Cramér-Rao bounds for multiple poles and coefficients of quasipolynomials in colored noise

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***Abstract*—In this paper, we provide analytical expressions of the Cramér-Rao bounds for the frequencies, damping factors, amplitudes and phases of complex exponentials in colored noise. These expressions show the explicit dependence of the bounds of each distinct parameter with respect to the amplitudes and phases, leading to readily interpretable formulae, which are then simplified in an asymptotic context. The results are presented in the general framework of the Polynomial Amplitude Complex Exponentials (PACE) model, also referred to as the quasipolynomial model in the literature, which accounts for systems involving multiple poles, and represents a signal as a mixture of complex exponentials modulated by polynomials. This work looks further and generalizes the studies previously undertaken on the exponential and the quasipolynomial models.**

***Index Terms*—Performance analysis, Cramér-Rao bound, complex exponentials, polynomial modulation, multiple eigenvalues.**

I. INTRODUCTION

ESTIMATING mixtures of complex exponentials in noise is a very classical problem in signal processing. Such models are used in a variety of applications, including spectral estimation, source localization, speech processing, deconvolution, radar and sonar signal processing [1]. For well separated frequencies, the Fourier analysis provides a simple, statistically optimal solution for this estimation problem. However, when the spectral separation of the components approaches the Fourier resolution, better performance can be achieved with the so-called High Resolution (HR) methods such as MUSIC [2] and ESPRIT [3]. The performance of these parameter estimation methods is generally analyzed by measuring the accuracy of the estimated poles locations, which contain information about the frequencies or directions of arrival of the signal components, and the estimated amplitudes and phases of these components. To this end, the Cramér-Rao bound (CRB) is a fundamental tool in estimation theory, because it can be used to quantify the performance of an estimator, by comparing its variance to an optimal value, which somewhat can be considered as a reference target [4]. An unbiased estimator will be said efficient when the bound is reached, *i.e.* when its efficiency, defined as the ratio between its variance and the CRB, equals 1. The CRB of the model parameters are obtained by calculating the diagonal coefficients of the inverse of the *Fisher information matrix* [4]. In the literature, a number of papers have been devoted to the analysis of the Cramér-Rao bounds of mixtures of exponentials. Several variants of the model have been investigated, involving either undamped [5]–[9] or damped [1], [10]–[12] exponentials, and either real [5], [8] or complex [1], [6], [7], [9]–[12] exponentials. Such bounds have been used to analyze the performance of some classical HR approaches, such as the Prony [13], [14] and Pisarenko [15], [16] methods, and the MUSIC, ESPRIT and Matrix Pencil [12] algorithms [17]–[19]. In this paper, we will refer to the most general model (involving complex and damped exponentials) as the Exponential Sinusoidal Model (ESM).

Calculating the CRB of each individual parameter of this model is a difficult task, because of the inversion of the Fisher information matrix. In the literature, analytic expressions were obtained in simplified situations: asymptotic expansions of the CRB's were calculated in [8], [9] for large observation lengths, and exact expressions were provided in [11] for models involving only one or two poles. However, the formulae obtained in the most general case generally consist of complicated expressions involving auxiliary matrices, which can be useful for proving some properties of the CRB [1], [12], but are difficult to interpret. In comparison, we propose in this paper new analytical expressions of the Cramér-Rao bounds for each distinct parameter, which show the explicit dependence of the bounds with respect to the amplitudes and phases, leading to readily interpretable formulae. This study is generalized to the Polynomial Amplitude Complex Exponentials (PACE) model [20], also referred to as the quasipolynomial model [21], which naturally emerges when analyzing the CRB of the classical ESM model. The PACE model is actually the most general model tractable by HR methods, which accounts for systems involving multiple poles, and represents a signal as a mixture of complex exponentials modulated by polynomials [20]. Moreover, systems involving multiple poles are encountered in a broad range of applications. For instance, critically damped harmonic oscillators involve a double pole [22]. Laguerre functions are a special case of signals with multiple poles (the exponentials are modulated by Laguerre polynomials), often used in the estimation of time delays [23], [24], and in biomedical engineering, for modeling fluorescence decay [25]. Signals with multiple poles also appear in quantum physics, as solutions of the Schrödinger equation for hydrogen-like atoms [26], in laser physics, as transverse laser modes [27], and in finance, for modeling the evolution of interest rates [28]. Other systems involving multiple poles are encountered in [29], [30], where they are successfully applied to ARMA filter synthesis, in the context

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of system conversion from continuous time to discrete time.

An analysis of the Cramér-Rao bounds for the frequencies and damping factors of complex quasipolynomials with multiple poles in white noise was first proposed in [21] (real quasipolynomials were addressed in [31]). Here the investigation is extended to the Cramér-Rao bounds for the amplitudes and phases of complex polynomials, and to the case of colored noise. The novelty also lies in the analytical expressions of the Cramér-Rao bounds for each distinct parameter, which are then simplified in an asymptotic context. We show that the Cramér-Rao bounds for the parameters associated to a multiple pole present an exponential increase with the order of the pole. Consequently, it appears that the practical estimation of the PACE model is only possible if the exponentials are modulated by polynomials of low order. This work on the Cramér-Rao bounds is applied in [29], [30] in order to analyze the performance of the generalized ESPRIT algorithm introduced in [20] to estimate the PACE model.

The paper is organized as follows. Section II introduces the general framework: the Cramér-Rao bounds for the ESM model are presented in section II-A, and the PACE model, which was presented in [20] as a generalization of the ESM model, is summarized in section II-B. The extension of these bounds to the PACE model is presented in section III. The general case is addressed in section III-A, and some asymptotic expansions are proposed in section III-B. Section IV illustrates the variation of these bounds with respect to some parameters of the PACE model. The main conclusions of this work are summarized in section V. Finally, the proofs of the various results presented below can be found in the appendix.

II. General framework

A. Cramér-Rao bounds for the ESM model

The general theorem of the Cramér-Rao bound [4] is summarized below. It relies on the hypothesis of a regular statistical model.

Definition II.1 (Regular statistical model)**.** Let $\boldsymbol{x}$ be a random vector of dimension N, and consider a statistical model which admits a probability density function with respect to a measure μ, and parameterized by $\boldsymbol{\vartheta} \in \Theta$, where Θ is an open set of $\mathbb{R}^q$. The parameterization is called regular if the following conditions hold:

1) the probability density function $p(\boldsymbol{x};\, \boldsymbol{\vartheta})$ is continuously differentiable, μ-almost everywhere, with respect to $\boldsymbol{\vartheta}$.
2) the Fisher information matrix
$$\boldsymbol{F}(\boldsymbol{\vartheta}) \triangleq \int \boldsymbol{l}(\boldsymbol{x};\, \boldsymbol{\vartheta})\, \boldsymbol{l}(\boldsymbol{x};\, \boldsymbol{\vartheta})^T\, p(\boldsymbol{x};\, \boldsymbol{\vartheta})\, \mathrm{d}\mu(\boldsymbol{x}),$$
where $\boldsymbol{l}(\boldsymbol{x};\, \boldsymbol{\vartheta}) \triangleq \boldsymbol{\nabla}_{\boldsymbol{\vartheta}} \ln p(\boldsymbol{x};\, \boldsymbol{\vartheta})\, \mathbf{1}_{(p(\boldsymbol{x};\, \boldsymbol{\vartheta})>0)}$ defines the score function[1], is positive definite for any value of the parameter $\boldsymbol{\vartheta}$ and continuous with respect to $\boldsymbol{\vartheta}$.

