Generation of bounded invariants via stroboscopic set-valued maps

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Friday 11th December, 2020

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2 Problematic and description of the method

3 Euler's method and error bounds

4 Systems with bounded uncertainty

5 Van der Pol example

6 Conclusion and Perspectives

Motivation

- Dynamical systems:
  - in which a function describes the time dependence of a point in a geometrical space.
  - we only know certain observed or calculated states of its past or present state.
  - dynamical systems have a direct impact on human development.

$\Rightarrow$ The importance of studying:

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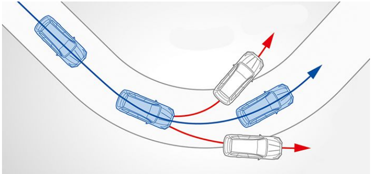
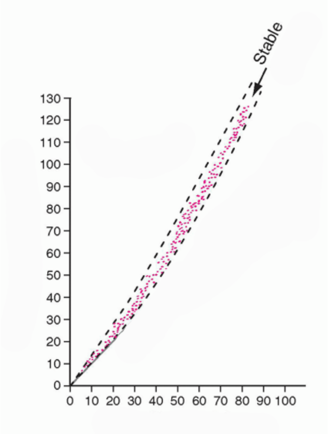


Electronic Stability Control (ESC)

Solar System

Stability

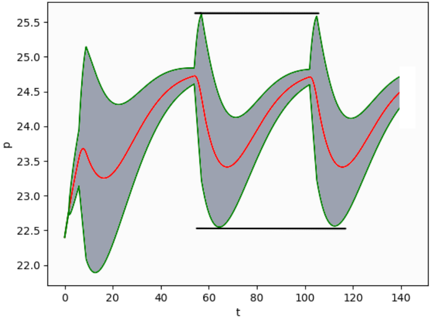
- A dynamical system is stable, if small perturbations to the solution lead to a new solution that stays close to the original solution forever.
- A stable system produces a bounded output for a given bounded input.



Stability

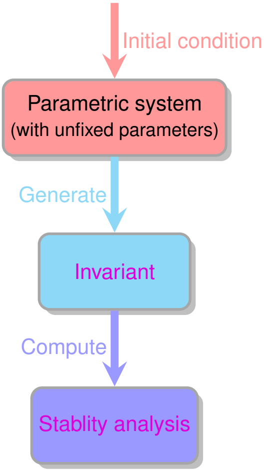
An invariant

- The bounded output of some periodic stable system can be considered as an invariant from certain t.
- An invariant is an unchanged object after operations applied to it.



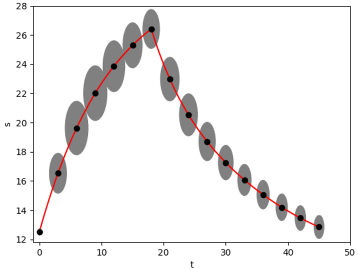
Invariant

Problematic



Description of the method

- Given a differential system $\Sigma : dx/dt = f(x)$ of dimension n, an initial point $x_0 \in \mathbb{R}^n$, a real $\varepsilon > 0$, and a ball $B_0 = B(x_0, \varepsilon)$[1]



- The center of each ball at time t is the Euler approximate solution $\tilde{x}(t)$ of the system starting at x_0, and the radius is a function $\delta_\varepsilon(t)$ bounding the distance between $\tilde{x}(t)$ and an exact solution $x(t)$ starting at B_0.

[1] $B(x_0, \varepsilon)$ is the set $\{z \in \mathbb{R}^n \mid \|z - x_0\| \leq \varepsilon\}$ where $\|\cdot\|$ denotes the Euclidean distance.

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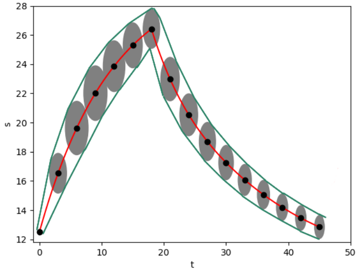

- The tube can be described as $\bigcup_{t \geq 0} B(t)$ where $B(t) \equiv B(\tilde{x}(t), \delta_\varepsilon(t))$.

