High resolution methods

Roland Badeau,
roland.badeau@telecom-paris.fr

M2 MVA
Audio signal analysis,
indexing and transformation

Sinusoidal modeling of audio signals

- Sounds that generate pitch perception have a quasi-periodic waveform

Sinusoidal modeling of audio signals

- Sounds that generate pitch perception have a quasi-periodic waveform
- Spectrum made of harmonic multiples of the fundamental frequency:

Sinusoidal modeling of audio signals

- Sounds that generate pitch perception have a quasi-periodic waveform
- Spectrum made of harmonic multiples of the fundamental frequency:
  - voiced speech sounds, produced by quasi-periodic vibration of the vocal cords

Sinusoidal modeling of audio signals

- Sounds that generate pitch perception have a quasi-periodic waveform
- Spectrum made of harmonic multiples of the fundamental frequency:
  - voiced speech sounds, produced by quasi-periodic vibration of the vocal cords
  - sounds produced by string or wind instruments

Sinusoidal modeling of audio signals

- Sounds that generate pitch perception have a quasi-periodic waveform
- Spectrum made of harmonic multiples of the fundamental frequency:
  - voiced speech sounds, produced by quasi-periodic vibration of the vocal cords
  - sounds produced by string or wind instruments
- The harmonicity property does not always hold:

Sinusoidal modeling of audio signals

- Sounds that generate pitch perception have a quasi-periodic waveform
- Spectrum made of harmonic multiples of the fundamental frequency:
  - voiced speech sounds, produced by quasi-periodic vibration of the vocal cords
  - sounds produced by string or wind instruments
- The harmonicity property does not always hold:
  - Some instruments are slightly inharmonic

TELECOM Paris

IP PARIS

Sinusoidal modeling of audio signals

- Sounds that generate pitch perception have a quasi-periodic waveform
- Spectrum made of harmonic multiples of the fundamental frequency:
  - voiced speech sounds, produced by quasi-periodic vibration of the vocal cords
  - sounds produced by string or wind instruments
- The harmonicity property does not always hold:
  - Some instruments are slightly inharmonic
  - Polyphony: overlap of harmonic combs

Sinusoidal modeling of audio signals

- Sounds that generate pitch perception have a quasi-periodic waveform
- Spectrum made of harmonic multiples of the fundamental frequency:
  - voiced speech sounds, produced by quasi-periodic vibration of the vocal cords
  - sounds produced by string or wind instruments
- The harmonicity property does not always hold:
  - Some instruments are slightly inharmonic
  - Polyphony: overlap of harmonic combs
  - Presence of pairs or triplets of close frequencies:

Sinusoidal modeling of audio signals

- Sounds that generate pitch perception have a quasi-periodic waveform
- Spectrum made of harmonic multiples of the fundamental frequency:
  - voiced speech sounds, produced by quasi-periodic vibration of the vocal cords
  - sounds produced by string or wind instruments
- The harmonicity property does not always hold:
  - Some instruments are slightly inharmonic
  - Polyphony: overlap of harmonic combs
  - Presence of pairs or triplets of close frequencies:
    - asymmetry in a bell geometry

Sinusoidal modeling of audio signals

- Sounds that generate pitch perception have a quasi-periodic waveform
- Spectrum made of harmonic multiples of the fundamental frequency:
  - voiced speech sounds, produced by quasi-periodic vibration of the vocal cords
  - sounds produced by string or wind instruments
- The harmonicity property does not always hold:
  - Some instruments are slightly inharmonic
  - Polyphony: overlap of harmonic combs
  - Presence of pairs or triplets of close frequencies:
    - asymmetry in a bell geometry
    - coupling between the strings and bridge (*chevalet*) in a guitar

Sinusoidal modeling of audio signals

- Sounds that generate pitch perception have a quasi-periodic waveform
- Spectrum made of harmonic multiples of the fundamental frequency:
  - voiced speech sounds, produced by quasi-periodic vibration of the vocal cords
  - sounds produced by string or wind instruments
- The harmonicity property does not always hold:
  - Some instruments are slightly inharmonic
  - Polyphony: overlap of harmonic combs
  - Presence of pairs or triplets of close frequencies:
    - asymmetry in a bell geometry
    - coupling between the strings and bridge (*chevalet*) in a guitar
    - pairs or triplets of strings in a piano, plus coupling of the vertical and horizontal vibration modes

Part I

Parametric signal model

Exponential Sinusoidal Model (ESM)

- Exponential amplitude modulation to model the natural damping of free vibrating systems

Exponential Sinusoidal Model (ESM)

- Exponential amplitude modulation to model the natural damping of free vibrating systems
- Real model: $s[t] = \sum_{k=0}^{K-1} a_k \, e^{\delta_k t} \cos(2\pi f_k t + \phi_k)$

Exponential Sinusoidal Model (ESM)

- Exponential amplitude modulation to model the natural damping of free vibrating systems
- Real model: $s[t] = \sum_{k=0}^{K-1} a_k \, e^{\delta_k t} \cos(2\pi f_k t + \phi_k)$
- Complex model: $s[t] = \sum_{k=0}^{K-1} a_k \, e^{\delta_k t} \, e^{i(2\pi f_k t + \phi_k)}$

Exponential Sinusoidal Model (ESM)

- Exponential amplitude modulation to model the natural damping of free vibrating systems
- Real model: $s[t] = \sum_{k=0}^{K-1} a_k \, e^{\delta_k t} \cos(2\pi f_k t + \phi_k)$
- Complex model: $s[t] = \sum_{k=0}^{K-1} a_k \, e^{\delta_k t} \, e^{i(2\pi f_k t + \phi_k)}$
- Compact form: $s[t] = \sum_{k=0}^{K-1} \alpha_k \, {z_k}^t$ where

Exponential Sinusoidal Model (ESM)

- Exponential amplitude modulation to model the natural damping of free vibrating systems
- Real model: $s[t] = \sum_{k=0}^{K-1} a_k \, e^{\delta_k t} \cos(2\pi f_k t + \phi_k)$
- Complex model: $s[t] = \sum_{k=0}^{K-1} a_k \, e^{\delta_k t} \, e^{i(2\pi f_k t + \phi_k)}$
- Compact form: $s[t] = \sum_{k=0}^{K-1} \alpha_k \, {z_k}^t$ where
  - $\alpha_k = a_k e^{i\phi_k}$ is a *complex amplitude*,

Exponential Sinusoidal Model (ESM)

- Exponential amplitude modulation to model the natural damping of free vibrating systems
- Real model: $s[t] = \sum_{k=0}^{K-1} a_k \, e^{\delta_k t} \cos(2\pi f_k t + \phi_k)$
- Complex model: $s[t] = \sum_{k=0}^{K-1} a_k \, e^{\delta_k t} \, e^{i(2\pi f_k t + \phi_k)}$
- Compact form: $s[t] = \sum_{k=0}^{K-1} \alpha_k \, {z_k}^t$ where
  - $\alpha_k = a_k e^{i\phi_k}$ is a *complex amplitude*,
  - $z_k = e^{\delta_k + i2\pi f_k}$ is a *complex pole*.