Theorem II.2 (Cramér-Rao bound)**.** *Consider a regular statistical model parameterized by $\boldsymbol{\vartheta} \in \Theta$. Let $\widehat{\boldsymbol{\vartheta}}$ be an unbiased estimator of $\boldsymbol{\vartheta}$ ($\forall \boldsymbol{\vartheta} \in \Theta$, $\mathbb{E}_{\boldsymbol{\vartheta}}[\widehat{\boldsymbol{\vartheta}}] = \boldsymbol{\vartheta}$). Then the dispersion matrix $\boldsymbol{D}(\boldsymbol{\vartheta}, \widehat{\boldsymbol{\vartheta}}) \triangleq \mathbb{E}_{\boldsymbol{\vartheta}}\left[\left(\widehat{\boldsymbol{\vartheta}} - \boldsymbol{\vartheta}\right)\left(\widehat{\boldsymbol{\vartheta}} - \boldsymbol{\vartheta}\right)^T\right]$ is such that the matrix $\boldsymbol{D}(\boldsymbol{\vartheta},\, \widehat{\boldsymbol{\vartheta}}) - \boldsymbol{F}(\boldsymbol{\vartheta})^{-1}$ is positive semidefinite.*

[1] In the whole paper, the function $\mathbf{1}_{(.)}$ is one if its argument is true and zero otherwise.

In particular, the diagonal coefficients of the matrix $\boldsymbol{D}(\boldsymbol{\vartheta},\, \widehat{\boldsymbol{\vartheta}}) - \boldsymbol{F}(\boldsymbol{\vartheta})^{-1}$ are non-negative. Consequently, the variances of the coefficients of $\widehat{\boldsymbol{\vartheta}}$ are greater than the diagonal coefficients of the matrix $\boldsymbol{F}(\boldsymbol{\vartheta})^{-1}$. Thus the Cramér-Rao bounds for the coefficients of $\widehat{\boldsymbol{\vartheta}}$ are obtained in three steps:

- calculation of the Fisher information matrix;
- inversion of this matrix;
- extraction of its diagonal coefficients.

From now on, suppose that the observed vector $\boldsymbol{x}$ is of the form $\boldsymbol{x} = \boldsymbol{s}(\boldsymbol{\vartheta}) + \boldsymbol{w}$, where $\boldsymbol{s}(\boldsymbol{\vartheta})$ is a deterministic vector, and $\boldsymbol{w}$ is a centered complex Gaussian random vector of covariance matrix $\boldsymbol{R}(\boldsymbol{\vartheta}) = \mathbb{E}\left[\boldsymbol{w}\,\boldsymbol{w}^H\right]$ (which we denote $\boldsymbol{w} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{R}(\boldsymbol{\vartheta}))$). Then $\boldsymbol{x} \sim \mathcal{N}(\boldsymbol{s}(\boldsymbol{\vartheta}), \boldsymbol{R}(\boldsymbol{\vartheta}))$. It is well known that the Fisher information matrix of a Gaussian random vector can be expressed as a function of the model parameters, as shown in the following proposition [4, pp. 525].

Proposition II.3 (Fisher matrix for a Gaussian density)**.** *For a family of complex Gaussian probability laws, whose covariance matrix is $\boldsymbol{R}(\boldsymbol{\vartheta})$ and whose expected value is $\boldsymbol{s}(\boldsymbol{\vartheta})$, where $\boldsymbol{R} \in \mathcal{C}^1(\Theta,\, \mathbb{C}^{N\times N})$ and $\boldsymbol{s} \in \mathcal{C}^1(\Theta,\, \mathbb{C}^N)$, the coefficients of the Fisher information matrix $\left\{F_{(i,j)}(\boldsymbol{\vartheta})\right\}_{1\leq i,\, j\leq k}$ are given by the extended Bangs-Slepian formula:*

$$\begin{aligned} F_{(i,j)}(\boldsymbol{\vartheta}) \quad = \quad & \mathrm{trace}\left(\boldsymbol{R}(\boldsymbol{\vartheta})^{-1}\tfrac{\partial \boldsymbol{R}(\boldsymbol{\vartheta})}{\partial \vartheta_i}\boldsymbol{R}(\boldsymbol{\vartheta})^{-1}\tfrac{\partial \boldsymbol{R}(\boldsymbol{\vartheta})}{\partial \vartheta_j}\right) \\ & +2\mathcal{R}e\left(\tfrac{\partial \boldsymbol{s}(\boldsymbol{\vartheta})}{\partial \vartheta_i}^H \boldsymbol{R}(\boldsymbol{\vartheta})^{-1}\,\tfrac{\partial \boldsymbol{s}(\boldsymbol{\vartheta})}{\partial \vartheta_j}\right). \end{aligned} \tag{1}$$

In the context of time series analysis, the vector $\boldsymbol{x} = [x(0)\ldots x(N-1)]^T$ contains the N successive samples of an observed signal $x(t)$, which is generally modeled as the sum of a deterministic signal $s(t)$, plus a complex white Gaussian noise $w(t)$ of variance σ^2 (in this case $\boldsymbol{R}(\boldsymbol{\vartheta}) = \sigma^2\boldsymbol{I}_N$). Moreover, the signal $s(t)$ is supposed to satisfy the ESM model of order $K \in \mathbb{N}^*$:

$$s(t) = \sum_{k=0}^{K-1} a_k e^{\delta_k t} e^{i(2\pi f_k t + \phi_k)},$$

where $f_k \in \left]-\frac{1}{2}, \frac{1}{2}\right]$ is the frequency, $\delta_k \in \mathbb{R}$ the damping (or amplifying) factor, $a_k \in \mathbb{R}_+^*$ the amplitude and $\phi_k \in]-\pi, \pi]$ the phase of the k^{th} component. This model can also be written in the form $s(t) = \sum\limits_{k=0}^{K-1} \alpha_k {z_k}^t$, where the complex amplitudes $\alpha_k = a_k e^{i\phi_k}$ are non-zero, and the poles $z_k = e^{\delta_k + i2\pi f_k}$ are supposed to be distinct. By applying formula (1) to the ESM model, one obtains an analytical expression of the Fisher information matrix. Then the following theorem can be derived, whose proof is developed in [12]:

Proposition II.4. *The Cramér-Rao bounds for the parameters $(\phi_k,\, \delta_k,\, f_k)$ do not depend on $a_{k'}$ for all $k' \neq k$, but they are proportional to $\frac{1}{a_k^2}$. The bound for the parameter a_k does not depend on any amplitude. Lastly, the bounds for all parameters do not depend on any phase, and they are unchanged by a translation of the full set of frequencies.*

Besides, asymptotic expansions of the Cramér-Rao bounds can be calculated analytically (the case of the complex sinusoidal model was addressed in [9], and that of the real sinusoidal model was addressed in [8]).

Proposition II.5. *The Cramér-Rao bound for the standard deviation of the noise is* $\mathrm{CRB}\{\sigma\} = \frac{\sigma^2}{4N}$. *Moreover, if all poles are on the unit circle, the Cramér-Rao bounds for the other ESM parameters admit the following first order expansions with respect to N:*

- $\mathrm{CRB}\{\delta_k\} = \frac{6\sigma^2}{N^3 a_k^2} + O\left(\frac{1}{N^4}\right)$;
- $\mathrm{CRB}\{f_k\} = \frac{6\sigma^2}{4\pi^2 N^3 a_k^2} + O\left(\frac{1}{N^4}\right)$;
- $\mathrm{CRB}\{a_k\} = \frac{2\sigma^2}{N} + O\left(\frac{1}{N^2}\right)$;
- $\mathrm{CRB}\{\phi_k\} = \frac{2\sigma^2}{N a_k^2} + O\left(\frac{1}{N^2}\right)$.

In particular, it can be noticed that the Cramér-Rao bounds for the frequencies f_k are of order $\frac{1}{N^3}$, which is seldom encountered in parametric estimation.

