Chapter 4
Fractional Brownian motion

In the nineties, statistical evidence, notably in finance and telecommunications, showed that Markov processes were to far away from the observations to be considered as viable models. In particular, there were strong suspicions that the data exhibit long range dependence. It is in this context that the fractional Brownian motion, introduced by B. Mandelbrot in the late sixties and almost forgotten since, enjoyed a new rise of interest. It is a Gaussian process with long range dependence. Consequently, it cannot be a semi-martingale and we cannot apply the theory of Itô calculus. As we have seen earlier, for the Brownian motion, the Malliavin divergence generalizes the Itô integral and can be constructed for the fBm, so it is tempting to view it as an ersatz of a stochastic integral. Actually, the situation is not simple and depends on what we call a stochastic integral.

4.1 Definition and sample-paths properties

Definition 4.1. For any H in $(0,1)$, the fractional Brownian motion of index (Hurst parameter) H, $\{B_H(t);\ t \in [0,1]\}$ is the centered Gaussian process whose covariance kernel is given by

$$R_H(s,t) = \mathbf{E}\left[B_H(s)B_H(t)\right] = \frac{V_H}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right)$$

where

$$V_H = \frac{\Gamma(2-2H)\cos(\pi H)}{\pi H(1-2H)}.$$

Note that for $H = 1/2$, we obtain

$$R_{1/2}(t,s) = \frac{1}{2}\left(t + s - |t-s|\right)$$

which is nothing but the sophisticated way to write

$$R_{1/2}(t, s) = \min(t, s).$$

Hence, $B_{1/2}$ is the ordinary Brownian motion.

Theorem 4.1. *Let $H \in (0, 1)$, the sample-paths of W^H are Hölder continuous of any order less than H (and no more) and belong to $W_{\alpha,p}$ for any $p \geq 1$ and any $\alpha \in (0, H)$.*

We denote by μ_H, the measure on $\mathcal{W}_{\alpha,p}$ which corresponds to the distribution of B_H.

Proof. Step 1. A simple calculations shows that, for any $\alpha \geq 0$, we have

$$\mathbf{E}\left[|B_H(t) - B_H(s)|^\alpha\right] = C_\alpha |t - s|^{H\alpha}.$$

Since B_H is Gaussian, its p-th moment can be expressed as a power the variance, hence we have

$$\mathbf{E}\left[\iint_{[0,1]^2} \frac{|B_H(t) - B_H(s)|^p}{|t - s|^{1+\alpha p}} \,\mathrm{d}t \,\mathrm{d}s\right] = C_\alpha \iint_{[0,1]^2} |t - s|^{-1+p(H-\alpha)} \,\mathrm{d}t \,\mathrm{d}s.$$

This integral is finite as soon as $\alpha < H$ hence, for any $\alpha < H$, any $p \geq 1$, B_H belongs to $W_{\alpha,p}$ with probability 1. Choose p arbitrary large and conclude that the sample-paths are Hölder continuous of any order less than H, in view of the Sobolev embeddings (see Theorem 1.4)

Step 2. As a consequence of the results in [Arc95], we have

$$\mu_H \left(\limsup_{u \to 0^+} \frac{B_H(u)}{u^H \sqrt{\log \log u^{-1}}} = \sqrt{V_H} \right) = 1.$$

Hence it is impossible for B_H to have sample-paths Hölder continuous of an order greater than H.

The difference of regularity is evident on simulations of sample-paths, see Figure 4.1.

As a consequence, B_H cannot be a semi-martingale as its quadratic variation is either null or infinite.

Lemma 4.1. *The process $(a^{-H} B_H(at), t \geq 0)$ has the same distribution as B_H.*

Proof. Consider the centered Gaussian process

$$Z(t) = a^{-H} B_H(at).$$

Its covariance kernel is given

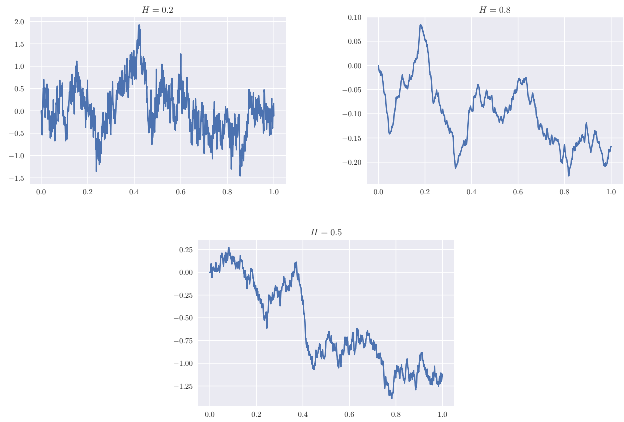


Fig. 4.1 Sample-path example for $H = 0.2$ (upper left), $H = 0.5$ (below) and $H = 0.8$ (upper right).

