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Exercises on high resolution methods

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Voir Page 4 M2 Mathématiques / Vision / Apprentissage - Audio signal analysis, indexing and transformation



Let us consider the *Exponential Sinusoidal Model* (ESM):

$$s[t] = \sum_{k=0}^{K-1} a_k e^{\delta_k t} e^{i(2\pi f_k t + \phi_k)},$$

which, to each frequency $f_k \in]-\frac{1}{2}, \frac{1}{2}]$, associates a real amplitude $a_k > 0$, a phase $\phi_k \in]-\pi, \pi]$, and a damping factor $\delta_k \in \mathbb{R}$. By defining the complex amplitudes $\alpha_k = a_k e^{i\phi_k}$ and the complex poles $z_k = e^{\delta_k + i2\pi f_k}$, this model can be rewritten in the more compact form

$$s[t] = \sum_{k=0}^{K-1} \alpha_k z_k^t.$$

In practice, the observed signal $x[t]$ never exactly fits this model. It is rather modeled as the sum of signal $s[t]$ plus a complex Gaussian white noise $b[t]$ of variance σ^2 :

$$x[t] = s[t] + b[t].$$

Remark: A complex Gaussian white noise of variance σ^2 is a complex process whose real part and imaginary part are two Gaussian white noises of same variance $\frac{\sigma^2}{2}$, independent from each other.

We assume that the signal $x[t]$ is observed on the time interval $\{0 \dots N-1\}$ of length $N > 2K$. We then consider two integers n and l such that $n > K$, $l > K$, and $N = n + l - 1$.

Finally, we define the $n \times l$ Hankel matrix which contains the N samples of the observed signal:

$$\mathbf{X} = \begin{bmatrix} x[0] & x[1] & \dots & x[l-1] \\ x[1] & x[2] & \dots & x[l] \\ \vdots & \vdots & \vdots & \vdots \\ x[n-1] & x[n] & \dots & x[N-1] \end{bmatrix}.$$

We define in the same way the Hankel matrices \mathbf{S} and \mathbf{B} of same dimension $n \times l$, from the samples of $s[t]$ and $b[t]$, respectively.

Notation:

- \mathbf{X}^T : transpose of matrix \mathbf{X} ,
- \mathbf{X}^* : conjugate of matrix \mathbf{X} ,
- \mathbf{X}^H : Hermitian transpose (conjugate transpose) of matrix \mathbf{X} .

1 Multiple Signal Classification (MUSIC)

Question 1 For all $k \in \{0 \dots K-1\}$, we consider the component $s_k[t] = \alpha_k z_k^t$. We then define the $n \times l$ Hankel matrix

$$\mathbf{S}_k = \begin{bmatrix} s_k[0] & s_k[1] & \dots & s_k[l-1] \\ s_k[1] & s_k[2] & \dots & s_k[l] \\ \vdots & \vdots & \vdots & \vdots \\ s_k[n-1] & s_k[n] & \dots & s_k[N-1] \end{bmatrix}$$



For all $z \in \mathbb{C}$, let us define the n -dimensional vector $\mathbf{v}^n(z) = [1, z, z^2, \dots, z^{n-1}]^T$, and the l -dimensional vector $\mathbf{v}^l(z) = [1, z, z^2, \dots, z^{l-1}]^T$. Then prove that $\mathbf{S}_k = \alpha_k \mathbf{v}^n(z_k) \mathbf{v}^l(z_k)^T$.

Question 2 Use the result of question 1 to prove that $\mathbf{S} = \sum_{k=0}^{K-1} \alpha_k \mathbf{v}^n(z_k) \mathbf{v}^l(z_k)^T$. Prove that this last equality can be rewritten in the form $\mathbf{S} = \mathbf{V}^n \mathbf{A} \mathbf{V}^{lT}$, where

- \mathbf{V}^n is an $n \times K$ Vandermonde matrix:

$$\mathbf{V}^n = \begin{bmatrix} 1 & 1 & \dots & 1 \\ z_0 & z_1 & \dots & z_{K-1} \\ z_0^2 & z_1^2 & \dots & z_{K-1}^2 \\ \vdots & \vdots & \vdots & \vdots \\ z_0^{n-1} & z_1^{n-1} & \dots & z_{K-1}^{n-1} \end{bmatrix}$$

- \mathbf{V}^l is an $l \times K$ Vandermonde matrix,
- $\mathbf{A} = \text{diag}(\alpha_0, \alpha_1, \dots, \alpha_{K-1})$ is a $K \times K$ diagonal matrix.

Question 3 Let us define matrix $\mathbf{R}_{ss} = \frac{1}{l} \mathbf{S} \mathbf{S}^H$. Prove that matrix \mathbf{R}_{ss} is Hermitian and positive semidefinite. Prove that \mathbf{R}_{ss} can be factorized in the form $\mathbf{R}_{ss} = \mathbf{V}^n \mathbf{P} \mathbf{V}^{nH}$, where \mathbf{P} is a $K \times K$ Hermitian and positive definite matrix. Conclude that matrix \mathbf{R}_{ss} has rank K (we remind that the poles z_k are pairwise distinct).

Question 4 Prove that matrix \mathbf{R}_{ss} is diagonalizable in an orthonormal basis, and that its eigenvalues $\{\lambda_i\}_{i=0..n-1}$ are non-negative. By assuming that they are sorted in decreasing order and by using the result of question 3, conclude that

- $\forall i \in \{0 \dots K-1\}, \lambda_i > 0$;
- $\forall i \in \{K \dots n-1\}, \lambda_i = 0$.

