



## Exercises on high resolution methods

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Let us consider the *Exponential Sinusoidal Model* (ESM):

$$s[t] = \sum_{k=0}^{K-1} a_k e^{\delta_k t} e^{i(2\pi f_k t + \phi_k)},$$

which, to each frequency  $f_k \in ]-\frac{1}{2}, \frac{1}{2}]$ , associates a real amplitude  $a_k > 0$ , a phase  $\phi_k \in ]-\pi, \pi]$ , and a damping factor  $\delta_k \in \mathbb{R}$ . By defining the complex amplitudes  $\alpha_k = a_k e^{i\phi_k}$  and the complex poles  $z_k = e^{\delta_k + i2\pi f_k}$ , this model can be rewritten in the more compact form

$$s[t] = \sum_{k=0}^{K-1} \alpha_k z_k^t.$$

In practice, the observed signal  $x[t]$  never exactly fits this model. It is rather modeled as the sum of signal  $s[t]$  plus a complex Gaussian white noise  $b[t]$  of variance  $\sigma^2$ :

$$x[t] = s[t] + b[t].$$

**Remark:** A complex Gaussian white noise of variance  $\sigma^2$  is a complex process whose real part and imaginary part are two Gaussian white noises of same variance  $\frac{\sigma^2}{2}$ , independent from each other.

We assume that the signal  $x[t]$  is observed on the time interval  $\{0 \dots N-1\}$  of length  $N > 2K$ . We then consider two integers  $n$  and  $l$  such that  $n > K$ ,  $l > K$ , and  $N = n + l - 1$ .

Finally, we define the  $n \times l$  Hankel matrix which contains the  $N$  samples of the observed signal:

$$\mathbf{X} = \begin{bmatrix} x[0] & x[1] & \dots & x[l-1] \\ x[1] & x[2] & \dots & x[l] \\ \vdots & \vdots & \vdots & \vdots \\ x[n-1] & x[n] & \dots & x[N-1] \end{bmatrix}.$$

We define in the same way the Hankel matrices  $\mathbf{S}$  and  $\mathbf{B}$  of same dimension  $n \times l$ , from the samples of  $s[t]$  and  $b[t]$ , respectively.

**Notation:**

- $\mathbf{X}^T$ : transpose of matrix  $\mathbf{X}$ ,
- $\mathbf{X}^*$ : conjugate of matrix  $\mathbf{X}$ ,
- $\mathbf{X}^H$ : Hermitian transpose (conjugate transpose) of matrix  $\mathbf{X}$ .

## 1 Multiple Signal Classification (MUSIC)

**Question 1** For all  $k \in \{0 \dots K-1\}$ , we consider the component  $s_k[t] = \alpha_k z_k^t$ . We then define the  $n \times l$  Hankel matrix

$$\mathbf{S}_k = \begin{bmatrix} s_k[0] & s_k[1] & \dots & s_k[l-1] \\ s_k[1] & s_k[2] & \dots & s_k[l] \\ \vdots & \vdots & \vdots & \vdots \\ s_k[n-1] & s_k[n] & \dots & s_k[N-1] \end{bmatrix}$$



For all  $z \in \mathbb{C}$ , let us define the  $n$ -dimensional vector  $\mathbf{v}^n(z) = [1, z, z^2, \dots, z^{n-1}]^T$ , and the  $l$ -dimensional vector  $\mathbf{v}^l(z) = [1, z, z^2, \dots, z^{l-1}]^T$ . Then prove that  $\mathbf{S}_k = \alpha_k \mathbf{v}^n(z_k) \mathbf{v}^l(z_k)^T$ .