Exponential Sinusoidal Model (ESM)

- Exponential amplitude modulation to model the natural damping of free vibrating systems
- Real model: $s[t] = \sum_{k=0}^{K-1} a_k\, e^{\delta_k t} \cos\left(2\pi f_k t + \phi_k\right)$
- Complex model: $s[t] = \sum_{k=0}^{K-1} a_k\, e^{\delta_k t}\, e^{i(2\pi f_k t + \phi_k)}$
- Compact form: $s[t] = \sum_{k=0}^{K-1} \alpha_k\, {z_k}^t$ where
  - $\alpha_k = a_k e^{i\phi_k}$ is a *complex amplitude*,
  - $z_k = e^{\delta_k + i2\pi f_k}$ is a *complex pole*.
- Hypotheses: for all $k \in \{0 \dots K-1\}$, $\alpha_k \neq 0$, $z_k \neq 0$, and all poles z_k are pairwise distinct

Exponential Sinusoidal Model (ESM)

- Exponential amplitude modulation to model the natural damping of free vibrating systems
- Real model: $s[t] = \sum_{k=0}^{K-1} a_k\, e^{\delta_k t} \cos(2\pi f_k t + \phi_k)$
- Complex model: $s[t] = \sum_{k=0}^{K-1} a_k\, e^{\delta_k t}\, e^{i(2\pi f_k t + \phi_k)}$
- Compact form: $s[t] = \sum_{k=0}^{K-1} \alpha_k\, {z_k}^t$ where
  - $\alpha_k = a_k e^{i\phi_k}$ is a *complex amplitude*,
  - $z_k = e^{\delta_k + i2\pi f_k}$ is a *complex pole*.
- Hypotheses: for all $k \in \{0 \dots K-1\}$, $\alpha_k \neq 0$, $z_k \neq 0$, and all poles z_k are pairwise distinct
- The observed signal $x[t]$ is modeled as the signal $s[t]$ plus a complex Gaussian white noise $b[t]$ of variance σ^2

Spectral estimation by Fourier analysis

- Peak detection in the Fourier transform

Spectral estimation by Fourier analysis

- Peak detection in the Fourier transform
- Advantages

Spectral estimation by Fourier analysis

- Peak detection in the Fourier transform
- Advantages
  - existence of a fast algorithm (FFT)

Spectral estimation by Fourier analysis

- Peak detection in the Fourier transform
- Advantages
  - existence of a fast algorithm (FFT)
  - robust estimation method

Spectral estimation by Fourier analysis

- Peak detection in the Fourier transform
- Advantages
  - existence of a fast algorithm (FFT)
  - robust estimation method
- Drawbacks

Spectral estimation by Fourier analysis

- Peak detection in the Fourier transform
- Advantages
  - existence of a fast algorithm (FFT)
  - robust estimation method
- Drawbacks
  - spectral resolution limited by the window length

Spectral estimation by Fourier analysis

- Peak detection in the Fourier transform
- Advantages
  - existence of a fast algorithm (FFT)
  - robust estimation method
- Drawbacks
  - spectral resolution limited by the window length
  - spectral precision limited by the length of the transform

Spectral estimation by Fourier analysis

- Peak detection in the Fourier transform
- Advantages
  - existence of a fast algorithm (FFT)
  - robust estimation method
- Drawbacks
  - spectral resolution limited by the window length
  - spectral precision limited by the length of the transform
  - trade-off between the width of the principal lobe and the height of the secondary lobes induced by the window shape

Spectral estimation by Fourier analysis

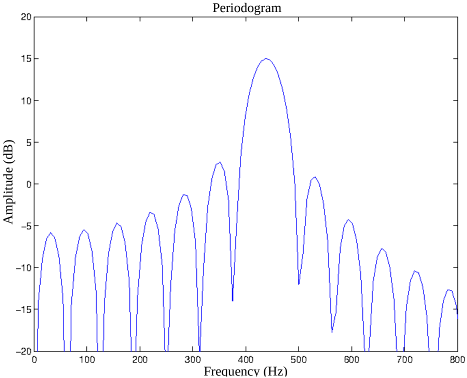
- Peak detection in the Fourier transform
- Advantages
  - existence of a fast algorithm (FFT)
  - robust estimation method
- Drawbacks
  - spectral resolution limited by the window length
  - spectral precision limited by the length of the transform
  - trade-off between the width of the principal lobe and the height of the secondary lobes induced by the window shape
  - widening of the peak in case of exponential damping

Resolution problems

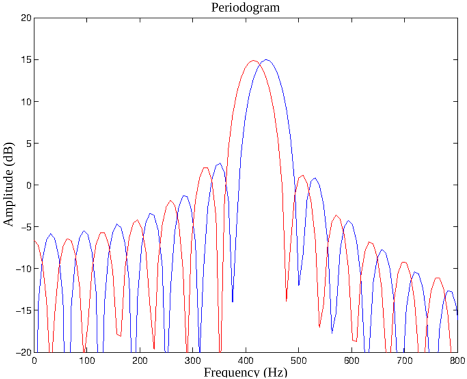
Test signal:

- Sampling frequency: 8000 Hz
- First sinusoid: 440 Hz (A)
- Second sinusoid: 415,3 Hz (G#)
- No damping, all amplitudes equal to 1
- Length of the rectangular window: $N = 128$ (16 ms)
- Length of the transform: 1024 samples

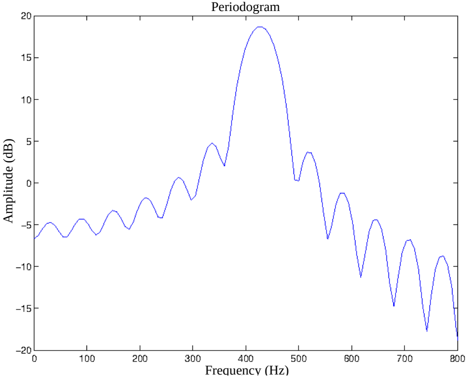
Resolution problems



Resolution problems



Resolution problems



Maximum likelihood method

- General parametric estimation principle, asymptotically unbiased, consistent and efficient

Maximum likelihood method

- General parametric estimation principle, asymptotically unbiased, consistent and efficient
- It leads to a 3-step estimation:

Maximum likelihood method

- General parametric estimation principle, asymptotically unbiased, consistent and efficient
- It leads to a 3-step estimation:
  - Estimation of the complex poles: numerical optimization of a function of K complex variables

Maximum likelihood method

- General parametric estimation principle, asymptotically unbiased, consistent and efficient
- It leads to a 3-step estimation:
  - Estimation of the complex poles: numerical optimization of a function of K complex variables
  - Estimation of the complex amplitudes: by means of the least squares method

Maximum likelihood method

- General parametric estimation principle, asymptotically unbiased, consistent and efficient
- It leads to a 3-step estimation:
  - Estimation of the complex poles: numerical optimization of a function of K complex variables
  - Estimation of the complex amplitudes: by means of the least squares method
  - Estimation of the variance: power of the residual signal

Maximum likelihood method

- General parametric estimation principle, asymptotically unbiased, consistent and efficient
- It leads to a 3-step estimation:
  - Estimation of the complex poles: numerical optimization of a function of K complex variables
  - Estimation of the complex amplitudes: by means of the least squares method
  - Estimation of the variance: power of the residual signal
- Difficulties of the first step:

Maximum likelihood method

- General parametric estimation principle, asymptotically unbiased, consistent and efficient
- It leads to a 3-step estimation:
  - Estimation of the complex poles: numerical optimization of a function of K complex variables
  - Estimation of the complex amplitudes: by means of the least squares method
  - Estimation of the variance: power of the residual signal
- Difficulties of the first step:
  - computational complexity

Maximum likelihood method

- General parametric estimation principle, asymptotically unbiased, consistent and efficient
- It leads to a 3-step estimation:
  - Estimation of the complex poles: numerical optimization of a function of K complex variables
  - Estimation of the complex amplitudes: by means of the least squares method
  - Estimation of the variance: power of the residual signal
- Difficulties of the first step:
  - computational complexity
  - presence of many local maxima

Maximum likelihood method

- General parametric estimation principle, asymptotically unbiased, consistent and efficient
- It leads to a 3-step estimation:
  - Estimation of the complex poles: numerical optimization of a function of K complex variables
  - Estimation of the complex amplitudes: by means of the least squares method
  - Estimation of the variance: power of the residual signal
- Difficulties of the first step:
  - computational complexity
  - presence of many local maxima
- Need for specific methods for the complex poles

Maximum likelihood method

- General parametric estimation principle, asymptotically unbiased, consistent and efficient
- It leads to a 3-step estimation:
  - Estimation of the complex poles: numerical optimization of a function of K complex variables
  - Estimation of the complex amplitudes: by means of the least squares method
  - Estimation of the variance: power of the residual signal
- Difficulties of the first step:
  - computational complexity
  - presence of many local maxima
- Need for specific methods for the complex poles
- High resolution parametric estimation methods overcome the limits of Fourier analysis

Part II

High resolution methods

Linear prediction methods

- Principle: any signal such that $s[t] - z_0\, s[t-1] = 0$ is of the form $s[t] = \alpha_0\, {z_0}^t$

Linear prediction methods

- Principle: any signal such that $s[t] - z_0\, s[t-1] = 0$ is of the form $s[t] = \alpha_0\, {z_0}^t$
- General case: let $P[z] \triangleq \prod_{k=0}^{K-1} (z - z_k) = \sum_{\tau=0}^{K} p_\tau\, z^{K-\tau}$.

