Estimation of the closest occurrence of a rare event

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We show how to estimate the distance d to the closest occurrence of a uniformly-distributed event in a 2D-space. Suppose that N events occur in an egocentred circular area of radius r.

$\Pr(d > x) = (1 - x^2/r^2)^N$

$E(d) = \int_0^r \Pr(d > x))$ (by Fubini's theorem)

$E(d) = \int_0^r (1 - x^2/R^2)^N \, dx$

Let $I_N = \int_0^1 (1 - x^2)^N \, dx$. We have $r \times I_N = E(d)$

Integrating by parts, we get:

$I_N = \int_0^1 -2Nx^2(1 - x^2)^{N-1} \, dx$

$I_N = 2N\, I_N - 2N\, I_{N-1}$

$I_N = I_{N-1}\; 2N/(2N+1)$

$I_N = (2\times4\times6\times\ldots\times2N) / (1\times3\times5\times\ldots\times(2N+1))$

$I_N = (2^N\, N!)^2 / (2N+1)!$

Applying Stirling's formula: $n! \sim (n/e)^n \sqrt{(2\pi n)}$ gives:

$I_N \approx 2\pi N\, 2^{2N}\, (N/e)^{2N} / (((2N+1)/e)^{2N+1}\sqrt{(2\pi(2N+1))})$

$I_N \approx \sqrt{\pi}/(2\sqrt{N})$, considering that $(1+1/(2N))^{2N+1} \approx e$.

If $D = N / (\pi r^2)$ designates the spatial density of events, then: $E(d) \sim 1/(2\sqrt{D})$

If R designates the closest observed occurrence of a uniformly-distributed event, then estimating the density of these events by $1/(\pi R^2)$ provides a good approximation of D.

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