T \,)\right|$$

The observation morphism

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$$\begin{array}{c} \mathbb{K}^{Q} \\ \Phi_{\mathcal{A}} \big\downarrow \\ \mathbb{K}\langle\!\langle A^* \rangle\!\rangle \end{array} \qquad\qquad \begin{array}{c} x \\ \Phi_{\mathcal{A}} \big\downarrow \\ t \end{array}$$

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$$\begin{tikzcd}
\mathbb{K}^Q \arrow[r, Rightarrow, "A^*"] \arrow[d, "\Phi_{\mathcal{A}}"'] & \mathbb{K}^Q \arrow[d, "\Phi_{\mathcal{A}}"] \\
\mathbb{K}\langle\langle A^* \rangle\rangle \arrow[r, Rightarrow, "A^*"] & \mathbb{K}\langle\langle A^* \rangle\rangle
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
x \arrow[r, mapsto] \arrow[d, "\Phi_{\mathcal{A}}"'] & x \cdot \mu(a) \arrow[d, "\Phi_{\mathcal{A}}"] \\
t \arrow[r, mapsto] & a^{-1} t
\end{tikzcd}$$

The observation morphism is a morphism of actions

The observation morphism

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\mathbb{K}\langle A^* \rangle \arrow[r, Rightarrow, "A^*"] \arrow[d, "\Psi_{\mathcal{A}}"'] & \mathbb{K}\langle A^* \rangle \arrow[d, "\Psi_{\mathcal{A}}"] \\
\mathbb{K}^Q \arrow[r, Rightarrow, "A^*"] \arrow[d, "\Phi_{\mathcal{A}}"'] & \mathbb{K}^Q \arrow[d, "\Phi_{\mathcal{A}}"] \\
\mathbb{K}\langle\langle A^* \rangle\rangle \arrow[r, Rightarrow, "A^*"] & \mathbb{K}\langle\langle A^* \rangle\rangle
\end{tikzcd}
\qquad
\begin{tikzcd}
w \arrow[r, mapsto] \arrow[d, "\Psi_{\mathcal{A}}"'] & w\,a \arrow[d, "\Psi_{\mathcal{A}}"] \\
x \arrow[r, mapsto] \arrow[d, "\Phi_{\mathcal{A}}"'] & x \cdot \mu(a) \arrow[d, "\Phi_{\mathcal{A}}"] \\
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The observation morphism is a morphism of actions

The representation theorem

$U \subseteq \mathbb{K}\langle\langle A^* \rangle\rangle$ submodule $\qquad U$ stable (by quotient)

Theorem (Fliess 71, Jacob 74)

$s \in \mathbb{K}\mathrm{Rec}\, A^* \iff \exists U$ *stable* *finitely generated* $s \in U$

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\mathbb{K}\langle\!\langle A^* \rangle\!\rangle \arrow[r, Rightarrow, "A^*"] & \mathbb{K}\langle\!\langle A^* \rangle\!\rangle
\end{tikzcd}
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$$\begin{tikzcd}
1_{A^*} \in & \mathbb{K}\langle A^* \rangle \arrow[r, Rightarrow, "A^*"] \arrow[d, "\Psi_{\mathcal{A}}"'] & \mathbb{K}\langle A^* \rangle \arrow[d, "\Psi_{\mathcal{A}}"] \\
I \in \mathrm{Im}\,\Psi_{\mathcal{A}} & \mathbb{K}^Q \arrow[r, Rightarrow, "A^*"] \arrow[d, "\Phi_{\mathcal{A}}"'] & \mathbb{K}^Q \arrow[d, "\Phi_{\mathcal{A}}"] \\
s \in \Phi_{\mathcal{A}}(\mathrm{Im}\,\Psi_{\mathcal{A}}) & \mathbb{K}\langle\langle A^* \rangle\rangle \arrow[r, Rightarrow, "A^*"] & \mathbb{K}\langle\langle A^* \rangle\rangle
\end{tikzcd}$$