Linear prediction methods

- Principle: any signal such that $s[t] - z_0\, s[t-1] = 0$ is of the form $s[t] = \alpha_0\, {z_0}^t$
- General case: let $P[z] \triangleq \prod_{k=0}^{K-1} (z - z_k) = \sum_{\tau=0}^{K} p_\tau\, z^{K-\tau}$.
- *A discrete signal* $\{s[t]\}_{t\in\mathbb{Z}}$ *is solution of the recursion* $\sum_{\tau=0}^{K} p_\tau\, s[t-\tau] = 0$ *if and only if it is of the form* $s[t] = \sum_{k=0}^{K-1} \alpha_k\, {z_k}^t$

TELECOM Paris IP PARIS

Linear prediction methods

- Principle: any signal such that $s[t] - z_0\, s[t-1] = 0$ is of the form $s[t] = \alpha_0\, {z_0}^t$
- General case: let $P[z] \triangleq \prod_{k=0}^{K-1} (z - z_k) = \sum_{\tau=0}^{K} p_\tau\, z^{K-\tau}$.
- *A discrete signal* $\{s[t]\}_{t\in\mathbb{Z}}$ *is solution of the recursion* $\sum_{\tau=0}^{K} p_\tau\, s[t-\tau] = 0$ *if and only if it is of the form* $s[t] = \sum_{k=0}^{K-1} \alpha_k\, {z_k}^t$
- Prony and Pisarenko methods:

Linear prediction methods

- Principle: any signal such that $s[t] - z_0\, s[t-1] = 0$ is of the form $s[t] = \alpha_0\, {z_0}^t$
- General case: let $P[z] \triangleq \prod_{k=0}^{K-1} (z - z_k) = \sum_{\tau=0}^{K} p_\tau\, z^{K-\tau}$.
- *A discrete signal* $\{s[t]\}_{t\in\mathbb{Z}}$ *is solution of the recursion* $\sum_{\tau=0}^{K} p_\tau\, s[t-\tau] = 0$ *if and only if it is of the form* $s[t] = \sum_{k=0}^{K-1} \alpha_k\, {z_k}^t$
- Prony and Pisarenko methods:
  - Estimate polynomial $P[z]$ by means of linear prediction

Linear prediction methods

- Principle: any signal such that $s[t] - z_0\, s[t-1] = 0$ is of the form $s[t] = \alpha_0\, {z_0}^t$
- General case: let $P[z] \triangleq \prod_{k=0}^{K-1} (z - z_k) = \sum_{\tau=0}^{K} p_\tau\, z^{K-\tau}$.
- *A discrete signal* $\{s[t]\}_{t\in\mathbb{Z}}$ *is solution of the recursion* $\sum_{\tau=0}^{K} p_\tau\, s[t-\tau] = 0$ *if and only if it is of the form* $s[t] = \sum_{k=0}^{K-1} \alpha_k\, {z_k}^t$
- Prony and Pisarenko methods:
  - Estimate polynomial $P[z]$ by means of linear prediction
  - Extract the roots of this polynomial

Linear prediction methods

- Principle: any signal such that $s[t] - z_0\, s[t-1] = 0$ is of the form $s[t] = \alpha_0\, {z_0}^t$
- General case: let $P[z] \triangleq \prod_{k=0}^{K-1}(z - z_k) = \sum_{\tau=0}^{K} p_\tau\, z^{K-\tau}$.
- *A discrete signal* $\{s[t]\}_{t\in\mathbb{Z}}$ *is solution of the recursion* $\sum_{\tau=0}^{K} p_\tau\, s[t-\tau] = 0$ *if and only if it is of the form*

$$s[t] = \sum_{k=0}^{K-1} \alpha_k\, {z_k}^t$$

- Prony and Pisarenko methods:
  - Estimate polynomial $P[z]$ by means of linear prediction
  - Extract the roots of this polynomial
- Drawback: mediocre performance in presence of noise

Matrix representation of the signal

- Observation horizon: $t \in \{0 \dots N-1\}$, where $N > 2K$

Matrix representation of the signal

- Observation horizon: $t \in \{0 \ldots N-1\}$, where $N > 2K$
- Data matrix ($n > K$, $l > K$ and $N = n + l - 1$):

$$\mathbf{S} = \begin{bmatrix} s[0] & s[1] & \ldots & s[l-1] \\ s[1] & s[2] & \ldots & s[l] \\ \vdots & \vdots & \vdots & \vdots \\ s[n-1] & s[n] & \ldots & s[N-1] \end{bmatrix}$$

Matrix representation of the signal

- Observation horizon: $t \in \{0 \dots N-1\}$, where $N > 2K$
- Data matrix ($n > K$, $l > K$ and $N = n + l - 1$):

$$\mathbf{S} = \begin{bmatrix} s[0] & s[1] & \dots & s[l-1] \\ s[1] & s[2] & \dots & s[l] \\ \vdots & \vdots & \vdots & \vdots \\ s[n-1] & s[n] & \dots & s[N-1] \end{bmatrix}$$

- Factorization of matrix $\mathbf{S}$: $\mathbf{S} = \mathbf{V}^n \mathbf{A} \mathbf{V}^{l\,T}$, where

Matrix representation of the signal

- Observation horizon: $t \in \{0 \dots N-1\}$, where $N > 2K$
- Data matrix ($n > K$, $l > K$ and $N = n + l - 1$):

$$\mathbf{S} = \begin{bmatrix} s[0] & s[1] & \dots & s[l-1] \\ s[1] & s[2] & \dots & s[l] \\ \vdots & \vdots & \vdots & \vdots \\ s[n-1] & s[n] & \dots & s[N-1] \end{bmatrix}$$

- Factorization of matrix $\mathbf{S}$: $\mathbf{S} = \mathbf{V}^n \mathbf{A} \mathbf{V}^{l\,T}$, where
  - $\mathbf{V}^n$ is the Vandermonde matrix of dimension $n \times K$,

$$\mathbf{V}^n = \begin{bmatrix} 1 & 1 & \dots & 1 \\ z_0 & z_1 & \dots & z_{K-1} \\ {z_0}^2 & {z_1}^2 & \dots & {z_{K-1}}^2 \\ \vdots & \vdots & \vdots & \vdots \\ {z_0}^{n-1} & {z_1}^{n-1} & \dots & {z_{K-1}}^{n-1} \end{bmatrix}$$

Matrix representation of the signal

- Observation horizon: $t \in \{0 \dots N-1\}$, where $N > 2K$
- Data matrix ($n > K$, $l > K$ and $N = n + l - 1$):

$$\mathbf{S} = \begin{bmatrix} s[0] & s[1] & \dots & s[l-1] \\ s[1] & s[2] & \dots & s[l] \\ \vdots & \vdots & \vdots & \vdots \\ s[n-1] & s[n] & \dots & s[N-1] \end{bmatrix}$$