B. Polynomial Amplitude Complex Exponentials (PACE)

The Polynomial Amplitude Complex Exponentials (PACE) [20], or complex quasipolynomial [21] model, is a generalization of the ESM model presented above. For all $k \in \{0 \dots K-1\}$, define the partial order $M_k \in \mathbb{N}^*$. A discrete signal $s(t)$ is a quasipolynomial of order $r = \sum_{k=0}^{K-1} M_k$ if it can be written in the form

$$s(t) = \sum_{k=0}^{K-1} \alpha_k[t]\, {z_k}^t \tag{2}$$

where the poles $z_k \in \mathbb{C}^*$ are distinct, and $\forall k \in \{0, \dots, K\text{-}1\}$, $\alpha_k[t]$ is a complex polynomial of order $M_k - 1$. Note that the polynomial $\alpha_k[t]$ in equation (2) can be represented on any polynomial basis. However there exists a particular basis, which leads to a parameterization of quasipolynomials such that the Cramér-Rao bounds admit a simple closed form, as will be shown in section III. This is the basis of falling factorials:

Definition II.6 (Falling factorial)**.** For all $m \in \mathbb{Z}$, the falling factorial of order m is the polynomial[2]

$$F_m[X] = \begin{cases} 0 & \text{if } m < 0 \\ 1 & \text{if } m = 0 \\ \frac{1}{m!} \prod\limits_{m'=0}^{m-1} (X - m') & \text{if } m > 0. \end{cases}$$

The polynomial $\alpha_k[t]$ is then parameterized as follows:

$$\alpha_k[t] = \sum_{m=0}^{M_k-1} \alpha_k^{(m)}\, {z_k}^{-m}\, F_m[t], \tag{3}$$

which defines the complex amplitudes $\alpha_k^{(m)}$ of the signal ($\forall k$, $\alpha_k^{(M_k-1)} \neq 0$ since the polynomial $\alpha_k[t]$ is of order $M_k - 1$). Note that $F_m[t]$ in equation (3) is zero for $t \in \{0 \dots m-1\}$. Finally, substituting equation (3) into equation (2), we obtain the complete definition of the PACE model:

Definition II.7 (Polynomial Amplitude Complex Exponentials model)**.** A discrete signal $s(t)$ satisfies the PACE model of order $r = \sum_{k=0}^{K-1} M_k$ if it can be written in the form

$$s(t) = \sum_{k=0}^{K-1} \sum_{m=0}^{M_k-1} s_k^{(m)}(t) \tag{4}$$

where the components $s_k^{(m)}$ are defined as

$$s_k^{(m)}(t) = \alpha_k^{(m)} F_m[t]\, {z_k}^{t-m}. \tag{5}$$

The real amplitudes and phases are then defined according to[3] $a_k^{(m)} = \left|\alpha_k^{(m)}\right|$, and $\phi_k^{(m)} = \Im\left(\log\left(\alpha_k^{(m)}\right)\right)$.

III. Cramér-Rao bounds for the PACE model

The Cramér-Rao theorem, which gives a lower bound for the variance of unbiased estimators, was summarized in section II-A. It is applied here to the PACE model defined in section II-B.

A. General Cramér-Rao bounds

First, let us calculate the expected value and the covariance matrix of the observed vector as a function of the model parameters. Let $\boldsymbol{\theta}_k$ be the vector containing the $2(M_k + 1)$ real-valued parameters associated to the pole z_k of order M_k:

$$\boldsymbol{\theta}_k \triangleq \left[a_k^{(0)}, \phi_k^{(0)}, \dots, a_k^{(M_k-1)}, \phi_k^{(M_k-1)}, \delta_k, f_k\right]^T. \tag{6}$$

The vector $\boldsymbol{\theta}_k$ belongs to the open subset $\Theta_k = \left(\mathbb{R}_+^* \times \mathbb{R}\right)^{M_k} \times (\mathbb{R} \times \mathbb{R})$ of $\mathbb{R}^{2(M_k+1)}$. Let $\boldsymbol{\theta}$ be the $2r + 2K$-dimensional vector containing the set of the model parameters:

$$\boldsymbol{\theta} \triangleq \left[\boldsymbol{\theta}_0^T, \dots, \boldsymbol{\theta}_{K-1}^T\right]^T.$$

In other respects, consider an additive colored noise, whose covariance matrix is $\boldsymbol{R}(\boldsymbol{\vartheta}) = \sigma^2 \boldsymbol{\Gamma}$, where $\boldsymbol{\Gamma}$ is an $N \times N$ positive definite Toeplitz matrix, whose diagonal coefficients are equal to 1 ($\boldsymbol{\Gamma} = \boldsymbol{I}_N$ in the case of a white noise). Throughout the paper, we will suppose that $\boldsymbol{\Gamma}$ is known, and that the variance σ^2 is the only unknown parameter related to the additive noise. Therefore the vector containing all the model parameters is $\boldsymbol{\vartheta} = \left[\sigma, \boldsymbol{\theta}^T\right]^T$. It belongs to the open subset $\Theta = \mathbb{R}_+^* \times \Theta_0 \times \dots \times \Theta_{K-1}$ of $\mathbb{R}^{1+2r+2K}$. Let $\boldsymbol{x}$ be the vector which contains the N samples of the observed signal. Then $\boldsymbol{x} \sim \mathcal{N}(\boldsymbol{s}(\boldsymbol{\vartheta}), \boldsymbol{R}(\boldsymbol{\vartheta}))$. The dependency of $\boldsymbol{s}$ and $\boldsymbol{R}$ with respect to the model parameters is explicitly defined below.

[2]Note that this definition does not exactly match the classical definition of the falling factorial [32], [33], from which the multiplicative factor $\frac{1}{m!}$ is missing.

[3]In the whole paper, the notation $\log(.)$ denotes the determination of the complex logarithm which corresponds to an angle lying in the range $]-\pi, \pi[$.

For all $z \in \mathbb{C}$, define the vector $\boldsymbol{v}(z) = \left[1,\, z, \ldots, z^{N-1}\right]^T$. Equations (4) and (5) show that the vector $\boldsymbol{s}(\boldsymbol{\vartheta}) = [s(0) \ldots s(N-1)]^T$ can be decomposed as

$$\boldsymbol{s}(\boldsymbol{\vartheta}) = \sum_{k=0}^{K-1} \sum_{m=0}^{M_k-1} \boldsymbol{s}_k^{(m)}(\boldsymbol{\theta}_k) \tag{7}$$

where the component $s_k^{(m)}(\boldsymbol{\theta}_k)$ can be written in the form

$$s_k^{(m)}(\boldsymbol{\theta}_k) = \alpha_k^{(m)} \frac{1}{m!} \frac{d^m \boldsymbol{v}}{dz^m}(z_k). \tag{8}$$

The application of proposition II.3 to the PACE model leads to an expression of the Fisher information matrix which can be analytically inverted (the developments are presented in the appendix, section A), from which the bounds given in proposition III.1 below are derived. Before that, it is necessary to introduce some notations.

For all pairs of poles (z_k, z'_k), where $k, k' \in \{0 \ldots K-1\}$, define the matrix $\boldsymbol{Z}_{(k,k')}$ of dimension $(M_k+1) \times (M_{k'}+1)$, whose coefficients are indexed by the indices $m \in \{0 \ldots M_k\}$ and $m' \in \{0 \ldots M_{k'}\}$:

$$Z_{(k,k')}(m, m') = \frac{1}{m!} \frac{1}{m'!} \frac{d^m \boldsymbol{v}(z_k)}{dz_k^m}^H \boldsymbol{\Gamma}^{-1} \frac{d^{m'} \boldsymbol{v}(z_{k'})}{dz_{k'}^{m'}}. \tag{9}$$

Finally, consider the matrix $\boldsymbol{Z}$ of dimension $(r+K) \times (r+K)$, defined by concatenating the blocks $\boldsymbol{Z}_{(k,k')}$ for all $k, k' \in \{0 \ldots K-1\}$. Its inverse $\boldsymbol{Z}^{-1}$ can then be decomposed into $K \times K$ blocks denoted $\boldsymbol{Z}^{-1}_{(k',k)}$ for all $k', k \in \{0 \ldots K-1\}$, of dimension $(M_{k'}+1) \times (M_k+1)$. The coefficients of the block $\boldsymbol{Z}^{-1}_{(k',k)}$, indexed by the indices $m' \in \{0 \ldots M_{k'}\}$ and $m \in \{0 \ldots M_k\}$, are denoted $Z^{-1}_{(k',k)}(m', m)$.