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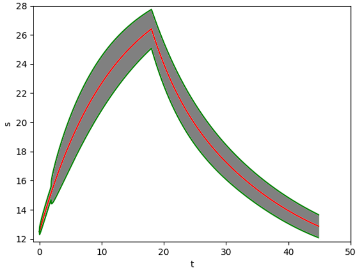


- To find a *bounded* invariant, we look for a positive real T such that $B((i+1)T) \subseteq B(iT)$ for some $i \in \mathbb{N}$. In case of success, the ball $B(iT)$ is guaranteed to contain the "stroboscopic" sequence $\{B(jT)\}_{j=i,i+1,...}$ of sets $B(t)$ at time $t = iT, (i+1)T, \ldots$. and thus constitutes the sought bounded invariant set.

[1] $B(x_0, \varepsilon)$ is the set $\{z \in \mathbb{R}^n \mid \|z - x_0\| \leq \varepsilon\}$ where $\|\cdot\|$ denotes the Euclidean distance.

Euler's method and error bounds

Let us consider the differential system:

$$\frac{dx(t)}{dt} = f(x(t)),$$

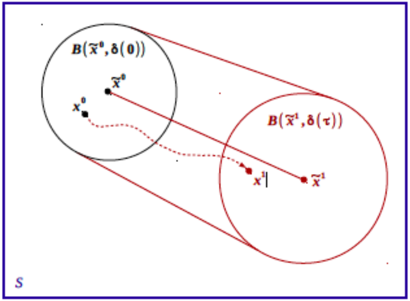
with states $x(t) \in \mathbb{R}^n$ and x_0 a given initial condition.

- $\tilde{x}(t; y_0)$ denotes Euler's approximate value of $x(t)$ (defined by $\tilde{x}(t; y_0) = y_0 + t \times f(y_0)$ for $t \in [0, \tau]$, where τ is the integration time-step).

Proposition

[LCDVCF17] Consider the solution $x(t; y_0)$ of $\frac{dx}{dt} = f(x)$ with initial condition y_0 and the approximate Euler solution $\tilde{x}(t; x_0)$ with initial condition x_0. For all $y_0 \in B(x_0, \varepsilon)$, we have:

$$\|x(t; y_0) - \tilde{x}(t; x_0)\| \leq \delta_\varepsilon(t).$$



[LCDVCF17] A. Le Coënt et al., "Control synthesis of nonlinear sampled switched systems using Euler's method," in SNR, (Apr. 22, 2017), ser. EPTCS, vol. 247, Uppsala, Sweden, 2017, pp. 18–33. DOI:

Definition

$\delta_\varepsilon(t)$ is defined as follows for $t \in [0, \tau]$:
if $\lambda < 0$:

$$\delta_\varepsilon(t) = \left(\varepsilon^2 e^{\lambda t} + \frac{C^2}{\lambda^2}\left(t^2 + \frac{2t}{\lambda} + \frac{2}{\lambda^2}\left(1 - e^{\lambda t}\right)\right)\right)^{\frac{1}{2}}$$

if $\lambda = 0$:

$$\delta_\varepsilon(t) = \left(\varepsilon^2 e^t + C^2(-t^2 - 2t + 2(e^t - 1))\right)^{\frac{1}{2}}$$

if $\lambda > 0$:

$$\delta_\varepsilon(t) = \left(\varepsilon^2 e^{3\lambda t} + \frac{C^2}{3\lambda^2}\left(-t^2 - \frac{2t}{3\lambda} + \frac{2}{9\lambda^2}\left(e^{3\lambda t} - 1\right)\right)\right)^{\frac{1}{2}}$$

where C and λ are real constants specific to function f, defined as follows:

$$C = \sup_{y \in \mathcal{S}} L\|f(y)\|,$$

Definition

L denotes the Lipschitz constant for f, and λ is the "one-sided Lipschitz constant" (or "logarithmic Lipschitz constant" [AS14]) associated to f, i. e., the minimal constant such that, for all $y_1, y_2 \in \mathcal{S}$:

$$\langle f(y_1) - f(y_2), y_1 - y_2 \rangle \leq \lambda \|y_1 - y_2\|^2, \qquad (H0)$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product of two vectors of $\mathcal{S}$.

The constant λ can be computed using a nonlinear optimization solver (e. g., CPLEX [Cpl09]) or using the Jacobian matrix of f.

[AS14] Z. Aminzare and E. D. Sontag, "Contraction methods for nonlinear systems: A brief introduction and some open problems," in **53rd IEEE Conference on Decision and Control, CDC 2014, Los Angeles, CA, USA, December 15-17, 2014**, 2014, pp. 3835–3847.

[Cpl09] I. I. Cplex, "V12. 1: User's manual for cplex," **International Business Machines Corporation**, vol. 46, no. 53, p. 157, 2009.