- Factorization of matrix $\mathbf{S}$: $\mathbf{S} = \mathbf{V}^n \mathbf{A} \mathbf{V}^{l\,T}$, where
  - $\mathbf{V}^n$ is the Vandermonde matrix of dimension $n \times K$,
  - $\mathbf{V}^l$ is the Vandermonde matrix of dimension $l \times K$,

Matrix representation of the signal

- Observation horizon: $t \in \{0 \dots N-1\}$, where $N > 2K$
- Data matrix ($n > K$, $l > K$ and $N = n + l - 1$):

$$\mathbf{S} = \begin{bmatrix} s[0] & s[1] & \dots & s[l-1] \\ s[1] & s[2] & \dots & s[l] \\ \vdots & \vdots & \vdots & \vdots \\ s[n-1] & s[n] & \dots & s[N-1] \end{bmatrix}$$

- Factorization of matrix $\mathbf{S}$: $\mathbf{S} = \mathbf{V}^n \mathbf{A} \mathbf{V}^{l\,T}$, where
  - $\mathbf{V}^n$ is the Vandermonde matrix of dimension $n \times K$,
  - $\mathbf{V}^l$ is the Vandermonde matrix of dimension $l \times K$,
  - $\mathbf{A} = \mathrm{diag}(\alpha_0, \alpha_1, \dots, \alpha_{K-1})$ is a diagonal matrix of dimension $K \times K$.

Empirical covariance matrix

- Let us define the empirical covariance matrix $\mathbf{R}_{ss} = \frac{1}{l}\mathbf{S}\,\mathbf{S}^H$

Empirical covariance matrix

- Let us define the empirical covariance matrix $\mathbf{R}_{ss} = \frac{1}{l}\mathbf{S}\,\mathbf{S}^H$
- Then $\mathbf{R}_{ss} = \mathbf{V}^n\,\mathbf{P}\,\mathbf{V}^{nH}$, where $\mathbf{P} = \frac{1}{l}\mathbf{A}\,\mathbf{V}^{l\,T}\mathbf{V}^{l\,*}\mathbf{A}^H$

TELECOM Paris IP PARIS

Empirical covariance matrix

- Let us define the empirical covariance matrix $\mathbf{R}_{ss} = \frac{1}{l}\mathbf{S}\,\mathbf{S}^H$
- Then $\mathbf{R}_{ss} = \mathbf{V}^n\,\mathbf{P}\,\mathbf{V}^{nH}$, where $\mathbf{P} = \frac{1}{l}\mathbf{A}\,\mathbf{V}^{l\,T}\,\mathbf{V}^{l\,*}\,\mathbf{A}^H$
- Matrix $\mathbf{R}_{ss}$ has rank K

Empirical covariance matrix

- Let us define the empirical covariance matrix $\mathbf{R}_{ss} = \frac{1}{l}\mathbf{S}\,\mathbf{S}^H$
- Then $\mathbf{R}_{ss} = \mathbf{V}^n\,\mathbf{P}\,\mathbf{V}^{nH}$, where $\mathbf{P} = \frac{1}{l}\mathbf{A}\,\mathbf{V}^{l\,T}\mathbf{V}^{l\,*}\mathbf{A}^H$
- Matrix $\mathbf{R}_{ss}$ has rank K
- $\mathbf{R}_{ss}$ is diagonalizable in an orthonormal basis $\{\mathbf{w}_0 \ldots \mathbf{w}_{n-1}\}$

Empirical covariance matrix

- Let us define the empirical covariance matrix $\mathbf{R}_{ss} = \frac{1}{l}\mathbf{S}\,\mathbf{S}^H$
- Then $\mathbf{R}_{ss} = \mathbf{V}^n\,\mathbf{P}\,\mathbf{V}^{nH}$, where $\mathbf{P} = \frac{1}{l}\mathbf{A}\,\mathbf{V}^{l\,T}\mathbf{V}^{l\,*}\mathbf{A}^H$
- Matrix $\mathbf{R}_{ss}$ has rank K
- $\mathbf{R}_{ss}$ is diagonalizable in an orthonormal basis $\{\mathbf{w}_0 \dots \mathbf{w}_{n-1}\}$
- Its eigenvalues $\lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_{n-1} \geq 0$ are such that

Empirical covariance matrix

- Let us define the empirical covariance matrix $\mathbf{R}_{ss} = \frac{1}{l}\mathbf{S}\,\mathbf{S}^H$
- Then $\mathbf{R}_{ss} = \mathbf{V}^n\mathbf{P}\,\mathbf{V}^{nH}$, where $\mathbf{P} = \frac{1}{l}\mathbf{A}\,\mathbf{V}^{l\,T}\mathbf{V}^{l\,*}\mathbf{A}^H$
- Matrix $\mathbf{R}_{ss}$ has rank K
- $\mathbf{R}_{ss}$ is diagonalizable in an orthonormal basis $\{\mathbf{w}_0 \dots \mathbf{w}_{n-1}\}$
- Its eigenvalues $\lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_{n-1} \geq 0$ are such that
  - $\forall i \in \{0 \dots K-1\}$, $\lambda_i > 0$;

Empirical covariance matrix

- Let us define the empirical covariance matrix $\mathbf{R}_{ss} = \frac{1}{l}\mathbf{S}\,\mathbf{S}^H$
- Then $\mathbf{R}_{ss} = \mathbf{V}^n\,\mathbf{P}\,\mathbf{V}^{nH}$, where $\mathbf{P} = \frac{1}{l}\mathbf{A}\,\mathbf{V}^{l\,T}\mathbf{V}^{l\,*}\mathbf{A}^H$
- Matrix $\mathbf{R}_{ss}$ has rank K
- $\mathbf{R}_{ss}$ is diagonalizable in an orthonormal basis $\{\mathbf{w}_0 \dots \mathbf{w}_{n-1}\}$
- Its eigenvalues $\lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_{n-1} \geq 0$ are such that
  - $\forall i \in \{0 \dots K-1\}$, $\lambda_i > 0$;
  - $\forall i \in \{K \dots n-1\}$, $\lambda_i = 0$.

Empirical covariance matrix

- Let us define the empirical covariance matrix $\mathbf{R}_{ss} = \frac{1}{l}\mathbf{S}\,\mathbf{S}^H$
- Then $\mathbf{R}_{ss} = \mathbf{V}^n\,\mathbf{P}\,\mathbf{V}^{nH}$, where $\mathbf{P} = \frac{1}{l}\mathbf{A}\,\mathbf{V}^{l\,T}\mathbf{V}^{l\,*}\mathbf{A}^H$
- Matrix $\mathbf{R}_{ss}$ has rank K
- $\mathbf{R}_{ss}$ is diagonalizable in an orthonormal basis $\{\mathbf{w}_0 \ldots \mathbf{w}_{n-1}\}$
- Its eigenvalues $\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_{n-1} \geq 0$ are such that
  - $\forall i \in \{0 \ldots K-1\}$, $\lambda_i > 0$;
  - $\forall i \in \{K \ldots n-1\}$, $\lambda_i = 0$.
- Let $\widehat{\mathbf{R}}_{bb} = \frac{1}{l}\mathbf{B}\,\mathbf{B}^H$ and $\mathbf{R}_{bb} = \mathbb{E}\left[\widehat{\mathbf{R}}_{bb}\right] = \sigma^2\mathbf{I}_n$.