Proposition III.1 (Cramér-Rao bounds for the PACE model)**.** *The Cramér-Rao bound (CRB) for the standard deviation of the noise is* $\mathrm{CRB}\{\sigma\} = \frac{\sigma^2}{4N}$*. For all* $k \in \{0 \ldots K-1\}$*, the CRB for the damping factors and the frequencies are*

$$\mathrm{CRB}\{\delta_k\} = \frac{\sigma^2 e^{-2\delta_k}}{2\left(M_k a_k^{(M_k-1)}\right)^2} Z^{-1}_{(k,k)}(M_k, M_k) \tag{10}$$

$$\mathrm{CRB}\{f_k\} = \frac{1}{4\pi^2} \mathrm{CRB}\{\delta_k\} \tag{11}$$

Moreover, the CRB for the amplitudes are

$$\mathrm{CRB}\left\{a_k^{(0)}\right\} = \frac{\sigma^2}{2} Z^{-1}_{(k,k)}(0,0)$$

and for all $m \in \{1 \ldots M_k - 1\}$*,*

$$\mathrm{CRB}\left\{a_k^{(m)}\right\} = \frac{\sigma^2}{2} \left(\frac{m}{M_k} \frac{a_k^{(m-1)}}{a_k^{(M_k-1)}}\right)^2 Z^{-1}_{(k,k)}(M_k, M_k)$$
$$+ \frac{\sigma^2}{2} Z^{-1}_{(k,k)}(m,m) - \sigma^2 \Re\left(\frac{m}{M_k} \frac{\alpha_k^{(m-1)}}{\alpha_k^{(M_k-1)}} Z^{-1}_{(k,k)}(M_k, m)\right).$$

Finally, for all $m \in \{0 \ldots M_k - 1\}$*, the CRB for the phases* $\phi_k^{(m)}$ *are defined only if* $a_k^{(m)} \neq 0$*:*

$$\mathrm{CRB}\left\{\phi_k^{(m)}\right\} = \frac{1}{{a_k^{(m)}}^2} \mathrm{CRB}\left\{a_k^{(m)}\right\}. \tag{12}$$

These formulae call for several comments:

- the bounds for δ_k and f_k do not depend on any phase,
- they are inversely proportional to $\left(a_k^{(M_k-1)}\right)^2$, where $a_k^{(M_k-1)}$ is the amplitude of highest index associated to the pole z_k, but they do not depend on any other amplitude,
- if the noise is white ($\boldsymbol{\Gamma} = \boldsymbol{I}_N$), they depend on the frequencies only by their differences (they are unchanged by a translation of the whole set of frequencies[4]).

Corollary III.2. *The Cramér-Rao bounds for the frequencies and damping factors presented in proposition III.1 can be rewritten in the following form:*

$$\begin{aligned} \mathrm{CRB}\{\delta_k\} &= \frac{\sigma^2 e^{-2\delta_k}}{2\left(M_k a_k^{(M_k-1)}\right)^2} \frac{F_k(z_0, \ldots, z_{K-1})}{\prod\limits_{k' \neq k} |z_{k'} - z_k|^{2(M_{k'}+1)}} \\ \mathrm{CRB}\{f_k\} &= \frac{1}{4\pi^2} \mathrm{CRB}\{\delta_k\} \end{aligned} \tag{13}$$

where $F_k(z_0, \ldots, z_{K-1})$ *is a continuous function, with finite and positive values.*

The proof of corollary III.2 is presented in the appendix (section A). This corollary shows the divergence of the Cramér-Rao bounds when two poles become arbitrarily close. Indeed, for any $k \in \{0 \ldots K-1\}$, if we let $z_{k'} = z_k + \varepsilon$ for some $k' \in \{0 \ldots K-1\}$, then $\mathrm{CRB}\{\delta_k\} \sim \frac{C}{|\varepsilon|^{2(M_{k'}+1)}}$ when $\varepsilon \to 0$ (and C does not depend on ε). In an asymptotic context, the expressions of the Cramér-Rao bounds given in proposition III.1 can be simplified, as shown in the following section.

B. Asymptotic Cramér-Rao bounds

In this section, simplified expressions of the Cramér-Rao bounds are proposed in the particular case where

- white noise ($\boldsymbol{\Gamma} = \boldsymbol{I}_N$) is considered;
- all poles are supposed to be on the unit circle ($\forall k \in \{0 \ldots K-1\}$, $\delta_k \to 0$);
- infinite observation length is assumed[5].

Proposition III.3 (Asymptotic Cramér-Rao bounds)**.** *In the particular case of white noise and all poles on the unit circle, the signal-to-noise ratio of the component* $s_k^{(m)}(t)$ *defined in equation* (5) *is* $\mathrm{SNR}_k^{(m)} = \frac{1}{\sigma^2} \frac{1}{N} \sum\limits_{t=0}^{N-1} |s_k^{(m)}(t)|^2$ *and admits the asymptotic expansion* $\mathrm{SNR}_k^{(m)} \sim \dfrac{\left(a_k^{(m)} N^m\right)^2}{(2m+1)\, m!^2\, \sigma^2}$.

[4]More precisely, it can be shown that the diagonal coefficients of the matrix $\boldsymbol{Z}^{-1}$ are unchanged by a translation of the whole set of frequencies.

[5]More precisely, it is supposed that $N >> \min\limits_{k \neq k'} \frac{1}{|f_k - f'_k|}$, which means that all spectral components are well separated.

For all $k \in \{0 \dots K-1\}$,

$$\mathrm{CRB}\{\delta_k\} \sim \frac{\sigma^2}{N^{2M_k+1} a_k^{(M_k-1)^2}} \frac{(2M_k+1)!^2}{2M_k^2(2M_k+1)M_k!^2} \tag{14}$$

$$\sim \frac{1}{N^3 \mathrm{SNR}_k^{(M_k-1)}} \frac{(2M_k+1)!^2}{2(4M_k^2-1)M_k!^4} \tag{15}$$

$$\mathrm{CRB}\{f_k\} \sim \frac{1}{4\pi^2} \mathrm{CRB}\{\delta_k\} \tag{16}$$

and for all $m \in \{0 \dots M_k - 1\}$,

$$\mathrm{CRB}\{a_k^{(m)}\} \sim \frac{\sigma^2}{N^{2m+1}} \frac{(M_k+1+m)!^2}{2(2m+1)m!^2(M_k-m)!^2} \tag{17}$$

$$\mathrm{CRB}\{\phi_k^{(m)}\} \sim \frac{1}{a_k^{(m)^2}} \mathrm{CRB}\left\{a_k^{(m)}\right\} \tag{18}$$

$$\sim \frac{1}{N\mathrm{SNR}_k^{(m)}} \frac{(M_k+1+m)!^2}{2(2m+1)^2(m!)^4(M_k\text{-}m)!^2}. \tag{19}$$

Proposition III.3 is proved in the appendix, in section B. It is important to note that:

- the bounds for δ_k and f_k are inversely proportional to the product of N^3 and the signal-to-noise ratio of the component $s_k^{(M_k-1)}$ (this is a well-known result in the case of the ESM model),
- these bounds rapidly increase with the order of the pole z_k. More precisely, it can be verified[6] that

$$\frac{(2M_k+1)!^2}{(4M_k^2-1)M_k!^4} \sim \frac{1}{\pi e^2} \frac{2^{4M_k}}{M_k}. \tag{20}$$

  Thus the estimation of a pole is all the more difficult as its order is high.

Note that in the case of single poles, the formulae given in proposition III.3 are identical to those provided in proposition II.5 (which was established in the framework of the ESM model).

IV. SIMULATION RESULTS

This section illustrates the variations of the Cramér-Rao bounds with respect to the parameters of the PACE model. Since the dependency on the amplitudes and the variance σ^2 is rather straightforward, as shown in propositions III.1 and III.3, we focus below on the dependency on the frequency gap between two components (section IV-A), the damping factor (section IV-B), the spectral flatness (section IV-C), and the order of a pole (section IV-D). In the figures below, only the CRB for the frequencies or the CRB for the damping factors is represented, since both are equivalent according to (11). In the same way, the *relative* CRB for each amplitude (*i.e.* the CRB normalized by the squared amplitude), also illustrated below, is equal to the CRB for the corresponding phase, according to equation (12).