The representation theorem

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\mathbb{K}^Q \arrow[r, Rightarrow, "A^*"] \arrow[d, "\Phi_{\mathcal{A}}"'] & \mathbb{K}^Q \arrow[d, "\Phi_{\mathcal{A}}"] \\
\mathbb{K}\langle\langle A^* \rangle\rangle \arrow[r, Rightarrow, "A^*"] & \mathbb{K}\langle\langle A^* \rangle\rangle
\end{tikzcd}$$

Lecture IV

Reduction and morphisms

Outline of Lecture IV

- An appetizing theorem
- Reduction of automata with weights in fields
- The decidability of equivalence problem
- The notion of conjugacy of automata
- Out-morphisms and In-morphisms of automata

An appetizing result

$\mathbb{K}$ semiring $\qquad\qquad$ A^* free monoid

Definition

The *Hadamard product* of $s, t \in \mathbb{K}\langle\langle A^* \rangle\rangle$ is

$$\forall w \in A^* \qquad \langle s \odot t, w \rangle = \langle s, w \rangle \langle t, w \rangle$$

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Theorem

If $\mathbb{K}$ *is commutative,*

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Theorem

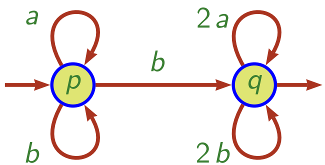
If $\mathbb{K}$ *is commutative,*

then $\mathbb{K}\mathrm{Rec}\, A^*$ *is closed under Hadamard product*

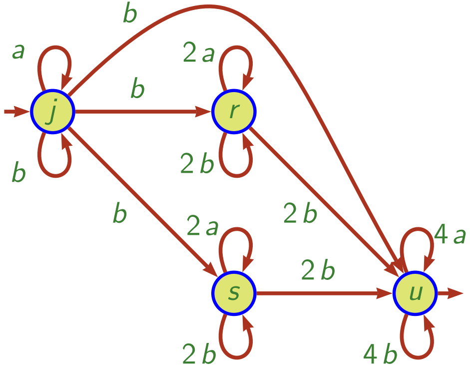
$$|(I, \mu, T) \odot (J, \kappa, U)| = |(I \otimes J, \mu \otimes \kappa, T \otimes U)|$$

An appetizing result

$\mathcal{C}_1$



$\mathcal{C}_2$



$\mathcal{C}_2 = \mathcal{C}_1 \otimes \mathcal{C}_1$

Reduced representation

$$\mathcal{A} = (I, \mu, T)$$

$\mathcal{A}$ is *reduced* if its *dimension* is minimal

(among all equivalent representations)

We suppose now that $\mathbb{K}$ is a (skew) field

Proposition

$\mathcal{A}$ is reduced iff $\Psi_{\mathcal{A}}$ is surjective and $\Phi_{\mathcal{A}}$ injective

Theorem

A reduced representation of $|\mathcal{A}|$ is effectively computable
(with cubic complexity)

Corollary

Equivalence of $\mathbb{K}$-recognisable series is decidable

Equivalence of weighted automata

Equivalence of weighted automata with weights in

the Boolean semiring $\mathbb{B}$	decidable
a subsemiring of a field	decidable
$(\mathbb{Z}, \min, +)$	undecidable
$\mathrm{Rat}\, B^*$	undecidable
$\mathbb{N}\mathrm{Rat}\, B^*$	decidable

Equivalence of weighted automata

Equivalence of weighted automata with weights in

the Boolean semiring $\mathbb{B}$	decidable
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Equivalence of

transducers	undecidable
transducers with multiplicity in $\mathbb{N}$	decidable
functional transducers	decidable
finitely ambiguous $(\mathbb{Z}, \mathsf{min}, +)$	decidable

Conjugacy of automata

Definition

Let $\mathcal{A} = \langle\, I, E, T\,\rangle$ and $\mathcal{B} = \langle\, J, F, U\,\rangle$ be two $\mathbb{K}$-automata.