Empirical covariance matrix

- Let us define the empirical covariance matrix $\mathbf{R}_{ss} = \frac{1}{l}\mathbf{S}\,\mathbf{S}^H$
- Then $\mathbf{R}_{ss} = \mathbf{V}^n\,\mathbf{P}\,\mathbf{V}^{nH}$, where $\mathbf{P} = \frac{1}{l}\mathbf{A}\,\mathbf{V}^{l\,T}\mathbf{V}^{l*}\mathbf{A}^H$
- Matrix $\mathbf{R}_{ss}$ has rank K
- $\mathbf{R}_{ss}$ is diagonalizable in an orthonormal basis $\{\mathbf{w}_0 \dots \mathbf{w}_{n-1}\}$
- Its eigenvalues $\lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_{n-1} \geq 0$ are such that
  - $\forall i \in \{0 \dots K-1\}$, $\lambda_i > 0$;
  - $\forall i \in \{K \dots n-1\}$, $\lambda_i = 0$.
- Let $\widehat{\mathbf{R}}_{bb} = \frac{1}{l}\mathbf{B}\,\mathbf{B}^H$ and $\mathbf{R}_{bb} = \mathbb{E}\left[\widehat{\mathbf{R}}_{bb}\right] = \sigma^2\mathbf{I}_n$.
- In the same way, let $\widehat{\mathbf{R}}_{xx} = \frac{1}{l}\mathbf{X}\,\mathbf{X}^H$ and $\mathbf{R}_{xx} = \mathbb{E}\left[\widehat{\mathbf{R}}_{xx}\right]$.

Empirical covariance matrix

- Let us define the empirical covariance matrix $\mathbf{R}_{ss} = \frac{1}{l}\mathbf{S}\,\mathbf{S}^H$
- Then $\mathbf{R}_{ss} = \mathbf{V}^n\,\mathbf{P}\,\mathbf{V}^{nH}$, where $\mathbf{P} = \frac{1}{l}\mathbf{A}\,\mathbf{V}^{l\,T}\mathbf{V}^{l\,*}\mathbf{A}^H$
- Matrix $\mathbf{R}_{ss}$ has rank K
- $\mathbf{R}_{ss}$ is diagonalizable in an orthonormal basis $\{\mathbf{w}_0 \ldots \mathbf{w}_{n-1}\}$
- Its eigenvalues $\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_{n-1} \geq 0$ are such that
  - $\forall i \in \{0 \ldots K-1\}$, $\lambda_i > 0$;
  - $\forall i \in \{K \ldots n-1\}$, $\lambda_i = 0$.
- Let $\widehat{\mathbf{R}}_{bb} = \frac{1}{l}\mathbf{B}\,\mathbf{B}^H$ and $\mathbf{R}_{bb} = \mathbb{E}\left[\widehat{\mathbf{R}}_{bb}\right] = \sigma^2\mathbf{I}_n$.
- In the same way, let $\widehat{\mathbf{R}}_{xx} = \frac{1}{l}\mathbf{X}\,\mathbf{X}^H$ and $\mathbf{R}_{xx} = \mathbb{E}\left[\widehat{\mathbf{R}}_{xx}\right]$.
- Then $\mathbf{R}_{xx} = \mathbf{R}_{ss} + \sigma^2\mathbf{I}_n$

Signal subspace and noise subspace

- For all $i \in \{0 \dots n-1\}$, $\mathbf{w}_i$ is also an eigenvector of $\mathbf{R}_{xx}$ corresponding to the eigenvalue $\lambda_i' = \lambda_i + \sigma^2$. Therefore,

Signal subspace and noise subspace

- For all $i \in \{0 \dots n-1\}$, $\mathbf{w}_i$ is also an eigenvector of $\mathbf{R}_{xx}$ corresponding to the eigenvalue $\lambda_i' = \lambda_i + \sigma^2$. Therefore,
  - $\forall i \in \{0 \dots K-1\}$, $\lambda_i' > \sigma^2$;

Signal subspace and noise subspace

- For all $i \in \{0 \dots n-1\}$, $\mathbf{w}_i$ is also an eigenvector of $\mathbf{R}_{xx}$ corresponding to the eigenvalue $\lambda_i' = \lambda_i + \sigma^2$. Therefore,
  - $\forall i \in \{0 \dots K-1\}$, $\lambda_i' > \sigma^2$;
  - $\forall i \in \{K \dots n-1\}$, $\lambda_i' = \sigma^2$.

Signal subspace and noise subspace

- For all $i \in \{0 \dots n-1\}$, $\mathbf{w}_i$ is also an eigenvector of $\mathbf{R}_{xx}$ corresponding to the eigenvalue $\lambda_i' = \lambda_i + \sigma^2$. Therefore,
  - $\forall i \in \{0 \dots K-1\}$, $\lambda_i' > \sigma^2$;
  - $\forall i \in \{K \dots n-1\}$, $\lambda_i' = \sigma^2$.
- Let $\mathbf{W} = [\mathbf{w}_0 \dots \mathbf{w}_{K-1}]$, and $\mathbf{W}_\perp = [\mathbf{w}_K \dots \mathbf{w}_{n-1}]$

TELECOM Paris IP PARIS

Signal subspace and noise subspace

- For all $i \in \{0 \dots n-1\}$, $\mathbf{w}_i$ is also an eigenvector of $\mathbf{R}_{xx}$ corresponding to the eigenvalue $\lambda_i' = \lambda_i + \sigma^2$. Therefore,
  - $\forall i \in \{0 \dots K-1\}$, $\lambda_i' > \sigma^2$;
  - $\forall i \in \{K \dots n-1\}$, $\lambda_i' = \sigma^2$.
- Let $\mathbf{W} = [\mathbf{w}_0 \dots \mathbf{w}_{K-1}]$, and $\mathbf{W}_\perp = [\mathbf{w}_K \dots \mathbf{w}_{n-1}]$
- Then $\mathrm{Span}(\mathbf{W}) = \mathrm{Span}(\mathbf{V}^n)$ is referred to as the signal subspace

Signal subspace and noise subspace

- For all $i \in \{0 \dots n-1\}$, $\mathbf{w}_i$ is also an eigenvector of $\mathbf{R}_{xx}$ corresponding to the eigenvalue $\lambda_i' = \lambda_i + \sigma^2$. Therefore,
  - $\forall i \in \{0 \dots K-1\}$, $\lambda_i' > \sigma^2$;
  - $\forall i \in \{K \dots n-1\}$, $\lambda_i' = \sigma^2$.
- Let $\mathbf{W} = [\mathbf{w}_0 \dots \mathbf{w}_{K-1}]$, and $\mathbf{W}_\perp = [\mathbf{w}_K \dots \mathbf{w}_{n-1}]$
- Then $\mathrm{Span}(\mathbf{W}) = \mathrm{Span}(\mathbf{V}^n)$ is referred to as the signal subspace
- In the same way, $\mathrm{Span}(\mathbf{W}_\perp)$ is referred to as the noise subspace

Signal subspace and noise subspace

- For all $i \in \{0 \dots n-1\}$, $\mathbf{w}_i$ is also an eigenvector of $\mathbf{R}_{xx}$ corresponding to the eigenvalue $\lambda_i' = \lambda_i + \sigma^2$. Therefore,
  - $\forall i \in \{0 \dots K-1\}$, $\lambda_i' > \sigma^2$;
  - $\forall i \in \{K \dots n-1\}$, $\lambda_i' = \sigma^2$.
- Let $\mathbf{W} = [\mathbf{w}_0 \dots \mathbf{w}_{K-1}]$, and $\mathbf{W}_\perp = [\mathbf{w}_K \dots \mathbf{w}_{n-1}]$
- Then $\mathrm{Span}(\mathbf{W}) = \mathrm{Span}(\mathbf{V}^n)$ is referred to as the signal subspace
- In the same way, $\mathrm{Span}(\mathbf{W}_\perp)$ is referred to as the noise subspace
- The poles $\{z_k\}_{k \in \{0 \dots K-1\}}$ are the solutions of equation $\left\| \mathbf{W}_\perp^H \mathbf{v}(z) \right\|^2 = 0$, where $\mathbf{v}(z) = [1, z, \dots, z^{n-1}]$

