Robust optimal periodic control using guaranteed Euler's method

Jawher Jerray [1] Laurent Fribourg[2] and Étienne André[3]

[1] Université Sorbonne Paris Nord, LIPN, CNRS, UMR 7030, F-93430, Villetaneuse, France
and [2] Université Paris-Saclay, LSV, CNRS, ENS Paris-Saclay
and [3] Université de Lorraine, CNRS, Inria, LORIA, F-54000 Nancy, France

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Outline

Motivation

- Dynamical systems:
  - in which a function describes the time dependence of a point in a geometrical space.
  - we only know certain observed or calculated states of its past or present state.
  - dynamical systems have a direct impact on human development.

$\Rightarrow$ The importance of studying:

- synchronization
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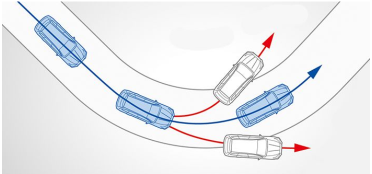
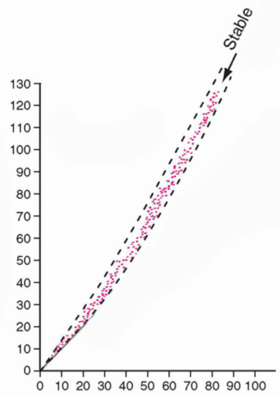


Electronic Stability Control (ESC)

Solar System

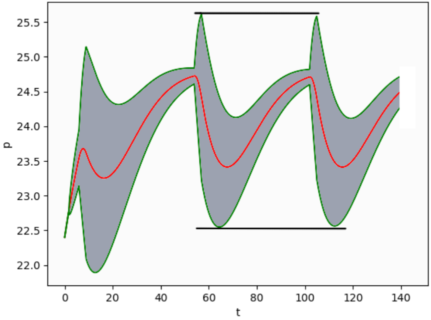
Robustness

- A control is considered robust if the dynamical system still stable, which means small perturbations to the solution using this control lead to a new solution that stays close to the original solution forever.
- A stable system produces a bounded output for a given bounded input.



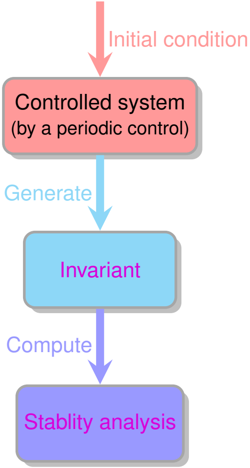
An invariant

- The bounded output of some periodic stable system that is generated by a periodic robust control can be considered as an invariant from certain t.
- An invariant is an unchanged object after operations applied to it.



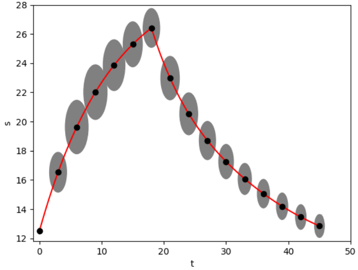
Invariant

Problematic



Description of the method

- Given a differential system $\Sigma : dy/dt = f_u(y)$ of dimension n controlled by u, an initial point $y_0 \in \mathbb{R}^n$, a real $\varepsilon > 0$, and a ball $B_0 = B(y_0, \varepsilon)$[1]



- The center of each ball at time t is the Euler approximate solution $\tilde{Y}^u_{y_0}(t)$ of the system starting at y_0, and the radius is a function $\delta^u_\varepsilon(t)$ bounding the distance between $\tilde{Y}^u_{y_0}(t)$ and an exact solution $Y^u_{y_0}(t)$ starting at B_0.

[1] $B(y_0, \varepsilon)$ is the set $\{z \in \mathbb{R}^n \mid \|z - y_0\| \leq \varepsilon\}$ where $\|\cdot\|$ denotes the Euclidean distance.

