



Solution of the exercises on high resolution methods

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Voir page 2 M2 Mathématiques / Vision / Apprentissage - Audio signal analysis, indexing and transformation



1 Multiple Signal Classification (MUSIC)

Question 1 It is sufficient to remark that

$$\mathbf{S}_k = \alpha_k \begin{bmatrix} 1 & z_k & \dots & z_k^{l-1} \\ z_k & z_k^2 & \dots & z_k^l \\ \vdots & \vdots & \ddots & \vdots \\ z_k^{n-1} & z_k^n & \dots & z_k^{N-1} \end{bmatrix} = \alpha_k \begin{bmatrix} 1 \\ z_k \\ \vdots \\ z_k^{n-1} \end{bmatrix} \begin{bmatrix} 1 & z_k & \dots & z_k^{l-1} \end{bmatrix}$$

Question 2 Since $s[t] = \sum_{k=0}^{K-1} x_k[t]$, we have $\mathbf{S} = \sum_{k=0}^{K-1} \mathbf{S}_k$, which leads to $\mathbf{S} = \sum_{k=0}^{K-1} \alpha_k \mathbf{v}^n(z_k) \mathbf{v}^l(z_k)^T$. The factorization $\mathbf{S} = \mathbf{V}^n \mathbf{A} \mathbf{V}^{lT}$ is a simple rewriting of this last equality.

Question 3 By using the equality $\mathbf{S} = \mathbf{V}^n \mathbf{A} \mathbf{V}^{lT}$, we get the factorization $\mathbf{R}_{ss} = \mathbf{V}^n \mathbf{P} \mathbf{V}^{nH}$, where $\mathbf{P} = \frac{1}{l} \mathbf{A} \mathbf{V}^{lT} \mathbf{V}^{l*} \mathbf{A}^H$. Thus matrices \mathbf{R}_{ss} and \mathbf{P} are Hermitian and positive semi-definite by construction. Moreover, matrix \mathbf{A} is invertible (because all α_k are non-zero) and \mathbf{V}^l has full rank (because all poles are distinct), therefore \mathbf{P} is invertible, which proves that \mathbf{P} is positive definite. Moreover, since matrix \mathbf{V}^n has full rank (because all poles are distinct), we have $\text{Span}(\mathbf{R}_{ss}) = \text{Span}(\mathbf{V}^n)$, thus $\text{rank}(\mathbf{R}_{ss}) = \text{rank}(\mathbf{V}^n) = K$.

Question 4 Since matrix \mathbf{R}_{ss} is Hermitian, it is diagonalizable in an orthonormal basis. Moreover, since it is positive semi-definite, its eigenvalues $\{\lambda_i\}_{i=0 \dots n-1}$ are non-negative. Lastly, the fact that it has exactly rank K implies that

- $\forall i \in \{0 \dots K-1\}, \lambda_i > 0$;
- $\forall i \in \{K \dots n-1\}, \lambda_i = 0$.

Question 5 $\mathbf{R}_{xx} = \mathbb{E} \left[\frac{1}{l} \mathbf{X} \mathbf{X}^H \right] = \frac{1}{l} \mathbb{E} \left[\mathbf{S} \mathbf{S}^H + \mathbf{S} \mathbf{B}^H + \mathbf{B} \mathbf{S}^H + \mathbf{B} \mathbf{B}^H \right]$. However, since the noise is centered, $\mathbb{E} \left[\mathbf{S} \mathbf{B}^H \right] = \mathbf{S} \mathbb{E} \left[\mathbf{B}^H \right] = \mathbf{0}$. Similarly, $\mathbb{E} \left[\mathbf{B} \mathbf{S}^H \right] = \mathbf{0}$, hence the result.

Question 6 $\mathbf{R}_{xx} \mathbf{w}_i = (\mathbf{R}_{ss} + \sigma^2 \mathbf{I}_n) \mathbf{w}_i = (\lambda_i + \sigma^2) \mathbf{w}_i$, therefore \mathbf{w}_i is an eigenvector of \mathbf{R}_{xx} corresponding to the eigenvalue $\lambda'_i = \lambda_i + \sigma^2$.