**Question 2** Use the result of question 1 to prove that  $\mathbf{S} = \sum_{k=0}^{K-1} \alpha_k \mathbf{v}^n(z_k) \mathbf{v}^l(z_k)^T$ . Prove that this last equality can be rewritten in the form  $\mathbf{S} = \mathbf{V}^n \mathbf{A} \mathbf{V}^{lT}$ , where

- $\mathbf{V}^n$  is an  $n \times K$  Vandermonde matrix:

$$\mathbf{V}^n = \begin{bmatrix} 1 & 1 & \dots & 1 \\ z_0 & z_1 & \dots & z_{K-1} \\ z_0^2 & z_1^2 & \dots & z_{K-1}^2 \\ \vdots & \vdots & \vdots & \vdots \\ z_0^{n-1} & z_1^{n-1} & \dots & z_{K-1}^{n-1} \end{bmatrix}$$

- $\mathbf{V}^l$  is an  $l \times K$  Vandermonde matrix,
- $\mathbf{A} = \text{diag}(\alpha_0, \alpha_1, \dots, \alpha_{K-1})$  is a  $K \times K$  diagonal matrix.

**Question 3** Let us define matrix  $\mathbf{R}_{ss} = \frac{1}{l} \mathbf{S} \mathbf{S}^H$ . Prove that matrix  $\mathbf{R}_{ss}$  is Hermitian and positive semidefinite. Prove that  $\mathbf{R}_{ss}$  can be factorized in the form  $\mathbf{R}_{ss} = \mathbf{V}^n \mathbf{P} \mathbf{V}^{nH}$ , where  $\mathbf{P}$  is a  $K \times K$  Hermitian and positive definite matrix. Conclude that matrix  $\mathbf{R}_{ss}$  has rank  $K$  (we remind that the poles  $z_k$  are pairwise distinct).

**Question 4** Prove that matrix  $\mathbf{R}_{ss}$  is diagonalizable in an orthonormal basis, and that its eigenvalues  $\{\lambda_i\}_{i=0..n-1}$  are non-negative. By assuming that they are sorted in decreasing order and by using the result of question 3, conclude that

- $\forall i \in \{0 \dots K-1\}, \lambda_i > 0$ ;
- $\forall i \in \{K \dots n-1\}, \lambda_i = 0$ .

**Question 5** Let  $\widehat{\mathbf{R}}_{xx} = \frac{1}{l} \mathbf{X} \mathbf{X}^H$  and  $\mathbf{R}_{xx} = \mathbb{E}[\widehat{\mathbf{R}}_{xx}]$ . Similarly, let  $\widehat{\mathbf{R}}_{bb} = \frac{1}{l} \mathbf{B} \mathbf{B}^H$  and  $\mathbf{R}_{bb} = \mathbb{E}[\widehat{\mathbf{R}}_{bb}]$ . By using equality  $\mathbf{X} = \mathbf{S} + \mathbf{B}$  and the fact that the noise is centered, prove that  $\mathbf{R}_{xx} = \mathbf{R}_{ss} + \mathbf{R}_{bb}$ . Then prove that for a complex Gaussian white noise,  $\mathbf{R}_{bb} = \sigma^2 \mathbf{I}_n$ .

**Question 6** For all  $i \in \{0 \dots n-1\}$ , let  $\mathbf{w}_i$  denote the eigenvector of matrix  $\mathbf{R}_{ss}$  corresponding to the eigenvalue  $\lambda_i$ . By using the result of question 5, prove that  $\mathbf{w}_i$  is also an eigenvector of  $\mathbf{R}_{xx}$  corresponding to the eigenvalue  $\lambda'_i = \lambda_i + \sigma^2$ . Conclude that

- $\forall i \in \{0 \dots K-1\}, \lambda'_i > \sigma^2$ ;
- $\forall i \in \{K \dots n-1\}, \lambda'_i = \sigma^2$ .

**Question 7** Let  $\mathbf{W}$  denote the matrix  $[\mathbf{w}_0 \dots \mathbf{w}_{K-1}]$ , and  $\mathbf{W}_\perp$  the matrix  $[\mathbf{w}_K \dots \mathbf{w}_{n-1}]$ . Prove that  $\text{Span}(\mathbf{W}) = \text{Span}(\mathbf{V}^n)$  (you can start by proving that  $\text{Span}(\mathbf{W}) \subset \text{Span}(\mathbf{V}^n)$ ).