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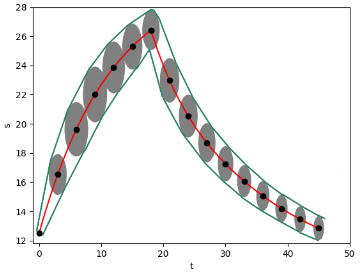

- The tube can be described as $\bigcup_{t \geq 0} B(t)$ where $B(t) \equiv B(\tilde{Y}^u_{y_0}(t), \delta_\varepsilon(t))$.

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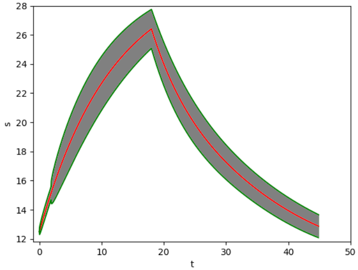

- To find a *bounded* invariant, we look for a positive real T such that $B((i+1)T) \subseteq B(iT)$ for some $i \in \mathbb{N}$. In case of success, the ball $B(iT)$ is guaranteed to contain the "stroboscopic" sequence $\{B(jT)\}_{j=i,i+1,\ldots}$ of sets $B(t)$ at time $t = iT, (i+1)T, \ldots$. and thus constitutes the sought bounded invariant set.

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Euler's method and error bounds

Let us consider the differential system controlled by u:

$$\frac{dy(t)}{dt} = f_u(y(t)).$$

where $f_u(y(t))$ stands for $f(\mathbf{u}(t), y(t))$ with $\mathbf{u}(t) = u$ for $t \in [0, \tau]$, and $y(t) \in \mathbb{R}^n$ denotes the state of the system at time t where τ is the integration time-step.

- $Y^u_{y_0}(t)$ denotes the exact continuous solution y of the system at time $t \in [0, \tau]$ under constant control u, with initial condition y_0.
- $\tilde{Y}^u_{y_0}(t) \equiv y_0 + tf_u(y_0)$ denotes Euler's approximate value of $Y^u_{y_0}(t)$ for $t \in [0, \tau]$.

Finite horizon and dynamic programming

Let us explain the principle of the method to find a control pattern $\pi \in U^k$ based on DP and Euler integration method used in [CF19a; CF19b].
Given $k \in \mathbb{N}$ and $\tau \in \mathbb{R}_{>0}$, we consider the following *finite time horizon optimal control problem*: Find for each $y \in \mathcal{S}$

- the *value* $\mathbf{v}_k(y)$, i. e.,

$$\mathbf{v}_k(y) = \min_{\pi \in U^k} \left\{ J_k(y, \pi) \right\} \equiv \min_{\pi \in U^k} \left\{ \| Y_y^\pi(k\tau) - y_{end} \| \right\}.$$

- and an *optimal pattern*:

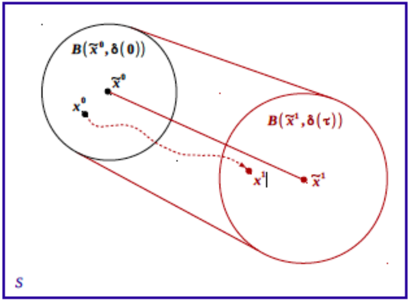
$$\pi_k(y) := \arg\min_{\pi \in U^k} \left\{ \| Y_y^\pi(k\tau) - y_{end} \| \right\}.$$

The space $\mathcal{S}$ is discretized by means of a grid $\mathcal{X}$ such that any point $y_0 \in \mathcal{S}$ has an "ε-representative" $z_0 \in \mathcal{X}$. This method is generated by a procedure $PROC_k^\varepsilon$ which, for any $y \in \mathcal{S}$, takes its representative $z \in \mathcal{X}$ as input, and returns a pattern $\pi_k^\varepsilon \in U^k$ corresponding to an approximate optimal value of $\mathbf{v}_k(y)$ (see [CF19b]).

