Rate equation modeling of the frequency noise and the intrinsic spectral linewidth in quantum cascade lasers

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Abstract: This work theoretically investigates the frequency noise (FN) characteristics of quantum cascade lasers (QCLs) through a three-level rate equation model, which takes into account both the carrier noise and the spontaneous emission noise through the Langevin approach. It is found that the power spectral density of the FN exhibits a broad peak due to the carrier noise induced carrier variation in the upper laser level, which is enhanced by the stimulated emission process. The peak amplitude is strongly dependent on the gain stage number and the linewidth broadening factor. In addition, an analytical formula of the intrinsic spectral linewidth of QCLs is derived based on the FN analysis. It is demonstrated that the laser linewidth can be narrowed by reducing the gain coefficient and/or accelerating the carrier scattering rates of the upper and the lower laser levels.

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References and links


1. Introduction

Quantum cascade lasers (QCLs) are intersubband semiconductor light sources emitting in the spectral range of mid-infrared (MIR) and terahertz (THz) [1,2]. The spectrum covers the molecular “fingerprints” of many gases like nitric oxide (NO), carbon dioxide (CO₂), methane (CH₄), and hence enables various gas sensing applications [3,4]. In addition, QCLs are promising laser sources for high-resolution spectroscopy [5,6], terahertz imaging [7], and free-space optical communications as well [8,9]. However, optical noise in QCLs including the relative intensity noise and the frequency/phase noise (FN) limits the sensitivity and the resolution for the above applications [10,11]. In comparison with interband semiconductor lasers, the relative intensity noise of QCLs does not exhibit any resonance peak owing to the ultra-fast carrier lifetimes (around 1.0 ps) [12]. In addition, it decreases more slowly with increasing optical power than that in interband lasers [13–15]. In order to suppress the relative intensity noise of QCLs, researchers have resorted to the optical injection and the optical feedback techniques [16–18].

The FN in semiconductor lasers consists of the spontaneous emission noise, the carrier generation and recombination noise, as well as the low-frequency flicker noise \(1/f\) noise, all of which determine the total spectral linewidth [19]. The former two noise sources are white noise and govern the lasers’ intrinsic spectral linewidth, which is broadened by the linewidth broadening factor (LBF) [20]. QCLs usually exhibit near-zero LBFs, leading to narrow intrinsic linewidth in the range of 0.1–1.0 kHz [21–23]. However, the latter flicker noise arising from the current source, the thermal fluctuation, and the internal electrical noise considerably broadens the total spectral linewidth of QCLs to the sub-MHz or MHz range [24–26]. In order to narrow the spectral linewidth of QCLs, a large variety of frequency stabilization schemes have been proposed, including electronic feedback to the current source.
[27], locking to an optical cavity [28], phase locking to a narrow-linewidth laser source [29],
as well as the popular optical injection locking to an optical frequency comb [30,31]. On the
other hand, there are only few theoretical studies on the FN characteristics of QCLs. M.
Yamanishi et al. derived an analytical formula of the intrinsic linewidth of QCLs by
introducing the concept of effective coupling efficiency of the spontaneous emission [32]. In
addition, it was demonstrated that the noisy stimulated emission due to thermal photons
considerably broadens the intrinsic linewidth of THz QCLs [33,34]. T. Liu et al. reported the
fundamental FN caused by intrinsic temperature fluctuations in QCLs, and developed a
quantum mechanical Langevin model for the calculation of the intrinsic linewidth [35,36]. In
this work, we theoretically investigate the FN characteristics of QCLs through a three-level
rate equation model, which includes all the Langevin noise sources for the carriers, the
photon, and the phase of the electric field. It is found that the power spectral density of the FN
exhibits a broad peak due to the carrier noise induced carrier variation of the upper laser level.
In addition, the intrinsic linewidth of QCLs is analytically obtained through the FN analysis.
It is proved that the intrinsic linewidth can be narrowed by reducing the gain coefficient
or/and increasing the carrier scattering rates of the upper and the lower laser levels.