$$\mathbf{E}\left[Z(t)Z(s)\right] = a^{-2H} R_H(at, as) = R_H(t, s).$$

Since a covariance kernel determines the distribution of a Gaussian process, Z and B_H have the same law.

Theorem 4.2. *With probability* 1*, we have:*

$$\lim_{n\to\infty} \sum_{j=1}^{n} \left| B_H\Big(\frac{j}{n}\Big) - B_H\Big(\frac{j-1}{n}\Big)\right|^2 = \begin{cases} 0 & \text{if } H > 1/2 \\ \infty & \text{if } H < 1/2. \end{cases}$$

Proof. Lemma 4.1 entails that

$$\sum_{j=1}^{n} \left| B_H\Big(\frac{j}{n}\Big) - B_H\Big(\frac{j-1}{n}\Big)\right|^{1/H}$$

has the same distribution as

$$\frac{1}{n} \sum_{j=1}^{n} \left| B_H\big(j\big) - B_H\big(j-1\big)\right|^{1/H}.$$

The ergodic theorem entails that this converges in $L^1(\mathcal{W} \to \mathbf{R};\, \mu_H)$ and almost-surely to $\mathbf{E}\left[|B_H(1)|^H\right]$. Hence the result.

4.2 Cameron-Martin space

The next step is to describe the Cameron-Martin space attached to the fBm of index H. The general theory of Gaussian processes says that we must consider the self-reproducing Hilbert space defined by the covariance kernel, see the appendix of Chapter 1.

Definition 4.2. Let

$$\mathcal{H}^0 = \mathrm{span}\{R_H(t,.),\, t \in [0,1]\},$$

equipped with the scalar product

$$\langle R_H(t,.),\, R_H(s,.)\rangle_{\mathcal{H}_H} = R_H(t,s). \tag{4.1}$$

The Cameron-Martin space of the fBm of Hurst index H, denoted by $\mathcal{H}_H$, is the completion in $L^2([0,1] \to \mathbf{R};\, \lambda)$ of $\mathcal{H}^0$ for the scalar product defined in (4.1).

This means that $f \in L^2([0,1] \to \mathbf{R};\, \lambda)$ belongs to $\mathcal{H}$ whenever there exists a sequence $(f_n,\, n \geq 0)$ of elements of $\mathcal{H}^0$ which is Cauchy for the norm induced by $\|\ \|_{\mathcal{H}_H}$, converges to f in $L^2([0,1] \to \mathbf{R};\, \lambda)$. Then,

$$f = \mathcal{H}_H - \lim_{n\to\infty} f_n, i.e. \lim_{n\to\infty} \|f - f_n\|_{\mathcal{H}_H} = \lim_{n\to\infty} \left(\lim_{m\to\infty} \|f_m - f_n\|_{\mathcal{H}_H} \right) = 0.$$

This is not a very practical definition but we can have a much better description of $\mathcal{H}_H$ thanks to the next theorems.

Lemma 4.2 (Representation of an RKHS). *Assume that there exists a function $K_H\, :\, [0,\,1] \times [0,\,1] \to \mathbf{R}$ such that*

$$R_H(s,t) = \int_{[0,\,1]} K_H(s,r)\, K_H(t,r)\ dr, \tag{4.2}$$

and that the linear map defined by K_H is one-to-one:

$$\left(\forall t \in [0,\,1], \int_{[0,\,1]} K_H(t,s) g(s)\ ds = 0\right) \Longrightarrow g = 0\ \lambda - a.s. \tag{4.3}$$

Then the Hilbert space $\mathcal{H}_H$ can be identified to $K_H\big(L^2([0,\,1] \to \mathbf{R};\, \lambda)\big)$: The space of functions of the form

$$f(t) = \int_{[0,\,1]} K_H(t,s) \dot{f}(s)\ ds$$

for some $\dot{f} \in L^2([0,\,1] \to \mathbf{R};\, \lambda)$, equipped with the inner product

$$\langle K_H f,\, K_H g\rangle_{K_H(L^2([0,\,1]\to\mathbf{R};\,\lambda)} = \langle f, g\rangle_{L^2([0,\,1]\to\mathbf{R};\,\lambda)}\,.$$

Note that we abused the notations by denoting $K_H^{-1}(f)$ as $\dot{f}$. We will be rewarded of this little infrigement below as all the formulas will look the same whatever the value of H.

