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Solution of the exercises on high resolution methods

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1 *Mu*ltiple *Si*gnal *C*lassification (MUSIC)

Question 1 It is sufficient to remark that

$$\boldsymbol{S_k} = \alpha_k \begin{bmatrix} 1 & z_k & \dots & {z_k}^{l-1} \\ z_k & {z_k}^2 & \dots & {z_k}^{l} \\ \vdots & \vdots & \vdots & \vdots \\ {z_k}^{n-1} & {z_k}^{n} & \dots & {z_k}^{N-1} \end{bmatrix} = \alpha_k \begin{bmatrix} 1 \\ z_k \\ \vdots \\ {z_k}^{n-1} \end{bmatrix} \begin{bmatrix} 1 & z_k & \dots & {z_k}^{l-1} \end{bmatrix}$$

Question 2 Since $s[t] = \sum_{k=0}^{K-1} x_k[t]$, we have $\boldsymbol{S} = \sum_{k=0}^{K-1} \boldsymbol{S}_k$, which leads to $\boldsymbol{S} = \sum_{k=0}^{K-1} \alpha_k\, \boldsymbol{v}^n(z_k)\, \boldsymbol{v}^l(z_k)^T$. The factorization $\boldsymbol{S} = \boldsymbol{V}^n\, \boldsymbol{A}\, \boldsymbol{V}^{l^T}$ is a simple rewriting of this last equality.

Question 3 By using the equality $\boldsymbol{S} = \boldsymbol{V}^n\, \boldsymbol{A}\, \boldsymbol{V}^{l^T}$, we get the factorization $\boldsymbol{R}_{ss} = \boldsymbol{V}^n\, \boldsymbol{P}\, \boldsymbol{V}^{n^H}$, where $\boldsymbol{P} = \frac{1}{l}\, \boldsymbol{A}\, \boldsymbol{V}^{l^T} \boldsymbol{V}^{l^*} \boldsymbol{A}^H$. Thus matrices $\boldsymbol{R}_{ss}$ and $\boldsymbol{P}$ are Hermitian and positive semi-definite by construction. Moreover, matrix $\boldsymbol{A}$ is invertible (because all α_k are non-zero) and $\boldsymbol{V}^l$ has full rank (because all poles are distinct), therefore $\boldsymbol{P}$ is invertible, which proves that $\boldsymbol{P}$ is positive definite. Moreover, since matrix $\boldsymbol{V}^n$ has full rank (because all poles are distinct), we have $\text{Span}(\boldsymbol{R}_{ss}) = \text{Span}(\boldsymbol{V}^n)$, thus $\text{rank}(\boldsymbol{R}_{ss}) = \text{rank}(\boldsymbol{V}^n) = K$.

Question 4 Since matrix $\boldsymbol{R}_{ss}$ is Hermitian, it is diagonalizable in an orthonormal basis. Moreover, since it is positive semi-definite, its eigenvalues $\{\lambda_i\}_{i=0\dots n-1}$ are non-negative. Lastly, the fact that it has exactly rank K implies that

- $\forall i \in \{0 \dots K-1\}$, $\lambda_i > 0$;
- $\forall i \in \{K \dots n-1\}$, $\lambda_i = 0$.

Question 5 $\boldsymbol{R}_{xx} = \mathbb{E}\left[\frac{1}{l} \boldsymbol{X}\, \boldsymbol{X}^H\right] = \frac{1}{l}\, \mathbb{E}\left[\boldsymbol{S}\boldsymbol{S}^H + \boldsymbol{S}\boldsymbol{B}^H + \boldsymbol{B}\boldsymbol{S}^H + \boldsymbol{B}\boldsymbol{B}^H\right]$. However, since the noise is centered, $\mathbb{E}\left[\boldsymbol{S}\boldsymbol{B}^H\right] = \boldsymbol{S}\, \mathbb{E}\left[\boldsymbol{B}^H\right] = \boldsymbol{0}$. Similarly, $\mathbb{E}\left[\boldsymbol{B}\boldsymbol{S}^H\right] = \boldsymbol{0}$, hence the result.

Question 6 $\boldsymbol{R}_{xx}\, \boldsymbol{w}_i = \left(\boldsymbol{R}_{ss} + \sigma^2 \boldsymbol{I}_n\right) \boldsymbol{w}_i = (\lambda_i + \sigma^2) \boldsymbol{w}_i$, therefore $\boldsymbol{w}_i$ is an eigenvector of $\boldsymbol{R}_{xx}$ corresponding to the eigenvalue $\lambda'_i = \lambda_i + \sigma^2$.

Question 7 For all $i \in \{0 \dots K-1\}$, $\boldsymbol{R}_{ss}\, \boldsymbol{w}_i = \lambda_i\, \boldsymbol{w}_i$. Therefore $\boldsymbol{w}_i = \frac{1}{\lambda_i}\left(\boldsymbol{V}^n\, \boldsymbol{P}\, \boldsymbol{V}^{n^H}\right) \boldsymbol{w}_i$. In particular, $\boldsymbol{w}_i \in \text{Span}(\boldsymbol{V}^n)$. Since this property holds for all $i \in \{0 \dots K-1\}$, we conclude that $\text{Span}(\boldsymbol{W}) \subset \text{Span}(\boldsymbol{V}^n)$. Moreover, $\text{rank}(\boldsymbol{W}) = K$ because $\boldsymbol{W}$ is an orthonormal matrix, hence linearly independent. In other respects, since $\text{rank}(\boldsymbol{V}^n) \leq K$, we conclude that $\text{Span}(\boldsymbol{W}) = \text{Span}(\boldsymbol{V}^n)$.