Signal subspace and noise subspace

- For all $i \in \{0 \dots n-1\}$, $\mathbf{w}_i$ is also an eigenvector of $\mathbf{R}_{xx}$ corresponding to the eigenvalue $\lambda'_i = \lambda_i + \sigma^2$. Therefore,
  - $\forall i \in \{0 \dots K-1\}$, $\lambda'_i > \sigma^2$;
  - $\forall i \in \{K \dots n-1\}$, $\lambda'_i = \sigma^2$.
- Let $\mathbf{W} = [\mathbf{w}_0 \dots \mathbf{w}_{K-1}]$, and $\mathbf{W}_\perp = [\mathbf{w}_K \dots \mathbf{w}_{n-1}]$
- Then $\mathrm{Span}(\mathbf{W}) = \mathrm{Span}(\mathbf{V}^n)$ is referred to as the signal subspace
- In the same way, $\mathrm{Span}(\mathbf{W}_\perp)$ is referred to as the noise subspace
- The poles $\{z_k\}_{k \in \{0 \dots K-1\}}$ are the solutions of equation $\left\|\mathbf{W}_\perp^H \mathbf{v}(z)\right\|^2 = 0$, where $\mathbf{v}(z) = [1, z, \dots, z^{n-1}]$
- The MUSIC method consists in solving this equation

Signal subspace and noise subspace

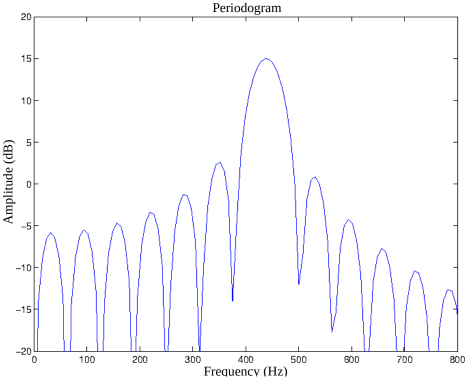
- For all $i \in \{0 \dots n-1\}$, $\mathbf{w}_i$ is also an eigenvector of $\mathbf{R}_{xx}$ corresponding to the eigenvalue $\lambda_i' = \lambda_i + \sigma^2$. Therefore,
  - $\forall i \in \{0 \dots K-1\}$, $\lambda_i' > \sigma^2$;
  - $\forall i \in \{K \dots n-1\}$, $\lambda_i' = \sigma^2$.
- Let $\mathbf{W} = [\mathbf{w}_0 \dots \mathbf{w}_{K-1}]$, and $\mathbf{W}_\perp = [\mathbf{w}_K \dots \mathbf{w}_{n-1}]$
- Then $\mathrm{Span}(\mathbf{W}) = \mathrm{Span}(\mathbf{V}^n)$ is referred to as the signal subspace
- In the same way, $\mathrm{Span}(\mathbf{W}_\perp)$ is referred to as the noise subspace
- The poles $\{z_k\}_{k \in \{0 \dots K-1\}}$ are the solutions of equation $\left\|\mathbf{W}_\perp^H \mathbf{v}(z)\right\|^2 = 0$, where $\mathbf{v}(z) = [1, z, \dots, z^{n-1}]$
- The MUSIC method consists in solving this equation
- The Spectral-MUSIC method consists in detecting the K highest peaks in function $z \mapsto \frac{1}{\left\|\mathbf{W}_\perp^H \mathbf{v}(z)\right\|^2}$.

Spectral MUSIC method

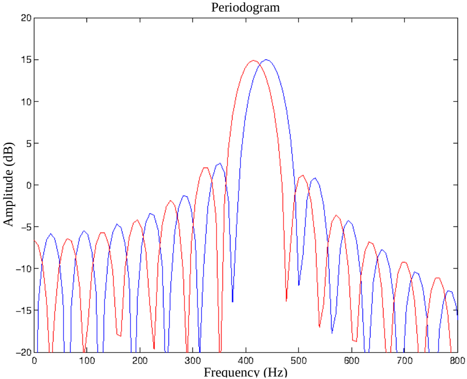
Test signal:

- Sampling frequency: 8000 Hz
- First sinusoid: 440 Hz (A)
- Second sinusoid: 415,3 Hz (G#)
- No damping, all amplitudes equal to 1
- Length of the rectangular window: $N = 128$ (16 ms)
- Length of the transform: 1024 samples

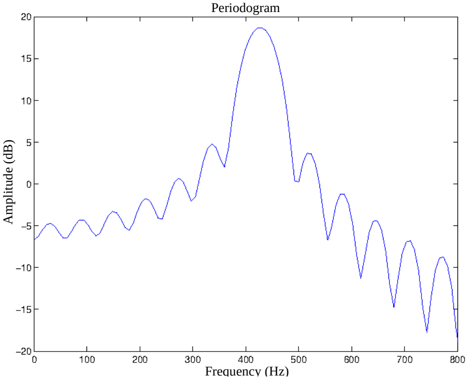
Spectral MUSIC method



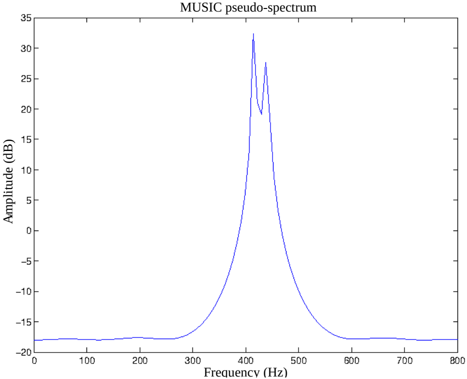
Spectral MUSIC method



Spectral MUSIC method



Spectral MUSIC method



ESPRIT method

- Rotational invariance property of $\mathbf{V}^n$:

$$\underbrace{\begin{bmatrix} 1 & \dots & 1 \\ z_0 & \dots & z_{K-1} \\ \vdots & \dots & \vdots \\ {z_0}^{n-2} & \dots & {z_{K-1}}^{n-2} \\ {z_0}^{n-1} & \dots & {z_{K-1}}^{n-1} \end{bmatrix}}_{\mathbf{V}^n}$$

$$n \times K$$

ESPRIT method

- Rotational invariance property of $\mathbf{V}^n$:

$$\underbrace{\begin{bmatrix} 1 & \dots & 1 \\ z_0 & \dots & z_{K-1} \\ \vdots & \dots & \vdots \\ {z_0}^{n-2} & \dots & {z_{K-1}}^{n-2} \\ {z_0}^{n-1} & \dots & {z_{K-1}}^{n-1} \end{bmatrix}}_{\substack{\mathbf{V}^n{}_{\uparrow} \\ (n-1)\times K}}$$

ESPRIT method

- Rotational invariance property of $\mathbf{V}^n$:

$$\underbrace{\begin{bmatrix} 1 & \dots & 1 \\ z_0 & \dots & z_{K-1} \\ \vdots & \dots & \vdots \\ {z_0}^{n-2} & \dots & {z_{K-1}}^{n-2} \\ {z_0}^{n-1} & \dots & {z_{K-1}}^{n-1} \end{bmatrix}}_{\substack{\mathbf{V}^n{}_{\uparrow} \\ (n-1)\times K}} \quad \underbrace{\begin{bmatrix} 1 & \dots & 1 \\ z_0 & \dots & z_{K-1} \\ \vdots & \dots & \vdots \\ {z_0}^{n-2} & \dots & {z_{K-1}}^{n-2} \\ {z_0}^{n-1} & \dots & {z_{K-1}}^{n-1} \end{bmatrix}}_{\substack{\mathbf{V}^n \\ n\times K}}$$