$\mathcal{A}$ is conjugate to $\mathcal{B}$ if

$\exists X$ $\mathbb{K}$-matrix $I\,X = J,$ $E\,X = X\,F,$ and $T = X\,U$

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Conjugacy of automata

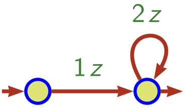
$$\mathcal{C}' = \left\langle \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & z & 0 \\ 0 & 0 & z \\ 0 & 0 & 2z \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\rangle \qquad \mathcal{A}' = \left\langle \begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & z \\ 0 & 2z \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$$

$$\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix},$$

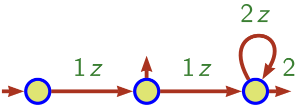
$$\begin{pmatrix} 0 & z & 0 \\ 0 & 0 & z \\ 0 & 0 & 2z \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 0 & z \\ 0 & 2z \end{pmatrix},$$

$$\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\mathcal{A}'$



$$\overset{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{pmatrix}}{\Longleftarrow}$$



$\mathcal{C}'$

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$I\;E\;E\;T$

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$I\,E\,E\,T = I\,E\,E\,X\,U$

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$$I\,E\,E\,T = I\,E\,E\,X\,U = I\,E\,X\,F\,U = I\,X\,F\,F\,U = J\,F\,F\,U$$

and then $\quad I\,E^*\,T = J\,F^*\,U$

Morphisms of weighted automata

Definition

A map $\varphi\colon Q \to R$ defines a $(Q \times R)$-amalgamation matrix H_φ

$$\varphi_2\colon \{j, r, s, u\} \to \{i, q, t\} \qquad \text{defines} \qquad H_{\varphi_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Morphisms of weighted automata

Definition

$\mathcal{A} = \langle\, I, E, T \,\rangle$ and $\mathcal{B} = \langle\, J, F, U \,\rangle$ $\mathbb{K}$-automata of dimension Q and R.

A map $\varphi\colon Q \to R$ defines an Out-morphism $\varphi\colon \mathcal{A} \to \mathcal{B}$

if $\mathcal{A}$ is conjugate to $\mathcal{B}$ by the matrix H_φ : $\mathcal{A} \overset{H_\varphi}{\Longrightarrow} \mathcal{B}$

$$I\, H_\varphi = J, \qquad E\, H_\varphi = H_\varphi\, F, \qquad T = H_\varphi\, U$$

$\mathcal{B}$ is a quotient of $\mathcal{A}$

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Directed notion

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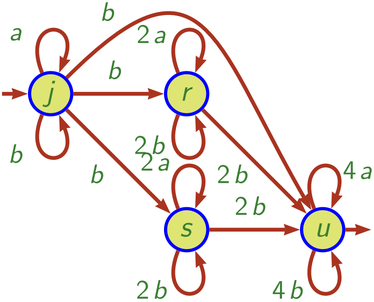
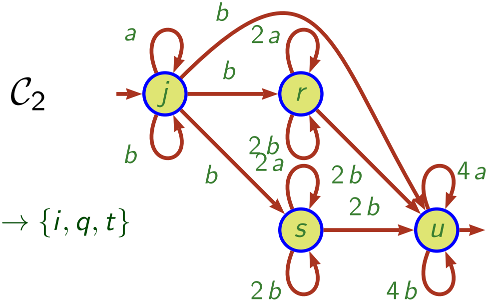
Directed notion Price to pay for the weight

Morphisms of weighted automata

$\mathcal{C}_2$



Morphisms of weighted automata

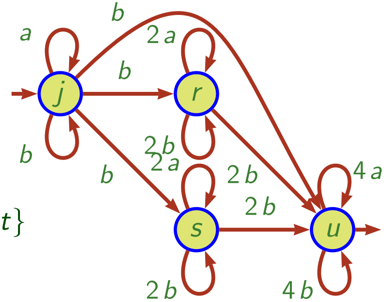


$\varphi_2 \colon \{j, r, s, u\} \to \{i, q, t\}$

$$H_{\varphi_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Morphisms of weighted automata

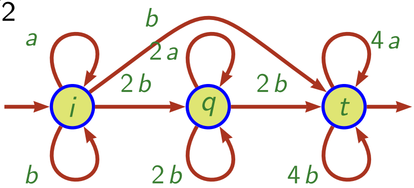
$\mathcal{C}_2$



$\varphi_2 \colon \{j, r, s, u\} \to \{i, q, t\}$

$$H_{\varphi_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\mathcal{V}_2$



$$\mathcal{C}_2 \xRightarrow{H_{\varphi_2}} \mathcal{V}_2$$

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Directed notion

Price to pay for the weight

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Directed notion Price to pay for the weight