[6]The proof of this result involves Stirling's approximation.

A. Variation of the Cramér-Rao bounds with respect to the frequency gap

It was shown above that in the case of white noise, the Cramér-Rao bound for a particular frequency depends on the whole set of frequencies only by their differences. Here we consider a signal of length $N = 200$, composed of two undamped components ($K = 2$) of same order $M_0 = M_1 = 1$, in white noise ($\boldsymbol{\Gamma} = \boldsymbol{I}_N$ and $\sigma^2 = 1$). These components have zero phases, and same amplitudes, such that $\mathrm{SNR}_0^{(0)}$ = $\mathrm{SNR}_1^{(0)}$ = 50 dB.

Figure 1 shows the variations of the Cramér-Rao bounds with respect to the frequency gap $\Delta f = |f_1 - f_0| \in]0, 0.5]$. It can be noticed that the variation rate of the bounds is broken at $\Delta f = \frac{1}{N} = 5\,10^{-3}$, which corresponds to the resolution limit of Fourier analysis. If $\Delta f >> \frac{1}{N}$, there is no resolution problem, and the bounds do not depend on the frequency gap. If $\Delta f << \frac{1}{N}$, the bounds are asymptotically proportional to $\frac{1}{\Delta f^4}$, as suggested by equation (13). Besides, if $\Delta f < 10^{-3}$, $\mathrm{CRB}\{f_k\} \geq \Delta f^2$, which means that both frequencies cannot be resolved (the bound is of the order of the value to be estimated), and $\mathrm{CRB}\left\{a_k^{(0)}\right\} \geq a_k^{(0)^2}$, which means that both amplitudes cannot be estimated correctly.

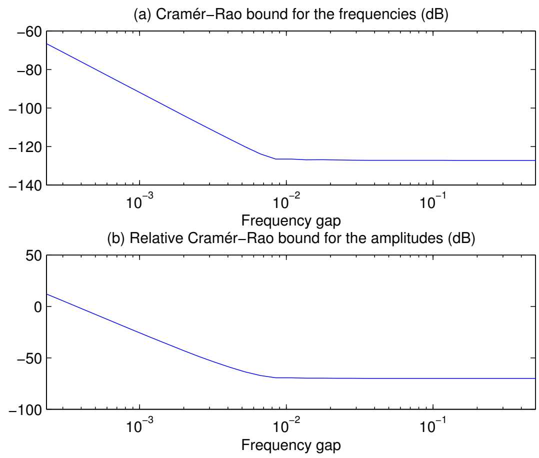

Fig. 1. Variation of the Cramér-Rao bounds with respect to the frequency gap

B. Variation of the Cramér-Rao bounds with respect to the damping factor

We consider a signal of length $N = 100$, composed of one component ($K = 1$) of order $M_0 = 1$, in white noise ($\boldsymbol{\Gamma} = \boldsymbol{I}_N$ and $\sigma^2 = 1$). This component has zero frequency and phase, and an amplitude such that $\mathrm{SNR}_0^{(0)}$ = 50 dB.

Figure 2 shows the variations of the Cramér-Rao bounds with respect to the damping factor δ_0. It can be noticed in figure 2-a that the lowest bound for δ_0 is obtained when the component is undamped, and that this bound increases with the magnitude of δ_0. Regarding the estimation of the amplitudes and phases, this symmetry is broken: figure 2-b shows that the lowest values of the bound are obtained when

$\delta_0 \leq 0$. This may be explained by the fact that the amplitude parameter corresponds to the amplitude at the beginning of the signal component. Therefore, when the noise level and the SNR of this component are constant, the amplitude parameter decreases when δ_0 increases. Consequently, the relative CRB of this parameter increases when δ_0 increases.

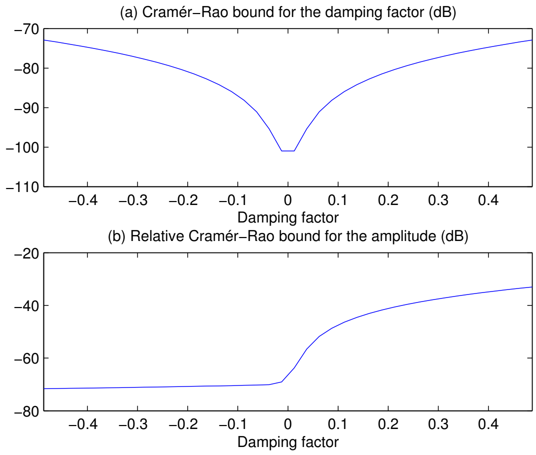

Fig. 2. Variation of the Cramér-Rao bounds with respect to the damping factor

C. Variation of the Cramér-Rao bounds with respect to the spectral flatness of the noise

We consider a signal of length $N = 100$, composed of one undamped component ($K = 1$) of order $M_0 = 1$, in colored noise. This component has a zero phase, a normalized frequency equal to 0.05, and an amplitude such that $\text{SNR}_0^{(0)}$ = 50 dB. The noise is obtained by filtering a white noise by the filter of transfer function $H_a(z) = \frac{1}{1-a\,z^{-1}}$ (where $0 \leq a < 1$), such that $\mathbf{\Gamma} = \text{Toeplitz}(1, a, a^2, \ldots a^{N-1})$. The Spectral Flatness (SF) measure of the noise is defined as

$$\text{SF}(a) = \frac{\exp\left(\int_0^1 \log\left(|H_a(e^{i2\pi f})|^2\right) \text{d}f\right)}{\int_0^1 |H_a(e^{i2\pi f})|^2 \text{d}f}.$$

By tuning the parameter a, it is possible to make the spectral flatness map the range $]0, 1]$ (the case SF= 1 corresponds to white noise).

Figure 3 shows the variations of the Cramér-Rao bounds with respect to the spectral flatness. It can be noticed in figures 3-a and 3-b that the Cramér-Rao bound admits a maximum when SF $\simeq 0.5$. When SF becomes very low, the decrease of the Cramér-Rao bound may be explained by the fact that the power spectral density of the noise becomes a sharp peak centered at the null frequency, and the noise level is lower around the sinusoidal component.

D. Variation of the Cramér-Rao bounds with respect to the pole order

We consider a signal of length $N = 20$, composed of one undamped component ($K = 1$) of order $M_0 \in \{1 \ldots 10\}$, in white noise ($\mathbf{\Gamma} = \boldsymbol{I}_N$ and $\sigma^2 = 1$). This component has zero phases, and amplitudes such that $\text{SNR}_0^{(M_0-1)}$ = 50 dB, and $\forall m < M_0 - 1$, $\text{SNR}_0^{(m)}$ = 0. The corresponding pole is $z_0 = 1$.

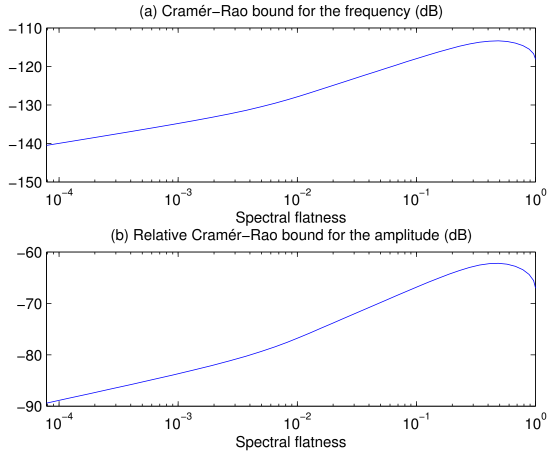

Fig. 3. Variation of the Cramér-Rao bounds with respect to the spectral flatness of the noise

Figure 4 shows the variations of the Cramér-Rao bounds with respect to the pole order M_0. It can be noticed that these bounds increase exponentially with M_0, which means that the estimation of the model parameters is all the more difficult as the model order is high. More precisely, equations (16) and (20) show that the bounds are asymptotically proportional to $\frac{16^{M_0}}{M_0}$. Besides, if $M_0 \geq 10$, the frequency cannot be estimated correctly, since figure 4-a shows that $\text{CRB}\{f_0\} \simeq 1$. The problem is even more critical for the amplitudes, since it can be noticed in figure 4-b that $\text{CRB}\{a_0^{(M_0-1)}\} \geq \left(a_0^{(M_0-1)}\right)^2$ as soon as $M_0 \geq 7$. As a conclusion, the practical estimation of the PACE model parameters is only possible for low order poles. This can be explained by the fact that the matrix $\boldsymbol{Z}$ defined in equation (9) is very badly conditioned for large orders M_k and high values of N.