**Remark:** The subspace spanned by  $\mathbf{W}_\perp$  is an eigen-subspace of matrix  $\mathbf{R}_{xx}$  corresponding to the eigenvalue  $\sigma^2$ . This is why it is called *noise subspace*. The orthonormal matrix  $\mathbf{W}$  and the Vandermonde matrix  $\mathbf{V}^n$  span the same subspace. It thus completely characterizes the  $K$  poles of the signal, This is why it is called *signal subspace*. However, all the eigenvalues of  $\mathbf{R}_{xx}$  corresponding to the signal subspace are increased by  $\sigma^2$ , which means that this subspace also contains noise.

**Question 8** Prove that the poles  $\{z_k\}_{k \in \{0 \dots K-1\}}$  are the solutions of equation  $\|\mathbf{W}_\perp^H \mathbf{v}^n(z)\|^2 = 0$ .

**Remark:** In practice, real signals do not rigorously fit the model, and this equation does never hold exactly. This is why the "spectral-MUSIC" method for estimating the poles consists in detecting the  $K$  highest peaks of function  $z \mapsto \frac{1}{\|\mathbf{W}_\perp^H \mathbf{v}^n(z)\|^2}$ . It is thus easier to implement than the maximum likelihood method, which requires the numerical optimization of a cost function of  $K$  complex variables.

## 2 Estimation of Signal Parameters via Rotational Invariance Techniques (ESPRIT)

Let  $\mathbf{V}_\downarrow^n$  be the  $(n-1) \times K$  matrix that contains the  $n-1$  first rows of  $\mathbf{V}^n$ , and  $\mathbf{V}_\uparrow^n$  the  $(n-1) \times K$  matrix that contains the  $n-1$  last rows of  $\mathbf{V}^n$ . Similarly, let  $\mathbf{W}_\downarrow$  be the  $(n-1) \times K$  matrix that contains the  $n-1$  first rows of  $\mathbf{W}$ , and  $\mathbf{W}_\uparrow$  the  $(n-1) \times K$  matrix that contains the  $n-1$  last rows of  $\mathbf{W}$ .

**Question 1** Prove that matrices  $\mathbf{V}_\downarrow^n$  and  $\mathbf{V}_\uparrow^n$  are such that  $\mathbf{V}_\uparrow^n = \mathbf{V}_\downarrow^n \mathbf{D}$ , where  $\mathbf{D}$  is a  $K \times K$  diagonal matrix. What are its diagonal entries?

**Question 2** Prove that there is a  $K \times K$  invertible matrix  $\mathbf{G}$  such that  $\mathbf{V}^n = \mathbf{W} \mathbf{G}$  (we do not ask to compute  $\mathbf{G}$ , but only to prove its existence). Then prove that  $\mathbf{V}_\downarrow^n = \mathbf{W}_\downarrow \mathbf{G}$  and  $\mathbf{V}_\uparrow^n = \mathbf{W}_\uparrow \mathbf{G}$ .

**Question 3** Conclude that there is an invertible matrix  $\mathbf{\Phi}$  such that  $\mathbf{W}_\uparrow = \mathbf{W}_\downarrow \mathbf{\Phi}$ . What are the eigenvalues of  $\mathbf{\Phi}$ ?

**Question 4** By assuming that matrix  $\mathbf{W}_\downarrow^H \mathbf{W}_\downarrow$  is invertible, compute  $\mathbf{\Phi}$  as a function of  $\mathbf{W}_\downarrow$  and  $\mathbf{W}_\uparrow$ .

**Question 5** Propose an estimation method of the poles  $\{z_k\}_{k \in \{0 \dots K-1\}}$ .

**Remark:** The principal advantage of this method with respect to spectral-MUSIC is that it does longer require to numerically optimize a cost function in order to determine the poles, which instead are obtained by a direct computation.





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