Question 5 Let $\widehat{\mathbf{R}}_{xx} = \frac{1}{l} \mathbf{X} \mathbf{X}^H$ and $\mathbf{R}_{xx} = \mathbb{E}[\widehat{\mathbf{R}}_{xx}]$. Similarly, let $\widehat{\mathbf{R}}_{bb} = \frac{1}{l} \mathbf{B} \mathbf{B}^H$ and $\mathbf{R}_{bb} = \mathbb{E}[\widehat{\mathbf{R}}_{bb}]$. By using equality $\mathbf{X} = \mathbf{S} + \mathbf{B}$ and the fact that the noise is centered, prove that $\mathbf{R}_{xx} = \mathbf{R}_{ss} + \mathbf{R}_{bb}$. Then prove that for a complex Gaussian white noise, $\mathbf{R}_{bb} = \sigma^2 \mathbf{I}_n$.

Question 6 For all $i \in \{0 \dots n-1\}$, let \mathbf{w}_i denote the eigenvector of matrix \mathbf{R}_{ss} corresponding to the eigenvalue λ_i . By using the result of question 5, prove that \mathbf{w}_i is also an eigenvector of \mathbf{R}_{xx} corresponding to the eigenvalue $\lambda'_i = \lambda_i + \sigma^2$. Conclude that

- $\forall i \in \{0 \dots K-1\}, \lambda'_i > \sigma^2$;
- $\forall i \in \{K \dots n-1\}, \lambda'_i = \sigma^2$.

Question 7 Let \mathbf{W} denote the matrix $[\mathbf{w}_0 \dots \mathbf{w}_{K-1}]$, and \mathbf{W}_\perp the matrix $[\mathbf{w}_K \dots \mathbf{w}_{n-1}]$. Prove that $\text{Span}(\mathbf{W}) = \text{Span}(\mathbf{V}^n)$ (you can start by proving that $\text{Span}(\mathbf{W}) \subset \text{Span}(\mathbf{V}^n)$).



Remark: The subspace spanned by \mathbf{W}_\perp is an eigen-subspace of matrix \mathbf{R}_{xx} corresponding to the eigenvalue σ^2 . This is why it is called *noise subspace*. The orthonormal matrix \mathbf{W} and the Vandermonde matrix \mathbf{V}^n span the same subspace. It thus completely characterizes the K poles of the signal, This is why it is called *signal subspace*. However, all the eigenvalues of \mathbf{R}_{xx} corresponding to the signal subspace are increased by σ^2 , which means that this subspace also contains noise.

Question 8 Prove that the poles $\{z_k\}_{k \in \{0 \dots K-1\}}$ are the solutions of equation $\|\mathbf{W}_\perp^H \mathbf{v}^n(z)\|^2 = 0$.

Remark: In practice, real signals do not rigorously fit the model, and this equation does never hold exactly. This is why the "spectral-MUSIC" method for estimating the poles consists in detecting the K highest peaks of function $z \mapsto \frac{1}{\|\mathbf{W}_\perp^H \mathbf{v}^n(z)\|^2}$. It is thus easier to implement than the maximum likelihood method, which requires the numerical optimization of a cost function of K complex variables.

2 Estimation of Signal Parameters via Rotational Invariance Techniques (ESPRIT)

Let \mathbf{V}_\downarrow^n be the $(n-1) \times K$ matrix that contains the $n-1$ first rows of \mathbf{V}^n , and \mathbf{V}_\uparrow^n the $(n-1) \times K$ matrix that contains the $n-1$ last rows of \mathbf{V}^n . Similarly, let \mathbf{W}_\downarrow be the $(n-1) \times K$ matrix that contains the $n-1$ first rows of \mathbf{W} , and \mathbf{W}_\uparrow the $(n-1) \times K$ matrix that contains the $n-1$ last rows of \mathbf{W} .

Question 1 Prove that matrices \mathbf{V}_\downarrow^n and \mathbf{V}_\uparrow^n are such that $\mathbf{V}_\uparrow^n = \mathbf{V}_\downarrow^n \mathbf{D}$, where \mathbf{D} is a $K \times K$ diagonal matrix. What are its diagonal entries?

Question 2 Prove that there is a $K \times K$ invertible matrix \mathbf{G} such that $\mathbf{V}^n = \mathbf{W} \mathbf{G}$ (we do not ask to compute \mathbf{G} , but only to prove its existence). Then prove that $\mathbf{V}_\downarrow^n = \mathbf{W}_\downarrow \mathbf{G}$ and $\mathbf{V}_\uparrow^n = \mathbf{W}_\uparrow \mathbf{G}$.

Question 3 Conclude that there is an invertible matrix $\mathbf{\Phi}$ such that $\mathbf{W}_\uparrow = \mathbf{W}_\downarrow \mathbf{\Phi}$. What are the eigenvalues of $\mathbf{\Phi}$?

Question 4 By assuming that matrix $\mathbf{W}_\downarrow^H \mathbf{W}_\downarrow$ is invertible, compute $\mathbf{\Phi}$ as a function of \mathbf{W}_\downarrow and \mathbf{W}_\uparrow .

Question 5 Propose an estimation method of the poles $\{z_k\}_{k \in \{0 \dots K-1\}}$.

Remark: The principal advantage of this method with respect to spectral-MUSIC is that it does longer require to numerically optimize a cost function in order to determine the poles, which instead are obtained by a direct computation.





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