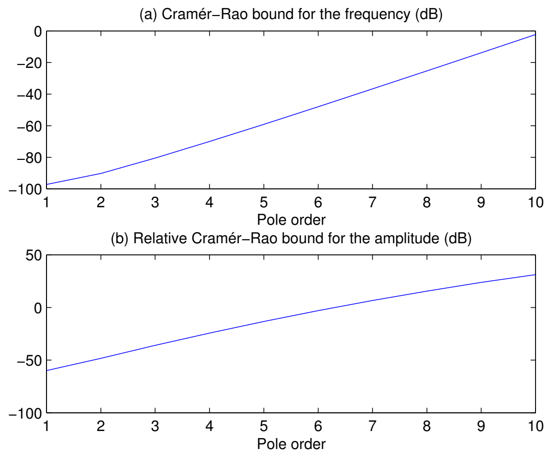

Fig. 4. Variation of the Cramér-Rao bounds with respect to the pole order

V. CONCLUSIONS

In this paper, the Cramér-Rao bounds for the quasipolynomial PACE model have been analytically calculated in the general case, and their expressions have been simplified under the hypothesis of infinite observation length (in the case of white noise and all poles on the unit circle). It was shown that the bounds for the frequencies and damping factors do not depend on any phase, and that they are inversely proportional to the squared amplitude of highest index associated to the corresponding pole. However they do not depend on any other amplitude. Besides, if the noise is white, they depend on the frequencies only by their differences (*i.e.* they are unchanged by a translation of the whole set of frequencies). In an asymptotic context, it was shown that they are inversely proportional to the product of the cubed length of the signal and the signal-to-noise ratio of the component of highest index associated to the corresponding pole. Our simulation results also showed that the Cramér-Rao bounds for the parameters associated to a multiple pole present an exponential increase with the order of the pole. More precisely, it appears that the practical estimation of the PACE amplitude parameters is only possible if the exponentials are modulated by polynomials of order lower than 5.

This work on the Cramér-Rao bounds is applied in [29], [30], in order to analyze the performance of the generalized ESPRIT algorithm introduced in [20] to estimate the PACE model. In the same paper, we present an application of this algorithm to the problem of ARMA filter synthesis, in the context of system conversion from continuous time to discrete time.

APPENDIX

A. General bounds for the PACE model

Proof of proposition III.1.: First, let us compute the Fisher information matrix related to the PACE model. We start from the expression given in proposition II.3, which involves the partial derivatives of the covariance matrix $\boldsymbol{R}$ and of the expected value $\boldsymbol{s}$ with respect to the model parameters.

First, remember that the vector containing the whole set of parameters is $\boldsymbol{\vartheta} = \left[\sigma, \boldsymbol{\theta}^T\right]^T$, where $\boldsymbol{\theta} = \left[\boldsymbol{\theta}_0^T, \ldots, \boldsymbol{\theta}_{K-1}^T\right]^T$ and $\boldsymbol{\theta}_k$ was defined in equation (6):

$$\boldsymbol{\theta}_k = \left[a_k^{(0)}, \phi_k^{(0)}, \ldots, a_k^{(M_k-1)}, \phi_k^{(M_k-1)}, \delta_k, f_k\right]^T.$$

The partial derivatives of the covariance matrix $\boldsymbol{R}(\vartheta) = \sigma^2\boldsymbol{\Gamma}$ with respect to all the model parameters are zero, except $\frac{\partial \boldsymbol{R}}{\partial \sigma} = 2\sigma\boldsymbol{\Gamma}$. Moreover, the partial derivative of the expected value $\boldsymbol{s}(\boldsymbol{\vartheta})$ with respect to σ is zero. Thus the matrix $\boldsymbol{F}(\boldsymbol{\vartheta})$ defined in equation (1), of dimension $(1+2r+2K)\times(1+2r+2K)$, can be written in the form

$$\boldsymbol{F}(\boldsymbol{\vartheta}) = \left[\begin{array}{c|c} \frac{4N}{\sigma^2} & \boldsymbol{0}_{1\times(2r+2K)} \\ \hline \boldsymbol{0}_{(2r+2K)\times 1} & \boldsymbol{F}'(\boldsymbol{\theta}) \end{array}\right],$$

where $\boldsymbol{F}'(\boldsymbol{\theta})$ is a $(2r+2K)\times(2r+2K)$ matrix. Thus we get $\boldsymbol{F}(\boldsymbol{\vartheta})^{-1} = \left[\begin{array}{c|c} \frac{\sigma^2}{4N} & 0 \\ \hline 0 & \boldsymbol{F}'(\boldsymbol{\theta})^{-1} \end{array}\right]$ from which we extract $\text{CRB}\{\sigma\} = \frac{\sigma^2}{4N}$. In order to obtain the bounds for the other parameters, it is necessary to calculate and inverse the matrix $\boldsymbol{F}'(\boldsymbol{\theta})$. First, equation (1) shows that

$$F'_{(i,j)}(\boldsymbol{\theta}) = \frac{2}{\sigma^2}\mathcal{R}e\left(\frac{\partial \boldsymbol{s}}{\partial \theta_i}^H \boldsymbol{\Gamma}^{-1}\frac{\partial \boldsymbol{s}}{\partial \theta_j}\right).$$

Following equation (6), the matrix $\boldsymbol{F}'(\boldsymbol{\theta})$ can be decomposed into $K\times K$ sub-blocks $\boldsymbol{F}'_{(k,k')}(\boldsymbol{\theta}_k, \boldsymbol{\theta}_{k'})$ of dimension $2(M_k+1)\times 2(M_{k'}+1)$:

$$\boldsymbol{F}'_{(k,k')}(\boldsymbol{\theta}_k, \boldsymbol{\theta}_{k'}) = \frac{2}{\sigma^2}\mathcal{R}e\left(\frac{\partial \boldsymbol{s}}{\partial \boldsymbol{\theta}_k}^H \boldsymbol{\Gamma}^{-1}\frac{\partial \boldsymbol{s}}{\boldsymbol{\theta}_{k'}}\right), \tag{21}$$

where the Jacobian matrix $\frac{\partial \boldsymbol{s}}{\partial \boldsymbol{\theta}_k}$ has dimension $N\times 2(M_k+1)$:

$$\frac{\partial \boldsymbol{s}}{\partial \boldsymbol{\theta}_k} = \left[\frac{\partial \boldsymbol{s}}{\partial a_k^{(0)}}, \frac{\partial \boldsymbol{s}}{\partial \phi_k^{(0)}} \cdots \frac{\partial \boldsymbol{s}}{\partial a_k^{(M_k-1)}}, \frac{\partial \boldsymbol{s}}{\partial \phi_k^{(M_k-1)}}, \frac{\partial \boldsymbol{s}}{\partial \delta_k}, \frac{\partial \boldsymbol{s}}{\partial f_k}\right].$$

Besides, the partial derivatives of the vector $\boldsymbol{s}$ defined in equations (7) and (8) with respect to the model parameters are

$$\begin{aligned} \frac{\partial \boldsymbol{s}}{\partial a_k^{(m)}} &= e^{i\,\phi_k^{(m)}}\ \frac{1}{m!}\frac{d^m \boldsymbol{v}(z_k)}{dz_k^m} \\ \frac{\partial \boldsymbol{s}}{\partial \phi_k^{(m)}} &= i\,\alpha_k^{(m)}\ \frac{1}{m!}\frac{d^m \boldsymbol{v}(z_k)}{dz_k^m} \\ \frac{\partial \boldsymbol{s}}{\partial \delta_k} &= z_k\sum_{m=1}^{M_k} m\,\alpha_k^{(m-1)}\ \frac{1}{m!}\frac{d^m \boldsymbol{v}(z_k)}{dz_k^m} \\ \frac{\partial \boldsymbol{s}}{\partial f_k} &= i\,2\pi z_k\sum_{m=1}^{M_k} m\,\alpha_k^{(m-1)}\ \frac{1}{m!}\frac{d^m \boldsymbol{v}(z_k)}{dz_k^m} \end{aligned} \tag{22}$$

where $\boldsymbol{v}(z) = \left[1, z, \ldots, z^{N-1}\right]^T$.