Proof. Step 1. Eqn. (4.3) means that

$$\mathfrak{K}_H = \operatorname{span}\{K_H(t, .),\, t \in [0,1]\}$$

is dense in $L^2([0,\,1] \to \mathbf{R};\, \lambda)$.
Step 2. Since $K_H\big(K_H(t, .)\big)(s) = R_H(t, s)$,

$$K_H\left(\sum_{k=1}^n \alpha_k K_H(t_k, .)\right) = \sum_{k=1}^n \alpha_k R_H(t_k, .).$$

On the one hand, we have

$$\left\|\sum_{k=1}^n \alpha_k R_H(t_k, .)\right\|_{\mathcal{H}_H}^2 = \sum_{k=1}^n \sum_{l=1}^n \alpha_k \alpha_l R_H(t_k, t_l) \tag{4.4}$$

and on the other hand, we observe that

$$\begin{aligned}\left\|\sum_{k=1}^n \alpha_k K_H(t_k, .)\right\|_{L^2([0,\,1]\to\mathbf{R};\,\lambda)}^2 &\\ &= \int_{[0,\,1]} \Big(\sum_{k=1}^n \alpha_k K_H(t_k, s)\Big)^2 \, \mathrm{d}s \\ &= \sum_{k=1}^n \sum_{l=1}^n \alpha_k \alpha_l \iint_{[0,\,1]\times[0,\,1]} K_H(t_k, s) K_H(t_l, s) \, \mathrm{d}s \\ &= \sum_{k=1}^n \sum_{l=1}^n \alpha_k \alpha_l R_H(t_k, t_l),\end{aligned} \tag{4.5}$$

in view of (4.2).
Step 3. Equations (4.4) and (4.5) mean that the map K_H:

$$\begin{aligned} K_H \,:\, \mathfrak{K}_H &\longrightarrow \mathcal{H}_H^0 \\ K_H(t, .) &\longrightarrow R_H(t, .) \end{aligned}$$

is a bijective isometry, when these spaces are equipped with the topology of $L^2([0,\,1] \to \mathbf{R};\, \lambda)$ and $\mathcal{H}_H$ respectively. By density, K_H is a bijective isometry from $L^2([0,\,1] \to \mathbf{R};\, \lambda)$ into $\mathcal{H}_H$. Otherwise stated, $K_H\big(L^2([0,\,1] \to \mathbf{R};\, \lambda)\big)$ is isometrically isomorphic, hence identified, to $\mathcal{H}_H$.

Example 4.1. For $H = 1/2$, we have

$$t \wedge s = \int_0^1 \mathbf{1}_{[0,t]}(r)\mathbf{1}_{[0,s]}(r)\ \mathrm{d}r.$$

This means that the RKHS of the Brownian motion is equal to $I_{1,2}$ since for $K_{1/2}(t,r) = \mathbf{1}_{[0,t]}(r)$,

$$K_{1/2}f(t) = \int_0^1 \mathbf{1}_{[0,t]}(r)\, f(r)\ \mathrm{d}r = I^1 f(t).$$

We now have to identify K_H for our kernel R_H.

Lemma 4.3. *For $H > 1/2$, Eqn.* (4.2) *is satisfied with*

$$K_H(t,r) = \frac{r^{1/2-H}}{\Gamma(H-1/2)} \int_r^t u^{H-1/2}(u-r)^{H-3/2}\ du\ \mathbf{1}_{[0,t]}(r). \tag{4.6}$$

Proof. According to the fundamental theorem of calculus, applied twice, we can write:

$$R_H(s,t) = \frac{V_H}{4H(2H-1)} \int_0^t \int_0^s |r-u|^{2H-2}\ \mathrm{d}u\ \mathrm{d}r \tag{4.7}$$

After a deep inspection of the handbooks of integrals or more simply, finding, with a bit of luck, the reference [BVP88], we see that

$$\begin{aligned}\frac{V_H}{4H(2H-1)}&|r-u|^{2H-2}\\ &= (ru)^{H-1/2} \int_0^{r\wedge u} v^{1/2-H}(r-v)^{H-3/2}(u-v)^{H-3/2}\ \mathrm{d}v. \end{aligned}\tag{4.8}$$

Plug (4.8) into (4.7) and apply Fubini to put the integration with respec to v in the outer most integral. This implies that (4.2) is satisfied with K_H given by (4.6).

