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Exercises on high resolution methods

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Let us consider the *Exponential Sinusoidal Model* (ESM):

$$s[t] = \sum_{k=0}^{K-1} a_k\, e^{\delta_k t} e^{i(2\pi f_k t + \phi_k)},$$

which, to each frequency $f_k \in \left]-\frac{1}{2}, \frac{1}{2}\right]$, associates a real amplitude $a_k > 0$, a phase $\phi_k \in]-\pi, \pi]$, and a damping factor $\delta_k \in \mathbb{R}$. By defining the complex amplitudes $\alpha_k = a_k e^{i\phi_k}$ and the complex poles $z_k = e^{\delta_k + i2\pi f_k}$, this model can be rewritten in the more compact form

$$s[t] = \sum_{k=0}^{K-1} \alpha_k\, z_k{}^t.$$

In practice, the observed signal $x[t]$ never exactly fits this model. It is rather modeled as the sum of signal $s[t]$ plus a complex Gaussian white noise $b[t]$ of variance σ^2:

$$x[t] = s[t] + b[t].$$

Remark: A complex Gaussian white noise of variance σ^2 is a complex process whose real part and imaginary part are two Gaussian white noises of same variance $\frac{\sigma^2}{2}$, independent from each other.

We assume that the signal $x[t]$ is observed on the time interval $\{0 \dots N-1\}$ of length $N > 2K$. We then consider two integers n and l such that $n > K$, $l > K$, and $N = n + l - 1$.

Finally, we define the $n \times l$ Hankel matrix which contains the N samples of the observed signal:

$$\boldsymbol{X} = \begin{bmatrix} x[0] & x[1] & \dots & x[l-1] \\ x[1] & x[2] & \dots & x[l] \\ \vdots & \vdots & \vdots & \vdots \\ x[n-1] & x[n] & \dots & x[N-1] \end{bmatrix}.$$

We define in the same way the Hankel matrices $\boldsymbol{S}$ and $\boldsymbol{B}$ of same dimension $n \times l$, from the samples of $s[t]$ and $b[t]$, respectively.

Notation:

- $\boldsymbol{X}^T$: transpose of matrix $\boldsymbol{X}$,
- $\boldsymbol{X}^*$: conjugate of matrix $\boldsymbol{X}$,
- $\boldsymbol{X}^H$: Hermitian transpose (conjugate transpose) of matrix $\boldsymbol{X}$.

1 *Mu*ltiple *Si*gnal *C*lassification (MUSIC)

Question 1 For all $k \in \{0 \dots K-1\}$, we consider the component $s_k[t] = \alpha_k\, z_k{}^t$. We then define the $n \times l$ Hankel matrix

$$\boldsymbol{S_k} = \begin{bmatrix} s_k[0] & s_k[1] & \dots & s_k[l-1] \\ s_k[1] & s_k[2] & \dots & s_k[l] \\ \vdots & \vdots & \vdots & \vdots \\ s_k[n-1] & s_k[n] & \dots & s_k[N-1] \end{bmatrix}$$

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For all $z \in \mathbb{C}$, let us define the n-dimensional vector $\boldsymbol{v}^n(z) = [1, z, z^2, \dots, z^{n-1}]^T$, and the l-dimensional vector $\boldsymbol{v}^l(z) = [1, z, z^2, \dots, z^{l-1}]^T$. Then prove that $\boldsymbol{S_k} = \alpha_k\, \boldsymbol{v}^n(z_k)\, \boldsymbol{v}^l(z_k)^T$.