Fig. 1. Schematic of three-level electronic structure of QCLs.

2. Rate equation model with Langevin noise sources

The rate equations are developed based on the three-level electronic structure of QCLs [37–
39]. As shown in Fig. 1, carriers are injected into the upper laser level of the gain region from
the injector region by resonant tunneling [2]. From the upper laser level, carriers are scattered
into the lower laser level with a time constant $\tau_{32}$, and into the bottom level with a time $\tau_{31}$
through longitudinal-optical phonon emissions [2]. The stimulated emission is enabled by the
population inversion between the upper and the lower laser levels. Carriers in the lower laser
level scatter into the bottom level with a time $\tau_{21}$, and finally escape the gain region with a
time $\tau_{\text{out}}$ into the subsequent injector and gain stages. Accordingly, the rate equations
describing the carrier numbers in the upper level ($N_3$), in the lower level ($N_2$), and in the
bottom level ($N_1$), the photon number ($S$), and the phase of the electric field ($\phi$) are given by

\[
\frac{dN_3}{dt} = \eta \frac{I}{q} \frac{N_3}{\tau_{32}} - \frac{N_3}{\tau_{31}} - G_0 S_\Delta N + F_3(t) 
\]

\[
\frac{dN_2}{dt} = \frac{N_3}{\tau_{32}} - \frac{N_2}{\tau_{21}} + G_0 S_\Delta N + F_2(t) 
\]

\[
\frac{dN_1}{dt} = \frac{N_3}{\tau_{31}} + \frac{N_2}{\tau_{21}} - \frac{N_1}{\tau_{\text{out}}} + F_1(t) 
\]

\[
\frac{dS}{dt} = \left( mG_0 S_\Delta N - \frac{1}{\tau_p} \right) S + m\beta \frac{N_1}{\tau_{\phi}} + F_3(t) 
\]
\[
\frac{d\phi}{dt} = \frac{\alpha_H}{2} \left( mG_0\Delta N - \frac{1}{\tau_p} \right) + F_\phi(t)
\]  

(5)

where \( I \) is the pump current, \( \eta \) is the current injection efficiency, \( G_0 \) is the gain coefficient, and \( \Delta N \) is the population inversion given by \( \Delta N = N_3 - N_2 \). \( \tau_p \) is the spontaneous emission lifetime, \( \alpha_H \) is the LBF, and \( m \) is number of gain stages. The time averages of all the carrier (\( F_{3,2,1} \)), photon (\( F_3 \)), and phase (\( F_\phi \)) Langevin noise sources are zero due to their random nature [40]. Following the method in [20], the auto- and cross-correlations of the noise sources are derived as

\[
\{F(t)F_j(t')\} = U_{ij}\delta(t-t')
\]  

(6)

with

\[
U_{33} = 2(G_0N_3S + N_3/\tau_{33} + N_3/\tau_{31}); \quad U_{22} = 2(G_0N_3S + N_3/\tau_{32})
\]

\[
U_{SS} = 2m(G_0N_3S + \beta N_3/\tau_{sp}); \quad U_{\phi\phi} = 2m(G_0N_3S + \beta N_3/\tau_{sp})/(4S^2)
\]

\[
U_{32} = -(G_0N_3S + G_0N_2S + N_3/\tau_{32}); \quad U_{3S} = -(G_0N_3S + G_0N_2S + \beta N_3/\tau_{sp})
\]

\[
U_{3\phi} = U_{2\phi} = U_{3\phi} = 0
\]

(7)

where correlations related to \( F_1 \) are not listed, since \( F_1 \) is not involved in the FN of QCLs as expressed in Eq. (11). It is remarked that the correlations in Eq. (7) are identical to those in [13], where the phase correlations were not reported.