Unfortunately, this integral is not defined for $H < 1/2$ because of the term $(u-r)^{H-3/2}$. Fortunately, the expression (4.6) can be expressed as an hypergeometric function. These somehow classical functions can be presented of different manners so that they are meaningful for a very wide range of parameters, including the domain which is of interest for us.

The Gauss hypergeometric function $F(a,b,c,z)$ (for details, see [NU88]) is defined for any a, b, any z, $|z| < 1$ and any $c \neq 0, -1, \ldots$ by

$$F(a,b,c,z) = \sum_{k=0}^{+\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k, \tag{4.9}$$

where $(a)_0 = 1$ and $(a)_k = \Gamma(a+k)/\Gamma(a) = a(a+1)\ldots(a+k-1)$ is the Pochhammer symbol. If a or b is a negative integer the series terminates after a finite number of terms and $F(a,b,c,z)$ is a polynomial in z. The radius of convergence of this series is 1 and there exists a finite limit when z tends to 1 ($z<1$) provided that $\Re(c-a-b)>0$. For any z such that $|\arg(1-z)|<\pi$, any a,b,c such that $\Re(c)>\Re(b)>0$, F can be defined by

$$F(a,b,c,z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 u^{b-1}(1-u)^{c-b-1}(1-zu)^{-a}\,du. \tag{4.10}$$

Given (a,b,c), consider Σ the set of triples (a',b',c') such that $|a-a'|=1$ or $|b-b'|=1$ or $|c-c'|=1$. Any hypergeometric function $F(a',b',c',z)$ with (a',b',c') in Σ is said to be contiguous to $F(a,b,c)$. For any two hypergeometric functions F_1 and F_2 contiguous to $F(a,b,c,z)$, there exists a relation of the type :

$$P_0(z)F(a,b,c,z)+P_1(z)F_1(z)+P_2(z)F_2(z)=0,\ \text{for } z,\ |\arg(1-z)|<\pi, \tag{4.11}$$

where for any i, P_i is a polynomial with respect to z. These relations permit to define the analytic continuation of $F(a,b,c,z)$ with respect to its four variables in the domain $\mathbb{C}\times\mathbb{C}\times(\mathbb{C}\backslash\{0,-1,-2,\dots\})\times\{z,|\arg(1-z)|<\pi\}$. We will also use other types of relations between different hypergeometric functions, namely :

$$\begin{aligned}F(a,b,c,z)&=\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)}(1-z)^{-a}F(a,c-b,1+a-b,1/(1-z))\\&\quad+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(c-b)\Gamma(a)}(1-z)^{-b}F(b,c-a,1-a+b,1/(1-z)),\end{aligned} \tag{4.12}$$

for any z such that $|\arg(1-z)|<\pi$ and $a-b\neq 0,\pm1,\pm2,\dots$.

Theorem 4.3. *For any $H\in(0,1)$, R_H can be factorized as in* (4.2) *with*

$$\begin{aligned}K_H\ :\ [0,1]^2&\longrightarrow\mathbf{R}\\(t,s)&\longmapsto\frac{(t-s)^{H-1/2}}{\Gamma(H+1/2)}\,F\big(H-1/2,1/2-H,H+1/2,1-t/s\big)\end{aligned} \tag{4.13}$$

If we identify integral operators and their kernel, this amounts to say that

$$R_H=K_H\circ K_H^*.$$

Proof. For $H>1/2$, a change of variable in (4.6) transforms the integral term in

$$(t-r)^{H-1/2}r^{H-1/2}\int_0^1 u^{H-3/2}\left(1-(1-t/r)u\right)^{H-1/2}\,du.$$

By the definition (4.10) of hypergeometric functions, we see that (4.13) holds true for $H > 1/2$. Using property (4.12), we have

$$\begin{aligned} K_H(t,r) = &\frac{2^{-2H}\sqrt{\pi}}{\Gamma(H)\sin(\pi H)} r^{H-1/2} \\ &+ \frac{1}{2\Gamma(H+1/2)}(t-r)^{H-1/2} F(1/2-H, 1, 2-2H, \frac{r}{t}). \end{aligned}$$