Systems with bounded uncertainty

A differential system with bounded uncertainty is of the form

$$\frac{dx(t)}{dt} = f(x(t), w(t)),$$

with $t \in \mathbb{R}^n_{\geq 0}$, states $x(t) \in \mathbb{R}^n$, and uncertainty $w(t) \in \mathcal{W} \subset \mathbb{R}^n$ ($\mathcal{W}$ is compact, i. e., closed and bounded).

- We suppose (see [LCADSC+17]) that there exist constants $\lambda \in \mathbb{R}$ and $\gamma \in \mathbb{R}_{\geq 0}$ such that, for all $y_1, y_2 \in \mathcal{S}$ and $w_1, w_2 \in \mathcal{W}$:

$$\langle f(y_1, w_1) - f(y_2, w_2), y_1 - y_2 \rangle \leq \lambda \|y_1 - y_2\|^2 + \gamma \|y_1 - y_2\| \|w_1 - w_2\| \quad (H1).$$

- Instead of computing λ and γ globally for $\mathcal{S}$, it is advantageous to compute them *locally* depending on the subregion of $\mathcal{S}$ occupied by the system state during a considered interval of time.

[LCADSC+17] A. Le Coënt **et al.**, "Distributed control synthesis using Euler's method," in **Proc. of International Workshop on Reachability Problems (RP'17)**, ser. Lecture Notes in Computer Science, vol. 247, Springer, 2017, pp. 118–131.

Proposition

$\delta_\varepsilon(t)$ is defined as follows for $t \in [0,\tau]$:

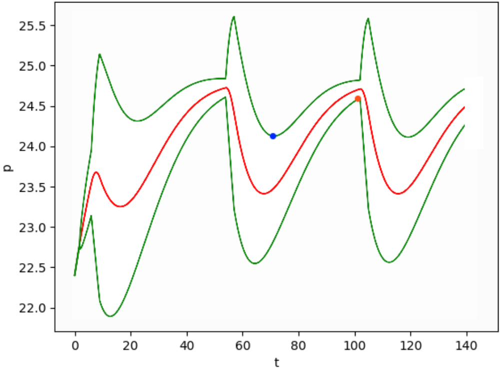
$$\textit{if } \lambda < 0: \quad \delta_{\varepsilon,\mathcal{W}}(t) = \left(\frac{C^2}{-\lambda^4}\left(-\lambda^2 t^2 - 2\lambda t + 2e^{\lambda t} - 2\right) + \frac{1}{\lambda^2}\left(\frac{C\gamma|\mathcal{W}|}{-\lambda}\left(-\lambda t + e^{\lambda t} - 1\right) + \lambda\left(\frac{\gamma^2(|\mathcal{W}|/2)^2}{-\lambda}(e^{\lambda t} - 1) + \lambda\varepsilon^2 e^{\lambda t}\right)\right)\right)^{1/2} \tag{1}$$

$$\textit{if } \lambda > 0: \quad \delta_{\varepsilon,\mathcal{W}}(t) = \frac{1}{(3\lambda)^{3/2}}\left(\frac{C^2}{\lambda}\left(-9\lambda^2 t^2 - 6\lambda t + 2e^{3\lambda t} - 2\right) + 3\lambda\left(\frac{C\gamma|\mathcal{W}|}{\lambda}\left(-3\lambda t + e^{3\lambda t} - 1\right) + 3\lambda\left(\frac{\gamma^2(|\mathcal{W}|/2)^2}{\lambda}(e^{3\lambda t} - 1) + 3\lambda\varepsilon^2 e^{3\lambda t}\right)\right)\right)^{1/2} \tag{2}$$

$$\textit{if } \lambda = 0: \quad \delta_{\varepsilon,\mathcal{W}}(t) = \left(C^2\left(-t^2 - 2t + 2e^t - 2\right) + \left(C\gamma|\mathcal{W}|\left(-t + e^t - 1\right) + \left(\gamma^2(|\mathcal{W}|/2)^2(e^t - 1) + \varepsilon^2 e^t\right)\right)\right)^{1/2} \tag{3}$$

Proposition

Suppose that, for some index $1 \leq j \leq n$, we have $m^j_+ < M^j_-$ where m^j_+ (resp. M^j_-) denotes the minimum (resp. maximum) of $\tilde{x}^j(t) + \delta_{\varepsilon,\mathcal{W}}(t)$ (resp. $\tilde{x}^j(t) - \delta_{\varepsilon,\mathcal{W}}(t)$) for $t \in [iT,(i+1)T]$. Then $B[iT,(i+1)T]$ contains no fixed point of Σ'.