Question 2 Use the result of question 1 to prove that $\boldsymbol{S} = \sum_{k=0}^{K-1} \alpha_k\, \boldsymbol{v}^n(z_k)\, \boldsymbol{v}^l(z_k)^T$. Prove that this last equality can be rewritten in the form $\boldsymbol{S} = \boldsymbol{V}^n\, \boldsymbol{A}\, \boldsymbol{V}^{l^T}$, where

- $\boldsymbol{V}^n$ is an $n \times K$ Vandermonde matrix:

$$\boldsymbol{V}^n = \begin{bmatrix} 1 & 1 & \dots & 1 \\ z_0 & z_1 & \dots & z_{K-1} \\ {z_0}^2 & {z_1}^2 & \dots & {z_{K-1}}^2 \\ \vdots & \vdots & \vdots & \vdots \\ {z_0}^{n-1} & {z_1}^{n-1} & \dots & {z_{K-1}}^{n-1} \end{bmatrix}$$

- $\boldsymbol{V}^l$ is an $l \times K$ Vandermonde matrix,
- $\boldsymbol{A} = \mathrm{diag}(\alpha_0, \alpha_1, \dots, \alpha_{K-1})$ is a $K \times K$ diagonal matrix.

Question 3 Let us define matrix $\boldsymbol{R}_{ss} = \frac{1}{l}\, \boldsymbol{S}\, \boldsymbol{S}^H$. Prove that matrix $\boldsymbol{R}_{ss}$ is Hermitian and positive semidefinite. Prove that $\boldsymbol{R}_{ss}$ can be factorized in the form $\boldsymbol{R}_{ss} = \boldsymbol{V}^n\, \boldsymbol{P}\, \boldsymbol{V}^{n^H}$, where $\boldsymbol{P}$ is a $K \times K$ Hermitian and positive definite matrix. Conclude that matrix $\boldsymbol{R}_{ss}$ has rank K (we remind that the poles z_k are pairwise distinct).

Question 4 Prove that matrix $\boldsymbol{R}_{ss}$ is diagonalizable in an orthonormal basis, and that its eigenvalues $\{\lambda_i\}_{i=0 \dots n-1}$ are non-negative. By assuming that they are sorted in decreasing order and by using the result of question 3, conclude that

- $\forall i \in \{0 \dots K-1\}$, $\lambda_i > 0$;
- $\forall i \in \{K \dots n-1\}$, $\lambda_i = 0$.

Question 5 Let $\widehat{\boldsymbol{R}}_{xx} = \frac{1}{l} \boldsymbol{X}\, \boldsymbol{X}^H$ and $\boldsymbol{R}_{xx} = \mathbb{E}\left[\widehat{\boldsymbol{R}}_{xx}\right]$. Similarly, let $\widehat{\boldsymbol{R}}_{bb} = \frac{1}{l} \boldsymbol{B}\, \boldsymbol{B}^H$ and $\boldsymbol{R}_{bb} = \mathbb{E}\left[\widehat{\boldsymbol{R}}_{bb}\right]$. By using equality $\boldsymbol{X} = \boldsymbol{S} + \boldsymbol{B}$ and the fact that the noise is centered, prove that $\boldsymbol{R}_{xx} = \boldsymbol{R}_{ss} + \boldsymbol{R}_{bb}$. Then prove that for a complex Gaussian white noise, $\boldsymbol{R}_{bb} = \sigma^2\, \boldsymbol{I}_n$.

Question 6 For all $i \in \{0 \dots n-1\}$, let $\boldsymbol{w}_i$ denote the eigenvector of matrix $\boldsymbol{R}_{ss}$ corresponding to the eigenvalue λ_i. By using the result of question 5, prove that $\boldsymbol{w}_i$ is also an eigenvector of $\boldsymbol{R}_{xx}$ corresponding to the eigenvalue $\lambda'_i = \lambda_i + \sigma^2$. Conclude that

- $\forall i \in \{0 \dots K-1\}$, $\lambda'_i > \sigma^2$;
- $\forall i \in \{K \dots n-1\}$, $\lambda'_i = \sigma^2$.

Question 7 Let $\boldsymbol{W}$ denote the matrix $[\boldsymbol{w}_0 \dots \boldsymbol{w}_{K-1}]$, and $\boldsymbol{W}_\perp$ the matrix $[\boldsymbol{w}_K \dots \boldsymbol{w}_{n-1}]$. Prove that $\mathrm{Span}(\boldsymbol{W}) = \mathrm{Span}(\boldsymbol{V}^n)$ (you can start by proving that $\mathrm{Span}(\boldsymbol{W}) \subset \mathrm{Span}(\boldsymbol{V}^n)$).