The small-signal Langevin noise sources perturb the laser system away from its steady-state condition, and the responses of the carriers, the photons and the phase are given by

\[
\delta N_{3,2,1}(t) = n_{3,2,1}e^{j\omega t}; \quad \delta S(t) = se^{j\omega t}; \quad \delta \phi(t) = \phi e^{j\omega t}
\]  

(8)

with \( \omega \) being the angular frequency. Taking the differentials of the rate Eqs. (1)-(5) and using Eq. (8), the differential rate equations in the frequency domain are obtained as

\[
\begin{bmatrix}
  j\omega + \gamma_{11} & -\gamma_{12} & 0 & \gamma_{14} & 0 \\
  -\gamma_{23} & j\omega + \gamma_{22} & 0 & -\gamma_{24} & 0 \\
  -\gamma_{31} & -\gamma_{32} & j\omega & 0 & 0 \\
  -\gamma_{41} & \gamma_{42} & 0 & j\omega + \gamma_{44} & 0 \\
  -\gamma_{51} & \gamma_{52} & 0 & 0 & j\omega \\
\end{bmatrix}
\begin{bmatrix}
  n_1(\omega) \\
  n_2(\omega) \\
  n_3(\omega) \\
  s(\omega) \\
  \phi(\omega) \\
\end{bmatrix}
= \begin{bmatrix}
  F_1(\omega) \\
  F_2(\omega) \\
  F_3(\omega) \\
  F_4(\omega) \\
  F_\phi(\omega) \\
\end{bmatrix}
\]  

(9)

with

\[
\gamma_{11} = G_0S + 1/\tau_{33} + 1/\tau_{31}; \quad \gamma_{12} = G_0S; \quad \gamma_{14} = G_0\Delta N \\
\gamma_{21} = G_0S + 1/\tau_{33}; \quad \gamma_{22} = G_0S + 1/\tau_{21}; \quad \gamma_{24} = G_0\Delta N \\
\gamma_{31} = 1/\tau_{31}; \quad \gamma_{32} = 1/\tau_{31}; \quad \gamma_{34} = mG_0S + m\beta/\tau_{sp} \\
\gamma_{42} = mG_0S; \quad \gamma_{44} = 1/\tau_p - mG_0\Delta N \\
\gamma_{51} = m\alpha_HG_0/2; \quad \gamma_{52} = m\alpha_HG_0/2
\]  

(10)

Following Cramer’s rule, the FN of QCLs is calculated by

\[
FN(\omega) = \left| j\omega\phi(2\pi) \right|^2
\]  

(11)

It is remarked that the above model is suitable for both MIR and THz QCLs to investigate the FN originating from the carrier noise and the spontaneous emission noise. However, it does
not include the thermal photon contribution in THz QCLs and the flicker noise in either laser, which is beyond the scope of this paper and will be studied in future work.

3. Results and discussion

The QCL parameters used for the simulations are listed in Table 1 [41], unless stated otherwise. The QCL under study exhibits a lasing threshold current of $I_{th} = 222$ mA. We first investigate the power spectral density of the FN, and then discuss the intrinsic spectral linewidth of the QCL.

Table 1. QCL material and optical parameters.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_0$</td>
<td>Gain coefficient</td>
<td>$5.3 \times 10^4$ s/$s$</td>
</tr>
<tr>
<td>$\tau_p$</td>
<td>Photon lifetime</td>
<td>3.7 ps</td>
</tr>
<tr>
<td>$\tau_s$</td>
<td>Spontaneous emission time</td>
<td>7.0 ns</td>
</tr>
<tr>
<td>$\beta$</td>
<td>Spontaneous emission factor</td>
<td>$1.0 \times 10^{-6}$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Linewidth broadening factor</td>
<td>0.5</td>
</tr>
<tr>
<td>$m$</td>
<td>Gain stage number</td>
<td>30</td>
</tr>
<tr>
<td>$\tau_{22}$</td>
<td>Scattering time upper to lower</td>
<td>2.0 ps</td>
</tr>
<tr>
<td>$\tau_{31}$</td>
<td>Scattering time upper to bottom</td>
<td>2.4 ps</td>
</tr>
<tr>
<td>$\tau_{21}$</td>
<td>Scattering time lower to bottom</td>
<td>0.5 ps</td>
</tr>
<tr>
<td>$\tau_{out}$</td>
<td>Tunneling out time</td>
<td>0.5 ps</td>
</tr>
</tbody>
</table>