Proposition

[LCDVCF17] Consider the solution $Y^u_{y_0}(t)$ of $\frac{dy}{dt} = f_u(y)$ with initial condition y_0 of ε-representative z_0 (hence such that $\|y_0 - z_0\| \leq \varepsilon$), and the approximate solution $\tilde{Y}^u_{z_0}(t)$ given by the explicit Euler scheme. For all $t \in [0, \tau]$, we have:

$$\|Y^u_{y_0}(t) - \tilde{Y}^u_{z_0}(t)\| \leq \delta^u_\varepsilon(t).$$



Definition

$\delta_\varepsilon^u(t)$ is defined as follows for $t \in [0, \tau]$:
if $\lambda_u < 0$:

$$\delta_\varepsilon^u(t) = \left(\varepsilon^2 e^{\lambda_u t} + \frac{C^2}{\lambda_u^2}\left(t^2 + \frac{2t}{\lambda_u} + \frac{2}{\lambda_u^2}\left(1 - e^{\lambda_u t}\right)\right)\right)^{\frac{1}{2}}$$

if $\lambda_u = 0$:

$$\delta_\varepsilon^u(t) = \left(\varepsilon^2 e^t + C^2(-t^2 - 2t + 2(e^t - 1))\right)^{\frac{1}{2}}$$

if $\lambda_u > 0$:

$$\delta_\varepsilon^u(t) = \left(\varepsilon^2 e^{3\lambda_u t} + \frac{C^2}{3\lambda_u^2}\left(-t^2 - \frac{2t}{3\lambda_u} + \frac{2}{9\lambda_u^2}\left(e^{3\lambda_u t} - 1\right)\right)\right)^{\frac{1}{2}}$$

where C_u and λ_u are real constants specific to function f_u, defined as follows:

$$C_u = \sup_{y \in \mathcal{S}} L_u \|f_u(y)\|,$$

Definition

L_u denotes the Lipschitz constant for f_u, and λ_u is the "one-sided Lipschitz constant" (or "logarithmic Lipschitz constant" [AS14]) associated to f_u, i. e., the minimal constant such that, for all $y_1, y_2 \in \mathcal{S}$:

$$\langle f_u(y_1) - f_u(y_2), y_1 - y_2 \rangle \leq \lambda_u \|y_1 - y_2\|^2, \qquad (H0)$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product of two vectors of $\mathcal{S}$.

The constant λ_u can be computed using a nonlinear optimization solver (e. g., CPLEX [Cpl09]) or using the Jacobian matrix of f.

[AS14] Z. Aminzare and E. D. Sontag, "Contraction methods for nonlinear systems: A brief introduction and some open problems," in **53rd IEEE Conference on Decision and Control, CDC 2014, Los Angeles, CA, USA, December 15-17, 2014**, 2014, pp. 3835–3847.

[Cpl09] I. I. Cplex, "V12. 1: User's manual for cplex," **International Business Machines Corporation**, vol. 46, no. 53, p. 157, 2009.

Systems with bounded uncertainty

A differential system with bounded uncertainty is of the form

$$\frac{dy(t)}{dt} = f_u(y(t), w(t)),$$

with $t \in \mathbb{R}^n \geq 0$, states $y(t) \in \mathbb{R}^n$, and uncertainty $w(t) \in \mathcal{W} \subset \mathbb{R}^n$ ($\mathcal{W}$ is compact, i. e., closed and bounded).

- We suppose (see [LCADSC+17]) that there exist constants $\lambda_u \in \mathbb{R}$ and $\gamma_u \in \mathbb{R}_{\geq 0}$ such that, for all $y_1, y_2 \in \mathcal{S}$ and $w_1, w_2 \in \mathcal{W}$:

$$\langle f_u(y_1, w_1) - f_u(y_2, w_2), y_1 - y_2 \rangle \leq \lambda_u \|y_1 - y_2\|^2 + \gamma_u \|y_1 - y_2\| \|w_1 - w_2\| \quad (H1).$$

- Instead of computing λ and γ globally for $\mathcal{S}$, it is advantageous to compute them *locally* depending on the subregion of $\mathcal{S}$ occupied by the system state during a considered interval of time.