If $H < 1/2$ then the hypergeometric function of the latter equation is continuous with respect to r on $[0,t]$ because $2-2H-1-1/2+H = 1/2-H$ is positive. Hence, for $H < 1/2$, $K_H(t,r)(t-r)^{1/2-H} r^{1/2-H}$ is continuous with respect to r on $[0,t]$. For $H > 1/2$, the hypergeometric function is no more continuous in t but we have [NU88] :

$$\begin{aligned} F(1/2-H, 1, 2-2H, \frac{r}{t}) = &C_1 F(1/2-H, 1, H+1/2, 1-r/t) \\ &+ C_2(1-r/t)^{1/2-H}(r/t)^{2H-1}. \end{aligned}$$

Hence, for $H \geq 1/2$, $K_H(t,r)r^{H-1/2}$ is continuous with respect to r on $[0,t]$. Fix $\delta \in [0, 1/2)$ and $t \in (0,1]$, we have :

$$|K_H(t,r)| \leq C r^{-|H-1/2|}(t-r)^{-(1/2-H)_+} \mathbf{1}_{[0,t]}(r)$$

where C is uniform with respect to $H \in [1/2-\delta, 1/2+\delta]$. Thus, the two functions defined on $\{H \in \mathbf{C},\ |H-1/2| < 1/2\}$ by

$$H \longmapsto R_H(s,t) \text{ and } H \longmapsto \int_0^1 K_H(s,r)K_H(t,r)\, \mathrm{d}r$$

are well defined, analytic with respect to H and coincide on $[1/2, 1)$, thus they are equal for any $H \in (0,1)$ and any s and t in $[0,1]$.

In the previous proof we proved a result which is so useful in its own that it deserves to be a theorem :

Theorem 4.4. *For any $H \in (0,1)$, for any t, the function*

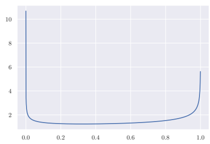
$$\begin{aligned} [0,t] &\longrightarrow \mathbf{R} \\ r &\longmapsto K_H(t,r) r^{|H-1/2|}(t-r)^{(1/2-H)_+} \end{aligned}$$

is continuous on $[0,t]$.

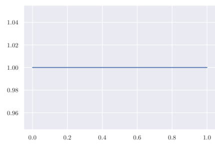
Moreover, there exists a constant c_H such for any $0 \leq r \leq t \leq 1$

$$|K_H(t,r)| \leq c_H\, r^{-|H-1/2|}(t-r)^{-(1/2-H)_+}. \tag{4.14}$$

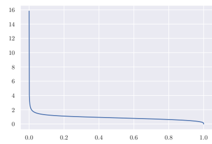
These continuity results are illustrated by the following pictures.



$K_{0.25}(1,s)$



$K_{0.5}(1,s)$



$K_{0.75}(1,s)$

We made some progress with this new description of $\mathcal{H}_H$. However, for a given element of $L^2([0,1 \to]; \mathbf{R})\lambda$, it is stil difficult to determine whether it belongs to $\mathcal{H}_H$. Since

$$\int_0^1 \int_0^1 K(t,r)^2 \, \mathrm{d}t \, \mathrm{d}r = \int_0^1 R_H(t,t) \, \mathrm{d}t < \infty,$$

we already known that the integral map of kernel K_H is Hilbert-Schmidt from $L^2([0,1] \to \mathbf{R};\, \lambda)$ into itself. Thanks to [SKM93, page 187], we are in position to give a fully satisfactory description of $\mathcal{H}_H$.

Theorem 4.5. *Consider the integral transform of kernel K_H, i.e.*

$$\begin{aligned} K_H \,:\, L^2([0,1] \to \mathbf{R};\, \lambda) &\longrightarrow L^2([0,1] \to \mathbf{R};\, \lambda) \\ f &\longmapsto \left(t \mapsto \int_0^t K_H(t,s) f(s) \, ds \right). \end{aligned}$$

The map K_H is an isomorphism from $L^2([0,1] \to \mathbf{R};\, \lambda)$ onto $I_{H+1/2,2}$ and

$$\begin{aligned} K_H f &= I_{0^+}^{2H} x^{1/2-H} I_{0^+}^{1/2-H} x^{H-1/2} f \quad \text{for } H \le 1/2, \\ K_H f &= I_{0^+}^{1} x^{H-1/2} I_{0^+}^{H-1/2} x^{1/2-H} f \quad \text{for } H \ge 1/2. \end{aligned}$$

Note that if $H \ge 1/2$, $r \to K_H(t,r)$ is continuous on $(0,t]$ so that we can include t in the indicator function.