3.1 Power spectral density of the frequency noise

Figure 2 shows the power spectral densities of the FN (solid lines) for the QCL biased at 1.5 × $I_{th}$, 2.0 × $I_{th}$, and 5.0 × $I_{th}$, respectively. Like interband lasers, the FN exhibits a plateau at both low frequencies (<1.0 GHz) and high frequencies (>10 THz), and it decreases with the pump current [20]. Surprisingly, the FN exhibits a peak, which is much broader than the common resonance peak in interband lasers [42], and the peak frequency increases with the pump current. In contrast, the relative intensity noise and the intensity modulation response of QCLs do not exhibit any resonance peak [14,43]. However, QCLs subject to optical injection can show a peak in the modulation response [37,38]. On the other hand, the FN peak disappears (dashed line) once the carrier noise $F_{1,2,3}$ in Eq. (9) is removed. This phenomenon suggests that the FN peak is due to the carrier noise in the upper and the lower laser levels, since the carrier noise in the bottom level does not contribute to the FN. The significant role of the carrier noise in QCLs differs from that in interband lasers, which is usually negligible in comparison with the spontaneous emission noise [44].

![Fig. 2. FN spectra at various pump currents. The dashed line is without carrier noise at 2.0 × $I_{th}$.](image-url)

The FN behavior can be understood through the Bode plot analysis, which describes the response of a system in the frequency domain using its zeros and poles [44]. Table 2 lists the zeros and the poles of the FN in Eq. (11) for the QCL biased at 2.0 × $I_{th}$. With the carrier...
noise, the smallest zero $z_3$ (non-zero absolute value) is less than the smallest pole $p_3$, leading to the appearance of the peak in Fig. 2. In contrast, $z_3$ becomes larger than $p_3$ without the carrier noise, resulting in the vanishing of the peak. However, there is no complex conjugate pair of poles in either case, proving that the FN exhibits no resonance.

<table>
<thead>
<tr>
<th>Zeros (10(^{11}) Hz)</th>
<th>Poles (10(^{11}) Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z_1, z_2$</td>
<td>$p_1, p_2$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>-0.12</td>
<td>-0.23</td>
</tr>
<tr>
<td>-1.9</td>
<td>-1.7</td>
</tr>
<tr>
<td>-9.5</td>
<td>-5.1</td>
</tr>
<tr>
<td>$z_3$</td>
<td>$p_3$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>-0.25</td>
<td>-0.23</td>
</tr>
<tr>
<td>-1.7</td>
<td>-1.7</td>
</tr>
<tr>
<td>-5.1</td>
<td>-5.1</td>
</tr>
</tbody>
</table>

Because the phase of the electric field is partly determined by the population inversion as expressed in Eq. (5), the physical origin of the FN peak can be explored by examining the small-signal carrier responses, which are defined as the square of the amplitude of the carrier variations $\left| n_{3,2,1}(\omega)\right|^2$ in the frequency domain. Figure 3 presents that the carrier response of the upper laser level ($n_3$) exhibits a peak while that of the lower laser level ($n_2$) does not, which leads to the appearance of the peak in the population inversion ($n_3-n_2$). The peak frequency of the population inversion is the same as that of the FN in Fig. 2. An analytical analysis shows that the peak frequency is related to the stimulated emission through the rate $G_0S (1.2 \times 10^{11} \text{ Hz at } 2.0 \times I_0)$ in Eq. (1), similar to the resonance frequency of interband lasers [20]. Following the discussions in Fig. 2 and in Table 2, the FN peak can be attributed to the carrier noise induced carrier variation of the upper laser level, which is enhanced by the stimulated emission process. However, deeper physical insights of the interaction process between the carrier variation and the stimulated emission process are still required in future work.