Substituting equation (22) into equation (21) shows that each block $\boldsymbol{F}'_{(k,k')}(\boldsymbol{\theta}_k, \boldsymbol{\theta}_{k'})$ can be factorized in the form

$$\boldsymbol{F}'_{(k,k')}(\boldsymbol{\theta}_k, \boldsymbol{\theta}_{k'}) = \frac{2}{\sigma^2}\boldsymbol{A}_k\,\boldsymbol{F}''_{(k,k')}(\boldsymbol{\theta}_k, \boldsymbol{\theta}_{k'})\ \boldsymbol{A}_{k'} \tag{23}$$

where

- $\forall k$, $\boldsymbol{A}_k$ is the $2(M_k+1)\times 2(M_k+1)$ diagonal matrix $\boldsymbol{A}_k = \text{diag}(1, a_k^{(0)}, 1, a_k^{(1)}, \ldots, 1, a_k^{(M_k-1)}, 1, 2\pi)$;
- $\boldsymbol{F}''_{(k,k')}(\boldsymbol{\theta}_k, \boldsymbol{\theta}_{k'})$ is a $2(M_k+1)\times 2(M_{k'}+1)$ matrix.

The next step consists in factorizing the matrix $\boldsymbol{F}''_{(k,k')}(\boldsymbol{\theta}_k, \boldsymbol{\theta}_{k'})$. For any complex matrix $\boldsymbol{M}$ of dimension $d\times d'$, define the real-valued matrix $\mathcal{R}_2(\boldsymbol{M})$, of dimension $(2d)\times(2d')$, which is formed of $d\times d$ sub-blocks $\mathcal{R}_2(\boldsymbol{M})_{(i,j)}$ of dimension 2×2, defined as

$$\mathcal{R}_2(\boldsymbol{M})_{(i,j)} = \left[\begin{array}{cc} \Re\left(M_{(i,j)}\right) & -\Im\left(M_{(i,j)}\right) \\ \Im\left(M_{(i,j)}\right) & \Re\left(M_{(i,j)}\right) \end{array}\right].$$

Then after some derivations, it can be verified that each block $\boldsymbol{F}''_{(k,k')}(\boldsymbol{\theta}_k, \boldsymbol{\theta}_{k'})$ can be factorized in the form

$$\boldsymbol{F}''_{(k,k')}(\boldsymbol{\theta}_k, \boldsymbol{\theta}_{k'}) = \mathcal{R}_2\left(\boldsymbol{\Phi}_k^H\boldsymbol{J}_k^H\boldsymbol{Z}_{(k,k')}\boldsymbol{J}_{k'}\boldsymbol{\Phi}_{k'}\right) \tag{24}$$

where

- $\forall k, k'$, the matrix $\boldsymbol{Z}_{(k,k')}$ is defined in equation (9);
- $\forall k$, $\boldsymbol{\Phi}_k$ is the $(M_k+1)\times(M_k+1)$ diagonal matrix $\boldsymbol{\Phi}_k = \text{diag}\left(e^{i\,\phi_k^{(0)}}, e^{i\,\phi_k^{(1)}}, \ldots, e^{i\,\phi_k^{(M_k-1)}}, z_k\right)$;

- $\forall k$, $\boldsymbol{J}_k$ is the $(M_k+1)\times(M_k+1)$ matrix

$$\boldsymbol{J}_k = \left[\begin{array}{c|c} \boldsymbol{I}_{M_k} & \begin{array}{c} 0 \\ \alpha_k^{(0)} \\ \vdots \\ (M_k-1)\,\alpha_k^{(M_k-2)} \end{array} \\ \hline \boldsymbol{0}_{(1\times M_k)} & M_k\,\alpha_k^{(M_k-1)} \end{array}\right].$$

Then define $\boldsymbol{A} = \mathrm{diag}(\boldsymbol{A}_0,\ldots,\boldsymbol{A}_{K-1})$, $\boldsymbol{\Phi} = \mathrm{diag}(\boldsymbol{\Phi}_0,\ldots,\boldsymbol{\Phi}_{K-1})$, and $\boldsymbol{J} = \mathrm{diag}(\boldsymbol{J}_0,\ldots,\boldsymbol{J}_{K-1})$. Substituting equation (24) into equation (23) shows that the full matrix $\boldsymbol{F}'(\boldsymbol{\theta})$ can be factorized in the form

$$\boldsymbol{F}'(\boldsymbol{\theta}) = \frac{2}{\sigma^2}\,\boldsymbol{A}\,\mathcal{R}_2\left(\boldsymbol{\Phi}^H\,\boldsymbol{J}^H\boldsymbol{Z}\,\boldsymbol{J}\,\boldsymbol{\Phi}\right)\boldsymbol{A}.$$

Besides, it can be verified that the operator $\mathcal{R}_2$ commutes with the matrix inversion operator: $\forall \boldsymbol{M}\in\mathbb{C}^{d\times d}$, $(\mathcal{R}_2(\boldsymbol{M}))^{-1} = \mathcal{R}_2(\boldsymbol{M}^{-1})$. Consequently, the inverse of the matrix $\boldsymbol{F}'(\boldsymbol{\theta})$ is

$$\boldsymbol{F}'(\boldsymbol{\theta})^{-1} = \frac{\sigma^2}{2}\,\boldsymbol{A}^{-1}\mathcal{R}_2\left(\boldsymbol{\Phi}^{-1}\,\boldsymbol{J}^{-1}\boldsymbol{Z}^{-1}\,\boldsymbol{J}^{-H}\boldsymbol{\Phi}^{-H}\right)\boldsymbol{A}^{-1}.$$

It can be decomposed into $K\times K$ sub-blocks of dimension $2(M_{k'}+1)\times 2(M_k+1)$, denoted $\boldsymbol{F}'^{-1}_{(k',k)}(\boldsymbol{\theta}_{k'},\boldsymbol{\theta}_k)$, which are equal to

$$\frac{\sigma^2}{2}\boldsymbol{A}_{k'}^{-1}\mathcal{R}_2\left(\boldsymbol{\Phi}_{k'}^{-1}\boldsymbol{J}_{k'}^{-1}\boldsymbol{Z}^{-1}_{(k',k)}\boldsymbol{J}_k^{-H}\boldsymbol{\Phi}_k^{-H}\right)\boldsymbol{A}_k^{-1}, \qquad (25)$$

where the $(M_k+1)\times(M_k+1)$ matrix $\boldsymbol{J}_k^{-1}$ satisfies

$$\boldsymbol{J}_k^{-1} = \left[\begin{array}{c|c} \boldsymbol{I}_{M_k} & \begin{array}{c} 0 \\ -\frac{\alpha_k^{(0)}}{M_k\alpha_k^{(M_k-1)}} \\ \vdots \\ -\frac{(M_k-1)\,\alpha_k^{(M_k-2)}}{M_k\alpha_k^{(M_k-1)}} \end{array} \\ \hline \boldsymbol{0}_{(1\times M_k)} & \frac{1}{M_k\alpha_k^{(M_k-1)}} \end{array}\right].$$

Finally, by developing the right member of equation (25), we obtain the diagonal coefficients of the sub-blocks $\boldsymbol{F}'^{-1}_{(k,k)}(\boldsymbol{\theta}_k,\boldsymbol{\theta}_k)$ of the inverse Fisher information matrix, which yield the expressions of the Cramér-Rao bounds for the model parameters presented in proposition III.1. ■

Proof of corollary III.2: The coefficients of the inverse matrix $\boldsymbol{Z}^{-1}$ can be expressed in terms of determinants. Indeed, $Z^{-1}_{(k,k)}(M_k,M_k) = \det(\boldsymbol{Z}_k^{(M_k)})/\det(\boldsymbol{Z})$, where $\boldsymbol{Z}_k^{(M_k)}$ is the matrix extracted from $\boldsymbol{Z}$ by deleting the row and the column of same indices (k,M_k).

