Exact solutions for linear propagation of chirped pulses using a chirped Gauss–Hermite orthogonal basis

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Received December 23, 1996

A generalized solution of the linear propagation equation is proposed in terms of chirped Gauss–Hermite orthogonal functions. Some well-known special cases are pointed out, and the usefulness of this approach in analyzing arbitrarily shaped chirped pulses in rapidly converging series is discussed. © 1997 Optical Society of America

Laser-generated pulses frequently suffer from the frequency chirping effect.[1–5] During their propagation in an optical fiber, in the linear regime chirped pulses give rise to an extra pulse broadening that is highly detrimental to the maximum bit-rate-length product achievable.[1,2] In some cases, however, the chirping effect can also be used to compress pulses temporally.[1,3] To study the propagation effects of chirped pulses and consequently to calculate optimum parameters (optimum compression length, minimum time duration, etc.), numerical convolution[3] or fast-Fourier-transform (FFT) methods[4] are generally used. A well-known exact solution is, however, available in the case of the chirped Gaussian pulse.[1,2,4] Here we generalize this result to an orthogonal set of chirped Gauss–Hermite functions, of which the chirped Gaussian solution is the first term. We then show that this orthogonal function expansion can be advantageously used to analyze arbitrarily shaped chirped pulses with rapid convergence and with an *a priori* fixed rms error.

The normalized linear propagation equation in optical fibers, including up to second-order dispersion,[1] is

$$j\frac{\partial \varphi(z,t)}{\partial z} = \frac{1}{2}\frac{\partial^2 \varphi(z,t)}{\partial t^2}, \tag{1}$$

where $\varphi(z, t)$ is the normalized electrical field envelope. Taking the Fourier transform of Eq. (1) and solving the resultant ordinary differential equation yield

$$\Phi(z,\omega) = \Phi(0,\omega)\exp\left(j\frac{\omega^2 z}{2}\right)\cdot \tag{2}$$

For $\Phi(0, \omega)$ we then choose, somewhat arbitrarily, the set of orthogonal functions

$$\Phi_m(0,\omega) = \frac{\sqrt{2\pi}(-j)^m}{(1+\alpha^2)^{1/4}}\exp\left(\frac{-\omega^2/2}{1+j\alpha}\right) \times \exp[-j(m+1/2)\arctan\alpha]\frac{H_m\left(\frac{\omega}{\sqrt{1+\alpha^2}}\right)}{(2^m m!\sqrt{\pi})^{1/2}}, \tag{3}$$

where α is the laser chirp factor, or phase–amplitude coupling factor, and $H_m(\omega)$ is the mth-order Hermite polynomial. Some of the coefficients in Eq. (3) are chosen to yield normalized functions in the time domain. We obtain the inverse Fourier transform of Eq. (2) by using Eq. (3) and the properties of the Gauss–Hermite functions, after some lengthy calculations, as follows:

$$\varphi_m(z,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\Phi_m(z,\omega)\exp(j\omega t)\mathrm{d}\omega = \frac{h_m\left\{\frac{t}{[(1+\alpha z)^2+z^2]^{1/2}}\right\}}{[(1+\alpha z)^2+z^2]^{1/4}}\exp\left[j(m+1/2)\times\arctan\left(\frac{z}{1+\alpha z}\right) - \frac{j}{2}\frac{\alpha+(1+\alpha^2)z}{(1+\alpha z)^2+z^2}t^2\right], \tag{4}$$

where

$$h_m(t) = \frac{1}{(2^m m!\sqrt{\pi})^{1/2}}\exp(-t^2/2)H_m(t), \quad \text{with} \int_{-\infty}^{+\infty}h_m(t)h_n(t)\mathrm{d}t = \delta_{mn} \tag{5}$$

are the orthonormal Gauss–Hermite functions. It can be proved, in a general way, that orthogonality for the $\Phi_m(0, \omega)$ functions leads to orthogonality for the $\varphi_m(z, t)$ functions by virtue of relation (2) and the Fourier-transform properties. Furthermore, the special choice of coefficients in Eq. (3) results in the orthonormality of chirped Gauss–Hermite functions of Eq. (4), namely,

$$\int_{-\infty}^{+\infty}\varphi_m(z,t)\varphi_n^*(z,t)\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\Phi_m(0,\omega)\Phi_n^*(0,\omega)\mathrm{d}\omega = \delta_{mn}. \tag{6}$$