Remark 4.1. We already know that the fBm is all the more regular than its Hurst index is close to 1. However, we see that the kernel K_H is more and more singular when H goes to 1. This means that it is probably a bad idea to devise properties of B_H using the properties of K_H. On the other hand, as an operator K_H is more and more regular as H increases. This indicates that the efficient approach is to work with K_H as an operator. We tried to illustrate this line of reasoning in the next results.

To summarize the previous considerations, we get

Theorem 4.6. *The Cameron-Martin of the fractional Brownian motion is $\mathcal{H}_H = \{K_H \dot{h};\, \dot{h} \in L^2([0,1] \to \mathbf{R};\, \lambda)\}$, i.e., any $h \in \mathcal{H}_H$ can be represented as*

$$h(t) = K_H \dot{h}(t) = \int_0^1 K_H(t,s)\dot{h}(s)\ ds,$$

where $\dot{h}$ belongs to $L^2([0,1] \to \mathbf{R};\ \lambda)$. For any $\mathcal{H}_H$–valued random variable u, we hereafter denote by $\dot{u}$ the $L^2([0,1] \to \mathbf{R};\ \lambda))$-valued random variable such that

$$u(w,t) = \int_0^t K_H(t,s)\dot{u}(w,s)\,ds.$$

The scalar product on $\mathcal{H}_H$ is given by

$$(h,g)_{\mathcal{H}_H} = (K_H\dot{h}, K_H\dot{g})_{\mathcal{H}_H} = (\dot{h},\dot{g})_{L^2([0,1]\to\mathbf{R};\ \lambda)}.$$

Remark 4.2. Theorem 4.5 implies that as a vector space, $\mathcal{H}_H$ is equal to $I_{H+1/2,2}$ but the norm on each of these spaces are different since

$$\|K_H\dot{h}\|_{\mathcal{H}_H} = \|\dot{h}\|_{L^2([0,1]\to\mathbf{R};\ \lambda)}$$
$$\text{and } \|K_H\dot{h}\|_{I_{H+1/2,2}} = \|(I_{0^+}^{-H-1/2} \circ K_H)\dot{h}\|_{L^2([0,1]\to\mathbf{R};\ \lambda)}.$$

4.3 Wiener space

We can now construct the fractional Wiener measure as we did for the ordinary Brownian motion.

Theorem 4.7. *Let $(\dot{h}_m,\ m \geq 0)$ be a complete orthonormal basis of $L^2([0,1] \to \mathbf{R};\ \lambda)$ and $h_m = K_H\dot{h}_m$. Consider the sequence*

$$S_n^H(t) = \sum_{m=0}^n X_m h_m(t)$$

where $(X_m,\ m \geq 0)$ is a sequence of independent standard Gaussian random variables. Then, $(S_n^H,\ n \geq 0)$ converges, with probability 1, in $W_{\alpha,p}$ for any $\alpha < H$ and any $p > 1$.

Proof. The proof proceeds exactly as the proof of Theorem 1.5. The trick is to note that

$$(h_m(t) - h_m(s))^2 = \langle K_H(t,.) - K_H(s,.),\ \dot{h}_m\rangle^2_{\mathcal{H}_H},$$

so that

$$\sum_{m=0}^\infty (h_m(t) - h_m(s))^2 = \|K_H(t,.) - K_H(s,.)\|^2_{L^2([0,1]\to\mathbf{R};\ \lambda)}$$
$$= R_H(t,t) - R_H(s,s) - 2R_H(t,s) = V_H|t-s|^{2H}.$$

Moreover,

$$\int_{[0,1]^2} |t-s|^{pH-1-\alpha p} \, \mathrm{d}s \, \mathrm{d}t < \infty \text{ if and only if } \alpha < H.$$

This means, by dominated convergence, that

$$\sup_{n \geq M} \mathbf{E}\left[\|S_n^H - S_M^H\|_{W_{\alpha,p}}^p \right] \\ = \iint_{[0,1]^2} \Big(\sum_{m=M+1}^{\infty} (h_m(t)-h_m(s))^2 \Big)^{p/2} |t-s|^{-1-\alpha p} \, \mathrm{d}s \, \mathrm{d}t \xrightarrow{M\to\infty} 0,$$

provided that $\alpha < H$. The proof is finished as in Theorem 1.5.