[LCADSC+17] A. Le Coënt **et al.**, "Distributed control synthesis using Euler's method," in **Proc. of International Workshop on Reachability Problems (RP'17)**, ser. Lecture Notes in Computer Science, vol. 247, Springer, 2017, pp. 118–131.

Proposition

$\delta^u_{\varepsilon,\mathcal{W}}(t)$ is defined as follows for $t \in [0,\tau]$:

$$\text{if } \lambda_u < 0: \quad \delta^u_{\varepsilon,\mathcal{W}}(t) = \left(\frac{C^2}{-\lambda_u^4}\left(-\lambda_u^2 t^2 - 2\lambda_u t + 2e^{\lambda_u t} - 2\right) + \frac{1}{\lambda_u^2}\left(\frac{C\gamma_u|\mathcal{W}|}{-\lambda_u}\left(-\lambda_u t + e^{\lambda_u t} - 1\right) + \lambda_u\left(\frac{\gamma_u^2(|\mathcal{W}|/2)^2}{-\lambda_u}(e^{\lambda_u t} - 1) + \lambda_u \varepsilon^2 e^{\lambda_u t}\right)\right)\right)^{1/2} \tag{1}$$

$$\text{if } \lambda_u > 0: \quad \delta^u_{\varepsilon,\mathcal{W}}(t) = \frac{1}{(3\lambda_u)^{3/2}}\left(\frac{C^2}{\lambda_u}\left(-9\lambda_u^2 t^2 - 6\lambda_u t + 2e^{3\lambda_u t} - 2\right) + 3\lambda_u\left(\frac{C\gamma_u|\mathcal{W}|}{\lambda_u}\left(-3\lambda_u t + e^{3\lambda_u t} - 1\right) + 3\lambda_u\left(\frac{\gamma_u^2(|\mathcal{W}|/2)^2}{\lambda_u}(e^{3\lambda_u t} - 1) + 3\lambda_u \varepsilon^2 e^{3\lambda_u t}\right)\right)\right)^{1/2}$$

$$\text{if } \lambda = 0: \quad \delta^u_{\varepsilon,\mathcal{W}}(t) = \left(C^2\left(-t^2 - 2t + 2e^t - 2\right) + \left(C\gamma|\mathcal{W}|\left(-t + e^t - 1\right) + \left(\gamma^2(|\mathcal{W}|/2)^2(e^t - 1) + \varepsilon^2 e^t\right)\right)\right)^{1/2} \tag{3}$$

Biochemical process example

Consider a biochemical process model Y of continuous culture fermentation (see [HLID09]) and initial condition in $B_0 = B(x_0, \varepsilon)$ for some $x_0 \in \mathbb{R}^2$ and $\varepsilon > 0$ (see [BQ20]):. Let $Y = (X, S, P) \in \mathbb{R}^3$ satisfies the differential system:

$$\begin{cases} \frac{dX}{dt} = -DX(t) + \mu(t)X(t) \\ \frac{dS}{dt} = D\big(S_f(t) - S(t)\big) - \frac{\mu(t)X(t)}{Y_{x/s}} \\ \frac{dP}{dt} = -DP + \big(\alpha\mu(t) + \beta\big)X(t) \end{cases}$$

[HLID09] B. Houska **et al.**, "Approximate robust optimization of time-periodic stationary states with application to biochemical processes," in **CDC**, (Dec. 16–18, 2009), Shanghai, China: IEEE, 2009, pp. 6280–6285. DOI: 10.1109/CDC.2009.5400684.