Some easily recognizable special cases of relation (4) are the following. For $\alpha = 0$ and $z = 0$, relation (4) reduces to the well-known property of the Gauss–Hermite functions that they are their own Fourier

0146-9592/97/100685-03$10.00/0

transforms multiplied by the proportionality factor $(-j)^m\sqrt{2\pi}$. For $\alpha = 0$ and arbitrary z, relation (4) reduces to the unidimensional Gauss–Hermite modal functions of optics, used in the product solution of the paraxial wave equation in rectangular coordinates.[6] When α and z are arbitrary and for $m = 0$, relation (4) reduces to the solution of chirped Gaussian pulse propagation in the linear regime.[1] This fexact solution has frequently been used for the estimation of bit-rate–length products in optical communications as well as for optimum parameter determination in chirped pulse compression schemes.[2]

The generalization proposed in relation (4) to arbitrary order m is in fact quite remarkable if we take into account that the functions $\varphi_m(z, t)$ are orthonormal to each other with respect to t for every z and α. This property, together with the energy conservation property of Eq. (1), permits an *a priori* rms error estimation for the orthogonal series expansions of arbitrarily shaped chirped pulses.

The series expansion of an arbitrary solution of Eq. (1) is written as

$$u(z, t) = \sum_{m=0}^{+\infty} c_m \varphi_m(z, t)\,, \tag{7}$$

where, because of orthonormality property (6), the coefficients c_m are given by

$$c_m = \int_{-\infty}^{+\infty} u(0, t)\varphi_m{}^*(0, t)\mathrm{d}t = \int_{-\infty}^{+\infty} u(0, t)\exp\left(j\frac{\alpha t^2}{2}\right)h_m(t)\mathrm{d}t\,, \tag{8}$$

and the relative rms error of the N-term approximation in Eq. (7), for every z, is

$$E_{\mathrm{rms}}^2 = 1 - \frac{\sum_{m=0}^{N-1}|c_m|^2}{\int_{-\infty}^{+\infty}|u(0, t)|^2\mathrm{d}t}\cdot \tag{9}$$

Especially for pulses of the form

$$u(0, t) = A(t)\exp\left(-j\frac{\alpha t^2}{2}\right), \tag{10}$$

where $A(t)$ is a real function, the coefficients c_m of Eq. (8) are real, independently of α, and expansion (7) converges rapidly for every α and z. In the special case that $A(t) = \exp(-t^2/2)$, the summation in Eq. (7) reduces to only one term. It is logically expected that similarly shaped (bell-shaped) functions[5] will be well represented with only a limited number of terms in Eq. (7).

To validate our approach we analyze a chirped hyperbolic secant pulse[1] of the form

$$u(0, t) = \mathrm{sech}\left(\sqrt{\pi/2}t\right)\exp\left(-j\frac{\alpha t^2}{2}\right)\cdot \tag{11}$$

Because of symmetry, only even-order terms in expansion (7) are nonzero. Furthermore, with the special choice of $\sqrt{\pi/2}$ as a scaling factor in Eq. (11), it turns out that only one of every four coefficients c_m is nonzero. In Table 1 the first four nonzero coefficients are shown, together with the relative rms error of each of the first four approximations. The first-order approximation corresponds to the Gaussian approximation of the hyperbolic secant pulse. In Table 2 the number of nonzero terms required in Eq. (7) for a predefined relative rms error to be obtained are shown. It should be stressed that, in this case, the order of the approximation N in Eq. (9) is higher than the number of nonzero terms used. Moreover, a graph of relative rms error versus number of nonzero basis functions in Fig. 1 depicts clearly the rapid convergence rate in this case. In fact, in this case, four terms are enough for a good working accuracy of 0.2% for every α and z. However, when the chirped pulse is not of the special form of Eq. (10), the expansion coefficients of Eq. (8) become complex and depend on the value of α. It is then advisable to use the simpler form of the expansion functions $\varphi_m(z, t)$ with $\alpha = 0$.