Remark: The subspace spanned by $\boldsymbol{W}_\perp$ is an eigen-subspace of matrix $\boldsymbol{R}_{xx}$ corresponding to the eigenvalue σ^2. This is why it is called *noise subspace*. The orthonormal matrix $\boldsymbol{W}$ and the Vandermonde matrix $\boldsymbol{V}^n$ span the same subspace. It thus completely characterizes the K poles of the signal, This is why it is called *signal subspace*. However, all the eigenvalues of $\boldsymbol{R}_{xx}$ corresponding to the signal subspace are increased by σ^2, which means that this subspace also contains noise.

Question 8 Prove that the poles $\{z_k\}_{k\in\{0\dots K-1\}}$ are the solutions of equation $\left\|\boldsymbol{W}_\perp^H \boldsymbol{v}^n(z)\right\|^2 = 0$.

Remark: In practice, real signals do not rigorously fit the model, and this equation does never hold exactly. This is why the "spectral-MUSIC" method for estimating the poles consists in detecting the K highest peaks of function $z \mapsto \frac{1}{\|\boldsymbol{W}_\perp^H \boldsymbol{v}^n(z)\|^2}$. It is thus easier to implement than the maximum likelihood method, which requires the numerical optimization of a cost function of K complex variables.

2 *E*stimation of *S*ignal *P*arameters via *R*otational *I*nvariance *T*echniques (ESPRIT)

Let $\boldsymbol{V}_\downarrow^n$ be the $(n-1)\times K$ matrix that contains the $n-1$ first rows of $\boldsymbol{V}^n$, and $\boldsymbol{V}_\uparrow^n$ the $(n-1)\times K$ matrix that contains the $n-1$ last rows of $\boldsymbol{V}^n$. Similarly, let $\boldsymbol{W}_\downarrow$ be the $(n-1)\times K$ matrix that contains the $n-1$ first rows of $\boldsymbol{W}$, and $\boldsymbol{W}_\uparrow$ the $(n-1)\times K$ matrix that contains the $n-1$ last rows of $\boldsymbol{W}$.

Question 1 Prove that matrices $\boldsymbol{V}_\downarrow^n$ and $\boldsymbol{V}_\uparrow^n$ are such that $\boldsymbol{V}_\uparrow^n = \boldsymbol{V}_\downarrow^n \boldsymbol{D}$, where $\boldsymbol{D}$ is a $K\times K$ diagonal matrix. What are its diagonal entries?

Question 2 Prove that there is a $K\times K$ invertible matrix $\boldsymbol{G}$ such that $\boldsymbol{V}^n = \boldsymbol{W}\boldsymbol{G}$ (we do not ask to compute $\boldsymbol{G}$, but only to prove its existence). Then prove that $\boldsymbol{V}_\downarrow^n = \boldsymbol{W}_\downarrow \boldsymbol{G}$ and $\boldsymbol{V}_\uparrow^n = \boldsymbol{W}_\uparrow \boldsymbol{G}$.

Question 3 Conclude that there is an invertible matrix $\boldsymbol{\Phi}$ such that $\boldsymbol{W}_\uparrow = \boldsymbol{W}_\downarrow \boldsymbol{\Phi}$. What are the eigenvalues of $\boldsymbol{\Phi}$?

Question 4 By assuming that matrix $\boldsymbol{W}_\downarrow^H \boldsymbol{W}_\downarrow$ is invertible, compute $\boldsymbol{\Phi}$ as a function of $\boldsymbol{W}_\downarrow$ and $\boldsymbol{W}_\uparrow$.

Question 5 Propose an estimation method of the poles $\{z_k\}_{k\in\{0\dots K-1\}}$.

Remark: The principal advantage of this method with respect to spectral-MUSIC is that it does longer require to numerically optimize a cost function in order to determine the poles, which instead are obtained by a direct computation.

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