![Fig. 3. Carrier responses due to the noise perturbation at 2.0 × $I_0$.](image-3)

![Fig. 4. (a) FN spectra with various gain stage numbers, and (b) with various LBFs at 2.0 × $I_0$.](image-4)
Figure 4(a) illustrates that a QCL with fewer gain stages exhibits a lower FN peak, and the peak is almost completely suppressed for only one gain stage. However, the gain stage number does not affect the low- or high-frequency part of the FN. The LBF in semiconductor lasers describes the phase-amplitude coupling effects of the refractive index and the gain, which enhances the low-frequency FN and thereby broadens the spectral linewidth [45]. Owing to the nearly symmetric homogeneous gain broadening of the intersubband transition, QCLs operating below the lasing threshold usually show near-zero LBFs [46], while QCLs operating above threshold show higher values ranging from 0.2 to 3.0 [47–49]. The non-zero LBF in QCLs has been attributed to the non-parabolicity of the band structure, the many-body effects, the resonant tunneling transport, and the counter-rotating wave contribution [50,51]. Figure 4(b) points out that the peak of the FN is strongly dependent on the LBF value. For a LBF of zero, the FN is constant over the entire frequency range because the carrier noise and the spontaneous emission noise are white, while the flicker noise is not taken into account. On the other hand, a non-zero LBF substantially raises the peak amplitude as well as the low-frequency FN, which determines the intrinsic spectral linewidth of QCLs as discussed in the following section.

3.2 Intrinsic spectral linewidth of the quantum cascade laser

For interband lasers, the intrinsic spectral linewidth is given by the well-known formula [20]

$$\Delta\nu_{IL}^{IIP} = \left( \frac{v_g^2 \alpha_T \alpha_m}{4 \pi P_0} n_{sp} \hbar \nu \right) \left( 1 + \alpha_n^2 \right)$$  \hspace{1cm} (12)

where $P_0$ is the output power, $v_g$ is the group velocity of light, $\alpha_T$ is the total cavity loss, $\alpha_m$ is the mirror loss, $n_{sp}$ is the population inversion factor, and $\hbar \nu$ is the photon energy. The first term on the right hand of the formula gives the Schawlow-Townes linewidth limit, and the second term suggests that the LBF broadens the laser linewidth.

For QCLs, the Schawlow-Townes limit can be obtained from its high-frequency FN plateau in Fig. 2, and from Eq. (11) we derive

$$\Delta\nu_{ST}^C = 2 \pi FN(\omega \to \infty) = m N_3 \frac{G_0 S + \beta / \tau_{sp}}{4 \pi S^2} = m G_0 N_3 \frac{1}{4 \pi S}$$  \hspace{1cm} (13)

On the other hand, the intrinsic spectral linewidth is determined by the low-frequency FN plateau as

$$\Delta\nu_{IL}^C = 2 \pi FN(\omega \to 0) = \Delta\nu_{ST}^C \left( 1 + \alpha_n^2 \right) = \frac{m G_0 N_3}{4 \pi S} \left( 1 + \alpha_n^2 \right)$$  \hspace{1cm} (14)