Some other computational methods frequently used for the same problem are the numerical convolution[3]

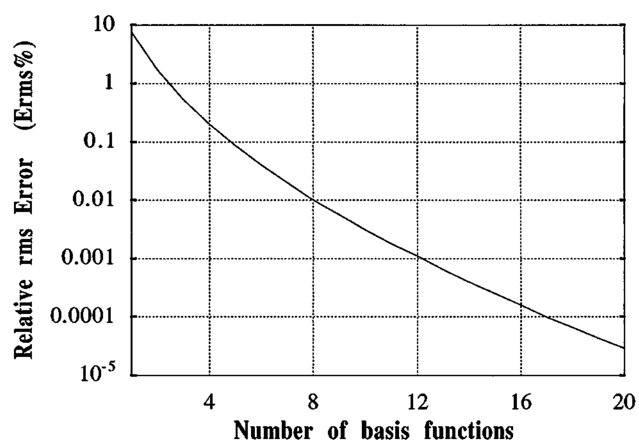

Fig. 1. Relative rms error (Erms) versus number of nonzero basis functions in the approximation of the pulse of Eq. (11).

Table 1. Nonzero Expansion Coefficients and Relative rms Error E_{rms} of the First Four Approximations to the Chirped Hyperbolic Secant Pulse of Eq. (11)

Basis Function	c_m	E_{rms} (%)
1	$c_0 = 1.259707$	7.47
2	$c_4 = 0.091934$	1.69
3	$c_8 = 0.020216$	0.53
4	$c_{12} = 0.006254$	0.20

Table 2. Number of Nonzero Basis Functions Required for a Predefined Relative rms Error E_{rms} in Approximating the Pulse of Eq. (11)

E_{rms} (%)	Basis Functions
0.01	8
0.001	12
0.0001	17

Table 3. Comparison between the Peak Amplitude Computed by the CGH and FFT Methods in the Case of the Pulse of Eq. (11) with $\alpha = -10$

		FFT (2048 Points)	
Distance z	CGH-4 Basis Functions	$t \in (-40, 40)$	$t \in (-200, 200)$
0.1	3.16	3.16	2.63
1.0	0.329	0.329	0.343
10.0	0.0994	0.0112	0.0974
100.0	0.0313	0.0682	0.0269

and the spectral FFT[4] methods. The main difficulty with the first method arises in the numerical calculation of the integral of a rapidly oscillating function. On the other hand, the spectral FFT method requires sufficiently large calculation windows in the time and frequency domains that are difficult to evaluate without some prior knowledge of the solution. Our method does not suffer from these problems, and once the expansion coefficients are known the solution has an analytical form, rendering itself easily amenable to further manipulations.

An example application of our chirped Gauss–Hermite (CGH) method and its comparison, in terms of accuracy, with the well-known FFT method in the linear case of chirped pulse compression follow. The initial pulse is of the form of Eq. (11) with $\alpha = -10$, and it is used to model a gain-switching laser pulse. Table 3 summarizes the results of the comparison for the peak amplitude of the propagating pulse $|u(z, t = 0)|$ for various normalized distances z. It is seen that the CGH method converges to the true solution with three-digit accuracy, with only four nonzero basis functions, for all z. The FFT method is seen to be highly sensitive to the correct choice of the computational window in the time domain, especially when the pulse undergoes significant changes in width, as is the case in pulse compression. As a result, different computational windows should be used for different propagation distances z, through an essentially trial-and-error procedure, to preserve a reasonable error bound. This procedure is avoided in the case of the CGH method, as its error properties remain invariable for all normalized distances z, even asymptotically for $z \to \infty$. Thus precious simulation time can be gained when multiple calculations are required, e.g., to determine the optimum compression distance as a function of the chirp factor α.

It should also be noted that the Fourier transform of the solution is analytically calculated with Eqs. (7), (4), and (5) in the form of a similar orthogonal function expansion:

$$U(z, \omega) = \frac{\sqrt{2\pi}}{(1 + \alpha^2)^{1/4}} \exp\left(-j\alpha \frac{\omega^2/2}{1 + \alpha^2}\right) \exp\left(j \frac{\omega^2 z}{2}\right) \times \sum_{m=0}^{+\infty} c_m (-j)^m \exp[-j(m + 1/2) \times \arctan \alpha] h_m\left(\omega/\sqrt{1 + \alpha^2}\right). \quad (12)$$

In conclusion, we have presented a generalized orthogonal function basis for the solution of the linear propagation equation, derived from an analytically calculated Fourier transform, using chirped Gauss–Hermite functions. Some well-known solutions of the linear propagation problem, such as chirped Gaussian pulse propagation, are shown to be special cases of our generalized solution. Rapid convergence, *a priori* rms error estimation, and analytical Fourier-transform calculation are some of the advantages of our orthogonal function expansion method.

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