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where X denotes the biomass concentration, S the substrate concentration, and P the product concentration of a continuous fermentation process. The model is controlled by $S_f \in [S_f^{min}, S_f^{max}]$. While the dilution rate D, the biomass yield $Y_{x/s}$, and the product yield parameters α and β are assumed to be constant and thus independent of the actual operating condition, the specific growth rate $\mu : \mathbb{R} \to \mathbb{R}$ of the biomass is a function of the states:

$$\mu(t) = \mu_m \frac{\left(1 - \frac{P(t)}{P_m}\right) S(t)}{K_m + S(t) + \frac{S(t)^2}{K_i}}$$

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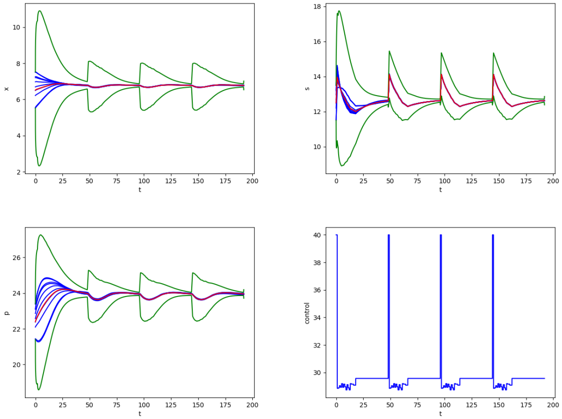
Biochemical process with uncertainty

Consider now the system Σ' with uncertainty $w(\cdot) \in \mathcal{W}_0 = [-0.005, 0.005]$ and initial condition Y_0:

$$\begin{cases} \frac{dX}{dt} = -DX(t) + \mu(t)X(t) + w \\ \frac{dS}{dt} = D\big(S_f(t) - S(t)\big) - \frac{\mu(t)X(t)}{Y_{x/s}} + w \\ \frac{dP}{dt} = -DP + \big(\alpha\mu(t) + \beta\big)X(t) + w \end{cases} \tag{4}$$

Biochemical process with uncertainty



Biochemical process with an additive perturbation $\|w\| \leq 0.005$ over 4 periods ($4T = 192$) for $\Delta t = 1/400$ and initial condition $(X(0), S(0), P(0)) = (6.52, 12.5, 22.4)$, with $X(t), S(t), P(t)$ and control $S_f(t)$.

- We have: $B((i_0+1)T_0) \subset B(i_0 T_0)$ for $i_0 = 1$.

Conclusion and Perspectives

Conclusion

- We presented a simple method to generate a bounded invariant for a differential system.
- We have given a simple condition which guarantees that, under a repeated control sequence the system with perturbation is robust.
- The method uses a simple algorithm to compute local rates of contraction in the framework of Euler's method.
- The method uses a very general criterion of inclusion of one set in another.

Perspectives

- Extend our method in order to take into account the specification of state constraints during the evolution of the system.

[AS14] Z. Aminzare and E. D. Sontag, "Contraction methods for nonlinear systems: A brief introduction and some open problems," in **53rd IEEE Conference on Decision and Control, CDC 2014, Los Angeles, CA, USA, December 15-17, 2014**, 2014, pp. 3835–3847.

[BQ20] J. B. van den Berg and E. Queirolo, "A general framework for validated continuation of periodic orbits in systems of polynomial ODEs," **Journal of Computational Dynamics**, vol. 0, no. 2158-2491-2019-0-10, 2020, ISSN: 2158-2491. DOI: 10.3934/jcd.2021004.

[CF19a] A. L. Coënt and L. Fribourg, "Guaranteed control of sampled switched systems using semi-Lagrangian schemes and one-sided Lipschitz constants," in **CDC**, (Dec. 11–13, 2019), Nice, France: IEEE, 2019, pp. 599–604. DOI: 10.1109/CDC40024.2019.9029376.

[CF19b] ——,"Guaranteed optimal reachability control of reaction-diffusion equations using one-sided Lipschitz constants and model reduction," in **WESE**, (Oct. 17–18, 2019), R. D. Chamberlain, M. Grimheden, and W. Taha, Eds., ser. LNCS, vol. 11971, New York City, NY, USA: Springer, 2019, pp. 181–202. DOI: 10.1007/978-3-030-41131-2_9.

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