By applying some judicious operators[7] to the left and the right side of $\boldsymbol{Z}$, it can be shown that $\det(\boldsymbol{Z})$ can be factorized in the form

$$\det(\boldsymbol{Z}) = \prod_{k_2>k_1}|z_{k_2}-z_{k_1}|^{2(M_{k_1}+1)(M_{k_2}+1)} \times D\left(\boldsymbol{\Gamma},\{z_{k'},\,M_{k'}+1\}_{k'\in\{0\ldots K-1\}}\right) \qquad (26)$$

where D is a function of the variables $z_0,\ldots,z_{K-1}$, with *positive* values. In the same way, $\det(\boldsymbol{Z}_k^{(M_k)})$ can be factorized in the form

$$\prod_{k_2>k_1}|z_{k_2}-z_{k_1}|^{2(M_{k_1}+\boldsymbol{1}_{\{k_1\neq k\}})(M_{k_2}+\boldsymbol{1}_{\{k_2\neq k\}})} \times D(\boldsymbol{\Gamma},\{z_{k'},\,M_{k'}+\boldsymbol{1}_{\{k'\neq k\}}\}_{k'\in\{0\ldots K-1\}}).$$

Thus $Z^{-1}_{(k,k)}(M_k,M_k) = \dfrac{F_k(z_0,\ldots,z_{K-1})}{\prod\limits_{k'\neq k}|z_{k'}-z_k|^{2(M_{k'}+1)}}$, where

$$F_k(z_0,\ldots,z_{K-1}) = \frac{D(\boldsymbol{\Gamma},\{z_{k'},\,M_{k'}+\boldsymbol{1}_{\{k'\neq k\}}\}_{k'\in\{0\ldots K-1\}})}{D(\boldsymbol{\Gamma},\{z_{k'},\,M_{k'}+1\}_{k'\in\{0\ldots K-1\}})}$$

is a continuous function, with finite and positive values. ■

B. Asymptotic bounds

Proof of proposition III.3: Suppose that $\boldsymbol{\Gamma} = \boldsymbol{I}_N$ and that all poles are on the unit circle. In this case, we show below that it is possible to calculate the asymptotic expansions of the coefficients of $\boldsymbol{Z}^{-1}$ with respect to N. Indeed, $\forall k,k'\in\{0\ldots K-1\}$, $\forall(m,m')\in\{0\ldots M_k\}\times\{0\ldots M_{k'}\}$,

- if $k\neq k'$, $Z_{(k,k')}(m,m') = O(N^{m+m'})$,
- if $k=k'$, $Z_{(k,k)}(m,m')$ is equal to

$$\frac{1}{m!(1+m+m')m'!}z_k^{m-m'}N^{m+m'+1} + O(N^{m+m'}).$$

Define the diagonal matrix $\boldsymbol{D}$ such that $\forall k,k'\in\{0\ldots K-1\}$, $\forall(m,m')\in\{0\ldots M_k\}\times\{0\ldots M_{k'}\}$,

- if $k\neq k'$, $D_{(k,k')}(m,m') = 0$,
- if $k=k'$, $D_{(k,k)}(m,m') = 0$ if $m\neq m'$, and $D_{(k,k)}(m,m) = z_k^m N^{-m-\frac{1}{2}}$ if not.

Define $\widetilde{\boldsymbol{Z}} = \boldsymbol{D}^*\boldsymbol{Z}\boldsymbol{D}$. Then $\widetilde{\boldsymbol{Z}} = \overline{\boldsymbol{Z}} + \boldsymbol{O}(\frac{1}{N})$, where $\forall k,k'\in\{0\ldots K-1\}$, $\forall(m,m')\in\{0\ldots M_k\}\times\{0\ldots M_{k'}\}$,

- if $k\neq k'$, $\overline{Z}_{(k,k')}(m,m') = 0$,
- if $k=k'$, $\overline{Z}_{(k,k)}(m,m') = \frac{1}{m!(1+m+m')m'!}$.

It can be proved that the inverse of $\overline{\boldsymbol{Z}}$ satisfies[8]

- if $k\neq k'$, $\overline{Z}^{-1}_{(k',k)}(m',m) = 0$,
- if $k=k'$, $\overline{Z}^{-1}_{(k,k)}(m',m)$ is equal to

$$\frac{(M_k+1+m')!}{(M_k-m')!}\,\frac{(-1)^{m'+m}}{m'!(1+m'+m)m!}\,\frac{(M_k+1+m)!}{(M_k-m)!}.$$

In particular, $\boldsymbol{Z}^{-1} = \boldsymbol{D}^{-1}\,\widetilde{\boldsymbol{Z}}^{-1}\,\boldsymbol{D}^{*-1}$, where $\widetilde{\boldsymbol{Z}}^{-1} = \overline{\boldsymbol{Z}}^{-1} + \boldsymbol{O}(\frac{1}{N})$. It can be deduced that $\forall k,k'\in\{0\ldots K-1\}$, $\forall(m,m')\in\{0\ldots M_k\}\times\{0\ldots M_{k'}\}$,

- if $k\neq k'$, $Z^{-1}_{(k',k)}(m',m) = O\left(N^{-(m'+m+2)}\right)$,
- if $k=k'$, $Z^{-1}_{(k,k)}(m',m)$ is asymptotic to

[7] Equation (9) implies a factorization of the matrix $\boldsymbol{Z}$ of the form $\boldsymbol{Z} = \overline{\boldsymbol{V}}^{N\,H}\,\boldsymbol{\Gamma}^{-1}\,\overline{\boldsymbol{V}}^N$, where the columns of the $N\times(r+K)$ matrix $\overline{\boldsymbol{V}}^N$ are the vectors $\frac{1}{m!}\frac{d^m\boldsymbol{v}(z_k)}{dz_k^m}$ for $0\leq k<K$ and $0\leq m\leq M_k$. It can be noticed that the matrix $\overline{\boldsymbol{V}}^N$ has a *Pascal-Vandermonde* structure [20]. Then the proof of equation (26) consists in applying to the right and the left sides of $\boldsymbol{Z} = \overline{\boldsymbol{V}}^{N\,H}\,\boldsymbol{\Gamma}^{-1}\,\overline{\boldsymbol{V}}^N$ the same operations as those which would be applied to the columns of the square Pascal-Vandermonde matrix $\overline{\boldsymbol{V}}^N_{(1:r+K,1:r+K)}$ in order to calculate its determinant.

[8] The proof of this result is tricky, and beyond the scope of this paper. It relies on number theory and combinatorics.

$$\frac{(M_k+1+m')!}{(M_k-m')!}\,\frac{(-1)^{m'+m}}{m'!(1+m'+m)m!}\,\frac{(M_k+1+m)!}{(M_k-m)!}\,\frac{z_k^{m'-m}}{N^{m'+m+1}}.$$

Finally, the formulae in proposition III.3 are obtained by substituting the expressions of the coefficients of the matrix $\boldsymbol{Z}^{-1}$ in the equations given in proposition III.1. ∎

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