In what follows, W may be taken either as $\mathcal{C}_0([0,1],\mathbf{R})$ or as any of the spaces $W_{\gamma,p}$ with

$$p \geq 1, \; 0 < \gamma < H.$$

For any $H \in (0,1)$, μ_H is the unique probability measure on W such that the canonical process $(B_H(s);\; s \in [0,1])$ is a centered Gaussian process with covariance kernel R_H :

$$\mathbf{E}_H[B_H(s)B_H(t)] = R_H(s,t).$$

The canonical filtration is given by $\mathcal{F}_t^H = \sigma\{W_s,\; s \leq t\} \vee \mathcal{N}_H$ and $\mathcal{N}_H$ is the set of the μ_H–negligible events. The analog of the diagram 1.1 reads as

$$\begin{tikzcd}
\mathcal{W}^* \arrow[r, "\mathfrak{e}^*"] & {\mathcal{H}_H}^* = (I_{H+1/2,2})^* \arrow[d, "\simeq"] & \\
L^2 \arrow[r, "K_H"] & \mathcal{H}_H = I_{H+1/2,2} \arrow[r, "\mathfrak{e}"] & \mathcal{W}
\end{tikzcd}$$

Fig. 4.2 Embeddings and identification for fractional Brownian motion.

We can as before, search for the image of ε_t by $\mathfrak{e}^*$. We have, for $h \in \mathcal{H}_H$, on the one hand,

$$h(t) = \langle \varepsilon_t, \mathfrak{e}(h) \rangle_{\mathcal{W}^*,\mathcal{W}} = \langle \mathfrak{e}^*(\epsilon_t),\, h \rangle_{\mathcal{H}_H}.$$

On the other hand,

$$h(t) = K_H \dot{h}(t) = \langle K_H(t,.),\, \dot{h} \rangle_{L^2([0,1]\to\mathbf{R};\,\lambda)} = \langle R_H(t,.), h \rangle_H.$$

Hence,

$$\mathfrak{e}^*(\varepsilon_t) = R_H(t,.) \text{ and } K_H^{-1}(\mathfrak{e}^*(\varepsilon_t)) = K_H(t,.).$$

Recall that for the ordinary Brownian motion, we have

$$\mathfrak{e}^*(\varepsilon_t) = t \wedge . = R_{1/2}(t, .) \text{ and } K_H^{-1}(\mathfrak{e}^*(\varepsilon_t)) = \mathbf{1}_{[0,t]}(.) = K_{1/2}(t, .).$$

Theorem 4.8. *For any z in $\mathcal{W}^*$,*

$$\int_{\mathcal{W}} e^{i\langle z,\omega\rangle_{\mathcal{W}^*,\mathcal{W}}}\ d\mu_H(\omega) = \exp\Big(-\frac{1}{2}\|\mathfrak{e}^*(z)\|^2_{\mathcal{H}_H}\Big). \tag{4.15}$$

Proof. By dominated convergence, we have

$$\begin{aligned}\int_{\mathcal{W}} e^{i\langle z,\omega\rangle_{\mathcal{W}^*,\mathcal{W}}}\ \mathrm{d}\mu_H(\omega) &= \lim_{n\to\infty} \mathbf{E}\left[\exp\left(i\sum_{m=0}^{n} X_m \left\langle z,\, \mathfrak{e}(K_H\dot{h}_m)\right\rangle_{\mathcal{W}^*,\mathcal{W}}\right)\right] \\ &= \lim_{n\to\infty} \exp\left(-\frac{1}{2}\sum_{m=0}^{n}\left\langle \mathfrak{e}^*(z),\, K_H\dot{h}_m\right\rangle^2_{\mathcal{H}}\right) \\ &= \exp\left(-\frac{1}{2}\sum_{m=0}^{\infty}\left\langle \mathfrak{e}^*(z),\, K_H\dot{h}_m\right\rangle^2_{\mathcal{H}}\right) \\ &= \exp\Big(-\frac{1}{2}\|\mathfrak{e}^*(z)\|^2_{\mathcal{H}_H}\Big),\end{aligned}$$

according to the Parseval identity.