Van der Pol System

Consider the Van der Pol (VdP) system Σ_p of dimension $n = 2$ with parameter $p \in \mathbb{R}$, and initial condition in $B_0 = B(x_0, \varepsilon)$ for some $x_0 \in \mathbb{R}^2$ and $\varepsilon > 0$ (see [BQ20]):

$$\begin{cases} \dfrac{du_1}{dt} = u_2 \\ \dfrac{du_2}{dt} = pu_2 - pu_1^2 u_2 - u_1 \end{cases} \tag{4}$$

[BQ20] J. B. van den Berg and E. Queirolo, "A general framework for validated continuation of periodic orbits in systems of polynomial ODEs," **Journal of Computational Dynamics**, vol. 0, no. 2158-2491-2019-0-10, 2020, ISSN: 2158-2491. DOI: 10.3934/jcd.2021004.

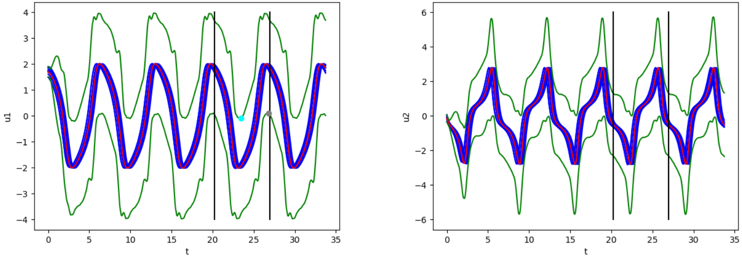
Van der Pol System with uncertainty

Consider now the system Σ' with uncertainty $w(\cdot) \in \mathcal{W}_0 = [-0.5, 0.5]$ and initial condition x_0:

$$\begin{cases} \dfrac{du_1}{dt} = u_2 \\ \dfrac{du_2}{dt} = (p_0 + w)u_2 - (p_0 + w)u_1^2 u_2 - u_1 \end{cases} \tag{5}$$

with $p_0 = 1.1$. It is easy to see that each solution of Σ_p with $p \in [p_0 - 0.5, p_0 + 0.5] = [0.6, 1.6]$ is a particular solution of system Σ'.

Van der Pol System with uncertainty



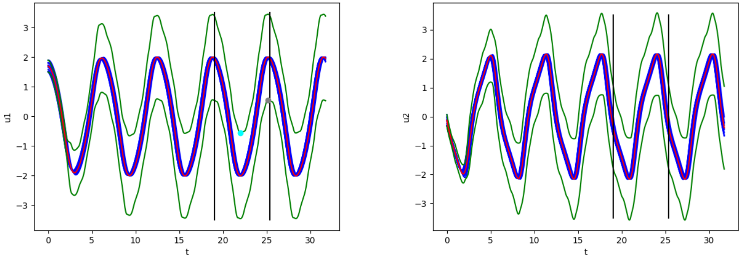
VdP system with parameter $p_0 = 1.1$, uncertainty $|\mathcal{W}_0| = 0.5$, initial radius $\varepsilon_0 = 0.2$, initial point $x_0 = (1.7018, -0.1284)$, period $T_0 = 6.746$, time-step $\tau = 10^{-3}$.

- We have: $B((i_0+1)T_0) \subset B(i_0 T_0)$ for $i_0 = 3$.
- The minimum m^1_+ of the upper green curve $\tilde{u}_1(t) + \delta_{\mathcal{W}}(t)$ is less than the maximum M^1_- of the lower green curve $\tilde{u}_1(t) - \delta_{\mathcal{W}}(t)$.
- Whatever the value of $p \in [p_0 - |\mathcal{W}_0|, p_0 + |\mathcal{W}_0|] = [0.6, 1.6]$, the solution of Σ_p never converges to a point of $\mathbb{R}^n$.
- Since the size of the system is $n = 2$, it follows by Poincaré-Bendixson's theorem that the solution of Σ_p converges always towards a limit circle

Consider now the system Σ' with uncertainty $w(\cdot) \in \mathcal{W}_1 = [-0.2, 0.2]$ and initial condition x_0:

$$\begin{cases} \dfrac{du_1}{dt} = u_2 \\ \dfrac{du_2}{dt} = (p_1 + w)u_2 - (p_1 + w)u_1^2 u_2 - u_1 \end{cases} \tag{5}$$

with $p_1 = 0.4$. It is easy to see that each solution of Σ_p with $p \in [p_1 - 0.2, p_1 + 0.2] = [0.2, 0.6]$ is a particular solution of system Σ'.