TELECOM Paris

IP PARIS

ESPRIT method

- Rotational invariance property of $\mathbf{V}^n$:

$$\underbrace{\begin{bmatrix} 1 & \dots & 1 \\ z_0 & \dots & z_{K-1} \\ \vdots & \dots & \vdots \\ {z_0}^{n-2} & \dots & {z_{K-1}}^{n-2} \\ {z_0}^{n-1} & \dots & {z_{K-1}}^{n-1} \end{bmatrix}}_{\substack{\mathbf{V}^n{}_{\uparrow} \\ (n-1)\times K}} \quad \underbrace{\begin{bmatrix} 1 & \dots & 1 \\ z_0 & \dots & z_{K-1} \\ \vdots & \dots & \vdots \\ {z_0}^{n-2} & \dots & {z_{K-1}}^{n-2} \\ {z_0}^{n-1} & \dots & {z_{K-1}}^{n-1} \end{bmatrix}}_{\substack{\mathbf{V}^n{}_{\downarrow} \\ (n-1)\times K}}$$

ESPRIT method

- Rotational invariance property of $\mathbf{V}^n$:

$$\underbrace{\begin{bmatrix} 1 & \dots & 1 \\ z_0 & \dots & z_{K-1} \\ \vdots & \dots & \vdots \\ {z_0}^{n-2} & \dots & {z_{K-1}}^{n-2} \\ {z_0}^{n-1} & \dots & {z_{K-1}}^{n-1} \end{bmatrix}}_{\substack{\mathbf{V}^n{}_{\uparrow} \\ (n-1)\times K}} = \underbrace{\begin{bmatrix} 1 & \dots & 1 \\ z_0 & \dots & z_{K-1} \\ \vdots & \dots & \vdots \\ {z_0}^{n-2} & \dots & {z_{K-1}}^{n-2} \\ {z_0}^{n-1} & \dots & {z_{K-1}}^{n-1} \end{bmatrix}}_{\substack{\mathbf{V}^n{}_{\downarrow} \\ (n-1)\times K}} \underbrace{\begin{bmatrix} z_0 & & (0) \\ & \ddots & \\ (0) & & z_{K-1} \end{bmatrix}}_{\substack{\mathbf{D} \\ K\times K}}$$

ESPRIT method

- Rotational invariance property of $\mathbf{V}^n$: $\mathbf{V}^n{}_{\uparrow} = \mathbf{V}^n{}_{\downarrow}\mathbf{D}$

ESPRIT method

- Rotational invariance property of $\mathbf{V}^n$: $\mathbf{V}^n{}_{\uparrow} = \mathbf{V}^n{}_{\downarrow}\,\mathbf{D}$
- Change of basis: $\mathbf{V}^n = \mathbf{W}\,\mathbf{G}$

ESPRIT method

- Rotational invariance property of $\mathbf{V}^n$: $\mathbf{V}^n{}_{\uparrow} = \mathbf{V}^n{}_{\downarrow}\,\mathbf{D}$
- Change of basis: $\mathbf{V}^n = \mathbf{W}\,\mathbf{G}$
- Rotational invariance of $\mathbf{W}$: $\mathbf{W}_{\uparrow} = \mathbf{W}_{\downarrow}\,\boldsymbol{\Phi}$
  where $\boldsymbol{\Phi} = \mathbf{G}\,\mathbf{D}\,\mathbf{G}^{-1}$ is referred to as the spectral matrix

ESPRIT method

- Rotational invariance property of $\mathbf{V}^n$: $\mathbf{V}^n{}_{\uparrow} = \mathbf{V}^n{}_{\downarrow}\,\mathbf{D}$
- Change of basis: $\mathbf{V}^n = \mathbf{W}\,\mathbf{G}$
- Rotational invariance of $\mathbf{W}$: $\mathbf{W}_{\uparrow} = \mathbf{W}_{\downarrow}\,\mathbf{\Phi}$
  where $\mathbf{\Phi} = \mathbf{G}\,\mathbf{D}\,\mathbf{G}^{-1}$ is referred to as the spectral matrix
- The eigenvalues of $\mathbf{\Phi}$ are the poles $\{z_k\}_{k\in\{0\ldots K-1\}}$

ESPRIT method

- Rotational invariance property of $\mathbf{V}^n$: $\mathbf{V}^n{}_\uparrow = \mathbf{V}^n{}_\downarrow \mathbf{D}$
- Change of basis: $\mathbf{V}^n = \mathbf{W}\,\mathbf{G}$
- Rotational invariance of $\mathbf{W}$: $\mathbf{W}_\uparrow = \mathbf{W}_\downarrow \boldsymbol{\Phi}$
  where $\boldsymbol{\Phi} = \mathbf{G}\,\mathbf{D}\,\mathbf{G}^{-1}$ is referred to as the spectral matrix
- The eigenvalues of $\boldsymbol{\Phi}$ are the poles $\{z_k\}_{k\in\{0\dots K-1\}}$
- Matrix $\boldsymbol{\Phi}$ is such that $\boldsymbol{\Phi} = \left(\mathbf{W}_\downarrow^H \mathbf{W}_\downarrow\right)^{-1} \mathbf{W}_\downarrow^H \mathbf{W}_\uparrow$

ESPRIT method

- Rotational invariance property of $\mathbf{V}^n$: $\mathbf{V}^n{}_{\uparrow} = \mathbf{V}^n{}_{\downarrow}\,\mathbf{D}$
- Change of basis: $\mathbf{V}^n = \mathbf{W}\,\mathbf{G}$
- Rotational invariance of $\mathbf{W}$: $\mathbf{W}_{\uparrow} = \mathbf{W}_{\downarrow}\,\mathbf{\Phi}$
  where $\mathbf{\Phi} = \mathbf{G}\,\mathbf{D}\,\mathbf{G}^{-1}$ is referred to as the spectral matrix
- The eigenvalues of $\mathbf{\Phi}$ are the poles $\{z_k\}_{k\in\{0\ldots K-1\}}$
- Matrix $\mathbf{\Phi}$ is such that $\mathbf{\Phi} = \left(\mathbf{W}_{\downarrow}^H\mathbf{W}_{\downarrow}\right)^{-1}\mathbf{W}_{\downarrow}^H\mathbf{W}_{\uparrow}$
- ESPRIT algorithm:

ESPRIT method

- Rotational invariance property of $\mathbf{V}^n$: $\mathbf{V}^n{}_{\uparrow} = \mathbf{V}^n{}_{\downarrow}\,\mathbf{D}$
- Change of basis: $\mathbf{V}^n = \mathbf{W}\,\mathbf{G}$
- Rotational invariance of $\mathbf{W}$: $\mathbf{W}_{\uparrow} = \mathbf{W}_{\downarrow}\,\mathbf{\Phi}$
  where $\mathbf{\Phi} = \mathbf{G}\,\mathbf{D}\,\mathbf{G}^{-1}$ is referred to as the spectral matrix
- The eigenvalues of $\mathbf{\Phi}$ are the poles $\{z_k\}_{k\in\{0\dots K-1\}}$
- Matrix $\mathbf{\Phi}$ is such that $\mathbf{\Phi} = \left(\mathbf{W}_{\downarrow}^H\mathbf{W}_{\downarrow}\right)^{-1}\mathbf{W}_{\downarrow}^H\mathbf{W}_{\uparrow}$
- ESPRIT algorithm:
  - compute the estimator $\widehat{\mathbf{R}}_{xx}$ of matrix $\mathbf{R}_{xx}$,