Using the steady-state solutions of the rate equations, Eq. (14) can be re-expressed as a function of the output power $P_0$ or the pump current $I_0$

$$\Delta\nu_{IL}^C = \frac{v_g^2 \alpha_T G_0}{4 \pi} \frac{\tau_{32}}{\tau_{32} - \tau_{21}} \left( \frac{h \nu v_g \alpha_m}{G_0 P_0} + \tau_{21} \right) \left( 1 + \alpha_n^2 \right)$$ \hspace{1cm} (15a)

$$\Delta\nu_{IL}^C = \frac{v_g^2 \alpha_T G_0}{4 \pi} \frac{\eta_r^{-1}}{\tau_{32} + \tau_{21}^{-1}} \left( \frac{1}{I_0 / I_\text{th} - 1} + 1 - \eta_r \right) \left( 1 + \alpha_n^2 \right)$$ \hspace{1cm} (15b)

with
Equation (15) clearly shows that increasing the pump current or the output power reduces the laser linewidth. When the pump current is high enough, the intrinsic linewidth of QCLs eventually saturates at a minimum level

\[ \Delta \nu_{l,\text{min}} = \frac{v_e \alpha_f G_0}{4\pi} \frac{\tau_{32} \tau_{21}}{\tau_{32} - \tau_{21}} \left(1 + \alpha^2_{\text{eff}}\right) \]  

(17)

In contrast, the intrinsic linewidth of interband lasers does not have this kind saturation behavior (see Eq. (12)), although external flicker noises can saturate or even rebroaden the total spectral linewidth [52]. Equation (17) is in agreement with that in [32], where the laser linewidth was derived using the Einstein relationship of the stimulated emission and the spontaneous emission. According to Eq. (15), the laser linewidth is not affected by the gain stage number as shown in Fig. 4(a), where the stage number does not modify the low-frequency FN. In contrast, a QCL with more gain stages exhibits a higher relative intensity noise [14].

Figure 5(a) shows that the intrinsic linewidth (solid line) of the QCL can be narrowed through reducing the gain coefficient. However, this is achieved at the cost of increasing the lasing threshold (dashed line). Equation (15) suggests that the intrinsic linewidth is dependent on the carrier scattering times. Indeed, a short time \( \tau_{32} \) in Fig. 5(b) and a short time \( \tau_{21} \) in Fig. 5(c) would result in a narrower intrinsic linewidth due to the reduced scattering times.
reduce the laser linewidth. At $5.0 \times I_{th}$, the linewidth slightly decreases from 1.8 kHz for $\tau_{32} = 4.0$ ps to 1.6 kHz for $\tau_{32} = 2.0$ ps, while it is reduced by about 40% from 1.6 kHz for $\tau_{21} = 0.5$ ps to 1.0 kHz for $\tau_{21} = 0.3$ ps. Therefore, the laser linewidth of QCLs can be effectively narrowed through accelerating the carrier scattering rate of the lower laser level. In addition, the FN peak amplitude is suppressed by the short scattering times as shown in the inset of Figs. 5(b) and 5(c). Finally, it is remarked that the reduction of $G_0$ and $\tau_{32}$ can also degrade the maximum available laser power and the dynamic performance such as the modulation bandwidth and the frequency chirp [20]. Therefore, there is a tradeoff in designing the gain coefficient and the carrier scattering times for achieving narrow-linewidth QCLs.

4. Conclusions

In conclusion, the FN characteristics and the intrinsic linewidth of QCLs have been modeled through a set of coupled rate equations including both the carrier noise and the spontaneous emission noise. It is found that the power spectral density of the FN exhibits a broad peak due to the stimulated-emission enhanced carrier variation of the upper laser level, which is driven by the carrier noise in both the upper and the lower laser levels. The peak amplitude is strongly dependent on the gain stage number and the LBF. In addition, we derive an analytical formula for the intrinsic linewidth of QCLs based on the FN analysis. It is proven that the laser linewidth can be narrowed by reducing the gain coefficient and/or accelerating the carrier scattering times of the upper and lower laser levels. In future work, we will include the thermal photon contribution to the FN of THz QCLs in the model, and design experiments to verify the theoretical predictions. In addition, recent work in [53,54] pointed out that the gain nonlinearity affected the performance of QCLs as for interband lasers. As such, the gain compression effects on the FN will be studied as well.

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