Question 8 For all $k \in \{0 \dots K-1\}$, $\boldsymbol{v}^n(z_k) \in \text{Span}(\boldsymbol{V}^n)$. However, $\text{Span}(\boldsymbol{V}^n) = \text{Span}(\boldsymbol{W}) \perp \text{Span}(\boldsymbol{W}_\perp)$. We conclude that $\boldsymbol{v}^n(z_k) \perp \text{Span}(\boldsymbol{W}_\perp)$, therefore $\left\|\boldsymbol{W}_\perp^H \boldsymbol{v}^n(z_k)\right\|^2 = 0$. Thus, all poles $\{z_k\}_{k\in\{0\dots K-1\}}$ are solutions. If there were another solution z distinct from the z_k, we would then get $\boldsymbol{v}^n(z) \perp \text{Span}(\boldsymbol{W}_\perp)$, therefore $\boldsymbol{v}^n(z_k) \in \text{Span}(\boldsymbol{V}^n)$. This is impossible because the set of vectors consisting of the columns of $\boldsymbol{V}^n$ and of $\boldsymbol{v}^n(z)$ would be linearly independent if z were different from the z_k. In conclusion, the poles $\{z_k\}_{k\in\{0\dots K-1\}}$ are the unique solutions of the equation $\left\|\boldsymbol{W}_\perp^H \boldsymbol{v}^n(z)\right\|^2 = 0$.

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2 *E*stimation of *S*ignal *P*arameters via *R*otational *I*nvariance *T*echniques (ESPRIT)

Question 1 We get $\boldsymbol{D} = \operatorname{diag}(z_0, z_1, \ldots, z_{K-1})$.

Question 2 Since matrices $\boldsymbol{V}^n$ and $\boldsymbol{W}$ are two bases of the same subspace, they satisfy the relation $\boldsymbol{V}^n = \boldsymbol{W}\boldsymbol{G}$, where $\boldsymbol{G}$ is the transfer matrix from basis $\boldsymbol{W}$ to basis $\boldsymbol{V}^n$. The equalities $\boldsymbol{V}^n_\downarrow = \boldsymbol{W}_\downarrow \boldsymbol{G}$ and $\boldsymbol{V}^n_\uparrow = \boldsymbol{W}_\uparrow \boldsymbol{G}$ are obtained by respectively extracting the $n-1$ first rows and the $n-1$ last rows of the equality $\boldsymbol{V}^n = \boldsymbol{W}\boldsymbol{G}$.

Question 3 $\boldsymbol{W}_\uparrow = \boldsymbol{V}^n_\uparrow \boldsymbol{G}^{-1} = \boldsymbol{V}^n_\downarrow \boldsymbol{D}\boldsymbol{G}^{-1} = \boldsymbol{W}_\downarrow \boldsymbol{G}\boldsymbol{D}\boldsymbol{G}^{-1}$. We thus get the equality $\boldsymbol{W}_\uparrow = \boldsymbol{W}_\downarrow \boldsymbol{\Phi}$, where $\boldsymbol{\Phi} = \boldsymbol{G}\boldsymbol{D}\boldsymbol{G}^{-1}$. This is the diagonalized form of matrix $\boldsymbol{\Phi}$. Thus the eigenvalues of $\boldsymbol{\Phi}$ are the diagonal entries of $\boldsymbol{D}$, i.e. the poles z_k.

Question 4 Since $\boldsymbol{W}_\uparrow = \boldsymbol{W}_\downarrow \boldsymbol{\Phi}$, we have $\boldsymbol{W}_\downarrow^H \boldsymbol{W}_\uparrow = \boldsymbol{W}_\downarrow^H \boldsymbol{W}_\downarrow \boldsymbol{\Phi}$, thus $\boldsymbol{\Phi} = \left(\boldsymbol{W}_\downarrow^H \boldsymbol{W}_\downarrow\right)^{-1} \boldsymbol{W}_\downarrow^H \boldsymbol{W}_\uparrow$.

Question 5 The ESPRIT method is composed of several steps :

- compute the estimator $\widehat{\boldsymbol{R}}_{xx}$ of matrix $\boldsymbol{R}_{xx}$,
- diagonalize it to obtain an estimation of matrix $\boldsymbol{W}$,
- extract from $\boldsymbol{W}$ the matrices $\boldsymbol{W}_\downarrow$ and $\boldsymbol{W}_\uparrow$,
- compute $\boldsymbol{\Phi} = \left(\boldsymbol{W}_\downarrow^H \boldsymbol{W}_\downarrow\right)^{-1} \boldsymbol{W}_\downarrow^H \boldsymbol{W}_\uparrow$,
- diagonalize matrix $\boldsymbol{\Phi}$ to obtain the poles $\{z_k\}_{k\in\{0\ldots K-1\}}$.

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