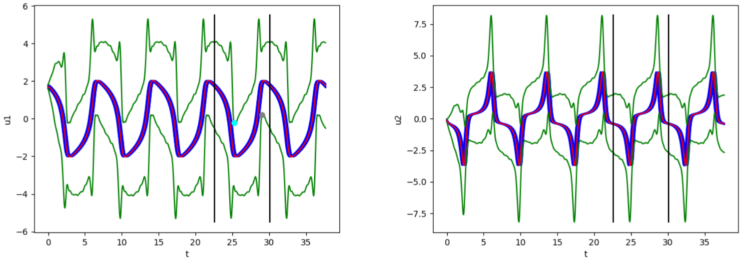
VdP system with parameter $p_1 = 0.4$, uncertainty $|\mathcal{W}_1| = 0.2$, initial radius $\varepsilon_1 = 0.2$, initial point $x_0 = (1.7018, -0.1284)$, period $T_1 = 6.347$, time-step $\tau = 10^{-3}$.

- We have: $B((i_1+1)T_1) \subset B(i_1 T_1)$ for $i_1 = 3$.
- We have $m_+^1 < M_-^1$, this shows that whatever the value of $p \in [p_1 - |\mathcal{W}_1|, p_1 + |\mathcal{W}_1|] = [0.2, 0.6]$, the solution of Σ_p never converges to a point of $\mathbb{R}^n$.
- It follows by Poincaré-Bendixson's theorem that the solution of Σ_p converges always towards a limit circle for any $p \in [0.2, 0.6]$ and initial condition in $B(x_0, \varepsilon_1)$.

Consider now the system Σ' with uncertainty $w(\cdot) \in \mathcal{W}_2 = [-0.3, 0.3]$ and initial condition x_0:

$$\begin{cases} \frac{du_1}{dt} = u_2 \\ \frac{du_2}{dt} = (p_2 + w)u_2 - (p_2 + w)u_1^2 u_2 - u_1 \end{cases} \quad (5)$$

with $p_2 = 1.9$. It is easy to see that each solution of Σ_p with $p \in [p_2 - 0.3, p_2 + 0.3] = [1.6, 2.2]$ is a particular solution of system Σ'.



VdP system with parameter $p_2 = 1.9$, uncertainty $|\mathcal{W}_2| = 0.3$, initial radius $\varepsilon_2 = 0.1$, initial point $x_0 = (1.7018, -0.1284)$, period $T_2 = 7.531$, time-step $\tau = 10^{-3}$.

- We have: $B((i_2+1)T_2) \subset B(i_2 T_2)$ for $i_2 = 3$.
- We have $m_+^1 < M_-^1$, then whatever the value of $p \in [p_2 - |\mathcal{W}_2|, p_2 + |\mathcal{W}_2|] = [1.6, 2.2]$, the solution of Σ_p never converges to a point of $\mathbb{R}^n$.
- It follows by Poincaré-Bendixson's theorem that the solution of Σ_p converges always towards a limit circle for any $p \in [1.6, 2.2]$ and initial condition in $B(x_0, \varepsilon_2)$.

Conclusion and Perspectives

Conclusion

- We presented a simple method to generate a bounded invariant for a differential system.
- The method shows that the solutions never converge to an equilibrium point for a parameterized differential system.
- The method uses a very general criterion of inclusion of one set in another.

Perspectives

- Adapt the method to solve the convergence to a limit cycle for complex systems.
- Extend our method in order to account for such an analysis.

Z. Aminzare and E. D. Sontag, "Contraction methods for nonlinear systems: A brief introduction and some open problems," in **53rd IEEE Conference on Decision and Control, CDC 2014, Los Angeles, CA, USA, December 15-17, 2014**, 2014, pp. 3835–3847.

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A. Le Coënt, J. Alexandre Dit Sandretto, A. Chapoutot, L. Fribourg, F. De Vuyst, and L. Chamoin, "Distributed control synthesis using Euler's method," in **Proc. of International Workshop on Reachability Problems (RP'17)**, ser. Lecture Notes in Computer Science, vol. 247, Springer, 2017, pp. 118–131.

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