Question 7 For all $i \in \{0 \dots K-1\}$, $\mathbf{R}_{ss} \mathbf{w}_i = \lambda_i \mathbf{w}_i$. Therefore $\mathbf{w}_i = \frac{1}{\lambda_i} (\mathbf{V}^n \mathbf{P} \mathbf{V}^{nH}) \mathbf{w}_i$. In particular, $\mathbf{w}_i \in \text{Span}(\mathbf{V}^n)$. Since this property holds for all $i \in \{0 \dots K-1\}$, we conclude that $\text{Span}(\mathbf{W}) \subset \text{Span}(\mathbf{V}^n)$. Moreover, $\text{rank}(\mathbf{W}) = K$ because \mathbf{W} is an orthonormal matrix, hence linearly independent. In other respects, since $\text{rank}(\mathbf{V}^n) \leq K$, we conclude that $\text{Span}(\mathbf{W}) = \text{Span}(\mathbf{V}^n)$.

Question 8 For all $k \in \{0 \dots K-1\}$, $\mathbf{v}^n(z_k) \in \text{Span}(\mathbf{V}^n)$. However, $\text{Span}(\mathbf{V}^n) = \text{Span}(\mathbf{W}) \perp \text{Span}(\mathbf{W}_\perp)$. We conclude that $\mathbf{v}^n(z_k) \perp \text{Span}(\mathbf{W}_\perp)$, therefore $\|\mathbf{W}_\perp^H \mathbf{v}^n(z_k)\|^2 = 0$. Thus, all poles $\{z_k\}_{k \in \{0 \dots K-1\}}$ are solutions. If there were another solution z distinct from the z_k , we would then get $\mathbf{v}^n(z) \perp \text{Span}(\mathbf{W}_\perp)$, therefore $\mathbf{v}^n(z) \in \text{Span}(\mathbf{V}^n)$. This is impossible because the set of vectors consisting of the columns of \mathbf{V}^n and of $\mathbf{v}^n(z)$ would be linearly independent if z were different from the z_k . In conclusion, the poles $\{z_k\}_{k \in \{0 \dots K-1\}}$ are the unique solutions of the equation $\|\mathbf{W}_\perp^H \mathbf{v}^n(z)\|^2 = 0$.



2 Estimation of Signal Parameters via Rotational Invariance Techniques (ESPRIT)

Question 1 We get $D = \text{diag}(z_0, z_1, \dots, z_{K-1})$.

Question 2 Since matrices V^n and W are two bases of the same subspace, they satisfy the relation $V^n = W G$, where G is the transfer matrix from basis W to basis V^n . The equalities $V_{\downarrow}^n = W_{\downarrow} G$ and $V_{\uparrow}^n = W_{\uparrow} G$ are obtained by respectively extracting the $n - 1$ first rows and the $n - 1$ last rows of the equality $V^n = W G$.

Question 3 $W_{\uparrow} = V_{\uparrow}^n G^{-1} = V_{\downarrow}^n D G^{-1} = W_{\downarrow} G D G^{-1}$. We thus get the equality $W_{\uparrow} = W_{\downarrow} \Phi$, where $\Phi = G D G^{-1}$. This is the diagonalized form of matrix Φ . Thus the eigenvalues of Φ are the diagonal entries of D , i.e. the poles z_k .

Question 4 Since $W_{\uparrow} = W_{\downarrow} \Phi$, we have $W_{\downarrow}^H W_{\uparrow} = W_{\downarrow}^H W_{\downarrow} \Phi$, thus $\Phi = (W_{\downarrow}^H W_{\downarrow})^{-1} W_{\downarrow}^H W_{\uparrow}$.

Question 5 The ESPRIT method is composed of several steps :

- compute the estimator \widehat{R}_{xx} of matrix R_{xx} ,
- diagonalize it to obtain an estimation of matrix W ,
- extract from W the matrices W_{\downarrow} and W_{\uparrow} ,
- compute $\Phi = (W_{\downarrow}^H W_{\downarrow})^{-1} W_{\downarrow}^H W_{\uparrow}$,
- diagonalize matrix Φ to obtain the poles $\{z_k\}_{k \in \{0 \dots K-1\}}$.



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