Morphisms of weighted automata

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A map $\varphi\colon Q \to R$ defines an In-morphism $\varphi\colon \mathcal{A} \to \mathcal{B}$

if $\mathcal{B}$ is conjugate to $\mathcal{A}$ by the matrix ${}^{\mathrm{t}}H_\varphi$: $\mathcal{B} \stackrel{{}^{\mathrm{t}}H_\varphi}{\Longrightarrow} \mathcal{A}$

$$J \; {}^{\mathrm{t}}H_\varphi = I, \qquad F \; {}^{\mathrm{t}}H_\varphi = {}^{\mathrm{t}}H_\varphi \; E, \qquad U = {}^{\mathrm{t}}H_\varphi \; T$$

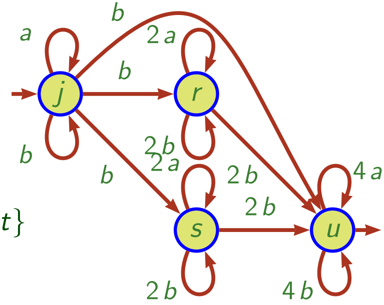
$\mathcal{B}$ is a co-quotient of $\mathcal{A}$

Directed notion

Price to pay for the weight

Morphisms of weighted automata

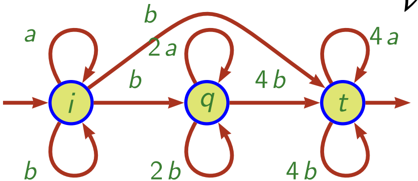
$\mathcal{C}_2$



$\varphi_2 \colon \{j, r, s, u\} \to \{i, q, t\}$

$$H_{\varphi_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

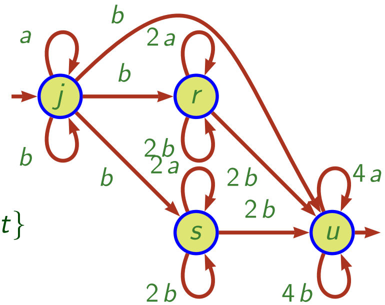
$\mathcal{V}_2'$



$$\mathcal{V}_2' \xRightarrow{{}^{\mathrm{t}}H_{\varphi_2}} \mathcal{C}_2$$

Morphisms of weighted automata

$\mathcal{C}_2$



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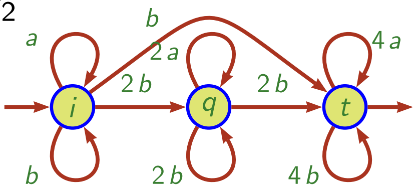
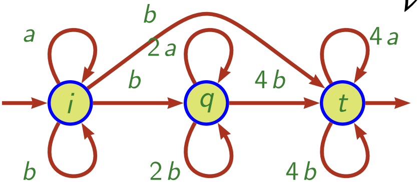


$\mathcal{V}'_2$



$\mathcal{C}_2 \xrightarrow{H_{\varphi_2}} \mathcal{V}_2$ $\qquad$ $\mathcal{V}'_2 \xrightarrow{{}^{\mathrm{t}}H_{\varphi_2}} \mathcal{C}_2$

Morphisms of weighted automata

Definition

$\mathcal{A} = \langle I, E, T \rangle$ and $\mathcal{B} = \langle J, F, U \rangle$ $\mathbb{K}$-automata
of dimension Q and R.

A map $\varphi\colon Q \to R$ defines an Out-morphism $\varphi\colon \mathcal{A} \to \mathcal{B}$

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$\mathcal{B}$ is a quotient of $\mathcal{A}$

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Theorem
Every $\mathbb{K}$-automaton has a minimal quotient that is effectively computable (by Moore algorithm).

Documents for these lectures

To be found at

`http://www.telecom-paristech.fr/~jsaka/EWSCS2014/`

In particular, a set of instructions for downloading

a $-\alpha$ release of a pre-experimental version of

the VAUCANSON 2 platform

implemented as a virtual machine interfaced with `IPython`