ESPRIT method

- Rotational invariance property of $\mathbf{V}^n$: $\mathbf{V}^n{}_{\uparrow} = \mathbf{V}^n{}_{\downarrow}\,\mathbf{D}$
- Change of basis: $\mathbf{V}^n = \mathbf{W}\,\mathbf{G}$
- Rotational invariance of $\mathbf{W}$: $\mathbf{W}_{\uparrow} = \mathbf{W}_{\downarrow}\,\mathbf{\Phi}$
  where $\mathbf{\Phi} = \mathbf{G}\,\mathbf{D}\,\mathbf{G}^{-1}$ is referred to as the spectral matrix
- The eigenvalues of $\mathbf{\Phi}$ are the poles $\{z_k\}_{k\in\{0\dots K-1\}}$
- Matrix $\mathbf{\Phi}$ is such that $\mathbf{\Phi} = \left(\mathbf{W}_{\downarrow}^H \mathbf{W}_{\downarrow}\right)^{-1} \mathbf{W}_{\downarrow}^H \mathbf{W}_{\uparrow}$
- ESPRIT algorithm:
  - compute the estimator $\widehat{\mathbf{R}}_{xx}$ of matrix $\mathbf{R}_{xx}$,
  - diagonalize it and extract matrix $\mathbf{W}$,

ESPRIT method

- Rotational invariance property of $\mathbf{V}^n$: $\mathbf{V}^n{}_\uparrow = \mathbf{V}^n{}_\downarrow \mathbf{D}$
- Change of basis: $\mathbf{V}^n = \mathbf{W}\,\mathbf{G}$
- Rotational invariance of $\mathbf{W}$: $\mathbf{W}_\uparrow = \mathbf{W}_\downarrow \mathbf{\Phi}$
  where $\mathbf{\Phi} = \mathbf{G}\,\mathbf{D}\,\mathbf{G}^{-1}$ is referred to as the spectral matrix
- The eigenvalues of $\mathbf{\Phi}$ are the poles $\{z_k\}_{k \in \{0 \dots K-1\}}$
- Matrix $\mathbf{\Phi}$ is such that $\mathbf{\Phi} = \left(\mathbf{W}_\downarrow^H \mathbf{W}_\downarrow\right)^{-1} \mathbf{W}_\downarrow^H \mathbf{W}_\uparrow$
- ESPRIT algorithm:
  - compute the estimator $\widehat{\mathbf{R}}_{xx}$ of matrix $\mathbf{R}_{xx}$,
  - diagonalize it and extract matrix $\mathbf{W}$,
  - compute $\mathbf{\Phi} = \left(\mathbf{W}_\downarrow^H \mathbf{W}_\downarrow\right)^{-1} \mathbf{W}_\downarrow^H \mathbf{W}_\uparrow$,

ESPRIT method

- Rotational invariance property of $\mathbf{V}^n$: $\mathbf{V}^n{}_\uparrow = \mathbf{V}^n{}_\downarrow \mathbf{D}$
- Change of basis: $\mathbf{V}^n = \mathbf{W}\,\mathbf{G}$
- Rotational invariance of $\mathbf{W}$: $\mathbf{W}_\uparrow = \mathbf{W}_\downarrow \boldsymbol{\Phi}$
  where $\boldsymbol{\Phi} = \mathbf{G}\,\mathbf{D}\,\mathbf{G}^{-1}$ is referred to as the spectral matrix
- The eigenvalues of $\boldsymbol{\Phi}$ are the poles $\{z_k\}_{k\in\{0\ldots K-1\}}$
- Matrix $\boldsymbol{\Phi}$ is such that $\boldsymbol{\Phi} = \left(\mathbf{W}_\downarrow^H \mathbf{W}_\downarrow\right)^{-1} \mathbf{W}_\downarrow^H \mathbf{W}_\uparrow$
- ESPRIT algorithm:
  - compute the estimator $\widehat{\mathbf{R}}_{xx}$ of matrix $\mathbf{R}_{xx}$,
  - diagonalize it and extract matrix $\mathbf{W}$,
  - compute $\boldsymbol{\Phi} = \left(\mathbf{W}_\downarrow^H \mathbf{W}_\downarrow\right)^{-1} \mathbf{W}_\downarrow^H \mathbf{W}_\uparrow$,
  - diagonalize $\boldsymbol{\Phi}$ and get the poles $\{z_k\}_{k\in\{0\ldots K-1\}}$.

Estimation of the amplitudes and phases

- Let $\mathbf{x}$ be the vector $[x[0], x[1], \ldots, x[N-1]]^T$ of dimension N

Estimation of the amplitudes and phases

- Let $\mathbf{x}$ be the vector $[x[0], x[1], \ldots, x[N-1]]^T$ of dimension N
- Let $\mathbf{V}^N$ denote the Vandermonde matrix with N rows

Estimation of the amplitudes and phases

- Let $\mathbf{x}$ be the vector $[x[0], x[1], \ldots, x[N-1]]^T$ of dimension N
- Let $\mathbf{V}^N$ denote the Vandermonde matrix with N rows
- Let $\alpha = [\alpha_0, \alpha_1, \ldots, \alpha_{K-1}]^T$ denote the vector of complex amplitudes that we aim to estimate

Estimation of the amplitudes and phases

- Let $\mathbf{x}$ be the vector $[x[0], x[1], \ldots, x[N-1]]^T$ of dimension N
- Let $\mathbf{V}^N$ denote the Vandermonde matrix with N rows
- Let $\alpha = [\alpha_0, \alpha_1, \ldots, \alpha_{K-1}]^T$ denote the vector of complex amplitudes that we aim to estimate
- The maximum likelihood principle leads to using the lest squares method: $\widehat{\alpha} = \underset{\beta}{\operatorname{argmin}} \left\| \mathbf{x} - \mathbf{V}^N \beta \right\|^2$

Estimation of the amplitudes and phases

- Let $\mathbf{x}$ be the vector $[x[0], x[1], \ldots, x[N-1]]^T$ of dimension N
- Let $\mathbf{V}^N$ denote the Vandermonde matrix with N rows
- Let $\alpha = [\alpha_0, \alpha_1, \ldots, \alpha_{K-1}]^T$ denote the vector of complex amplitudes that we aim to estimate
- The maximum likelihood principle leads to using the lest squares method: $\widehat{\alpha} = \underset{\beta}{\operatorname{argmin}} \left\| \mathbf{x} - \mathbf{V}^N \beta \right\|^2$
- The solution is $\widehat{\alpha} = \left({\mathbf{V}^N}^H \mathbf{V}^N \right)^{-1} {\mathbf{V}^N}^H \mathbf{x}$

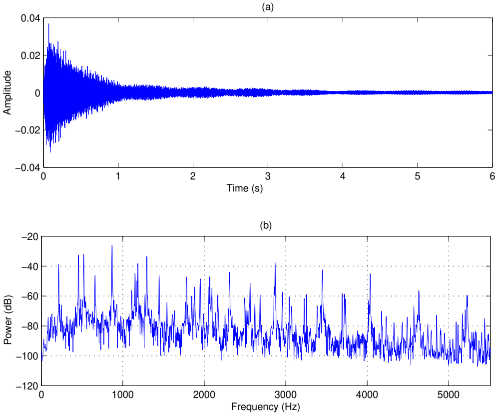
Estimation of the amplitudes and phases

- Let $\mathbf{x}$ be the vector $[x[0], x[1], \dots, x[N-1]]^T$ of dimension N
- Let $\mathbf{V}^N$ denote the Vandermonde matrix with N rows
- Let $\alpha = [\alpha_0, \alpha_1, \dots, \alpha_{K-1}]^T$ denote the vector of complex amplitudes that we aim to estimate
- The maximum likelihood principle leads to using the lest squares method: $\widehat{\alpha} = \underset{\beta}{\operatorname{argmin}} \left\| \mathbf{x} - \mathbf{V}^N \beta \right\|^2$
- The solution is $\widehat{\alpha} = \left({\mathbf{V}^N}^H \mathbf{V}^N \right)^{-1} {\mathbf{V}^N}^H \mathbf{x}$
- We finally get $\widehat{a}_k = |\widehat{\alpha}_k|$ and $\widehat{\phi}_k = \arg(\widehat{\alpha}_k)$

Part III

Signals to be processed

Bell sound



(a) Signal waveform
(b) Fourier transform (dB)