The Wiener integral is constructed as before as the extension of the map

$$\begin{aligned}\delta_H\ :\ \mathcal{W}^* \subset I_{1,2} &\longrightarrow L^2(\mu_H) \\ z &\longmapsto \langle z, B_H\rangle_{\mathcal{W}^*,\mathcal{W}}\,.\end{aligned}$$

By construction of the Wiener measure, the random variable $\langle z, B_H\rangle_{\mathcal{W}^*,\mathcal{W}}$ is Gaussian with mean 0 and variance $\|R_H(z)\|^2_{\mathcal{H}_H}$. For $z = \varepsilon_t$, we have

$$B_H(t) = \langle \varepsilon_t, B_H\rangle_{\mathcal{W}^*,\mathcal{W}} = \delta_H\big(R_H(t, .)\big).$$

Eqn. (4.15) is the exact analog of Eqn. (1.7) hence the Cameron-Martin Theorem can be proved identically:

Theorem 4.9. *For any $h \in \mathcal{H}_H$, for any bounded $F\ :\ \mathcal{W} \to \mathbf{R}$,*

$$\mathbf{E}\left[F(B_H + \mathfrak{e}(h)\right] = \mathbf{E}\left[F(B_H)\exp\left(\delta_H(h) - \frac{1}{2}\|h\|^2_{\mathcal{H}_H}\right)\right]. \tag{4.16}$$

For the Brownian motion, it is often easier to work with elements of $L^2([0,1] \to \mathbf{R};\, \lambda)$ instead of their image by $K_{1/2}$, which belongs to $I_{1,2}$. If we try to mimick this approach for the fractional Brownian motion, we should write:

$$B_H(t) = \delta_H\big(R_H(t,.)\big) = \delta_H\big(K_H(K_H(t,.))\big) = \int_0^1 K_H(t,s)\ \delta B_H(s),$$

which has to be compared to

$$B(t) = W^{1/2}(t) = \int_0^1 \mathbf{1}_{[0,t]}(s)\ \mathrm{d}W^{1/2}(s),$$

where the integral is taken in the Itô sense. Remark that these two equations are coherent since $K_{1/2}(t,.) = \mathbf{1}_{[0,t]}$.

Lemma 4.4. *The process $B = (\delta_H(K_H(\mathbf{1}_{[0,t]})), t \in [0,1])$ is a standard Brownian motion. For $u \in L^2([0,1] \to \mathbf{R};\, \lambda)$,*

$$\int_0^1 u(s)\ dB(s) = \delta_H(K_H u). \tag{4.17}$$

In particular,

$$B_H(t) = \int_0^t K_H(t,s)\ dB(s). \tag{4.18}$$

Proof. It is a Gaussian process by the definition of the Wiener integral. We just have to verify that it has the correct covariance kernel. For, it suffices to see that $\|K_H(\mathbf{1}_{[0,t]})\|^2_{\mathcal{H}_H} = t$. But,

$$\|K_H(\mathbf{1}_{[0,t]})\|^2_{\mathcal{H}_H} = \|\mathbf{1}_{[0,t]}\|^2_{L^2([0,1]\to\mathbf{R};\,\lambda)} = t.$$

This means that (4.17) holds for $u = \mathbf{1}_{[0,t]}$, hence for all piecewise constant functions u and by density, for all $u \in L^2$.

Remark 4.3. Eqn. (4.18) is known as the Karuhnen-Loeve representation. We could have started by considering a process defined by the right-hand-side of (4.18) and called it fractional Brownian motion. Actually, (4.18) is a stronger result: It says that starting from an fBm, one can construct a Brownian motion on the same probability space such that the representation (4.18) holds.

The following theorem is an easy consequence of the properties of the maps K_H.

Theorem 4.10. *The operator $\mathcal{K}_H = K_H \circ K_{1/2}^{-1}$ is continuous and invertible from $I_{\alpha,p}$ into $W_{\alpha+H-1/2,p}$, for any $\alpha > 0$.*

Formally, we have $B_H = K_H(\dot{B}) = K_H \circ K_{1/2}^{-1}(B)$ so we can expect that

Theorem 4.11. *For any H, we have*

$$B_H \overset{dist}{=} \mathcal{K}_H(B) \text{ and } B \overset{dist}{=} \mathcal{K}_H^{-1}(B_H) \tag{4.19}$$

Proof. Let $(\dot{h}_m, m \geq 0)$ be a complete orthonormal basis of $L^2([0,1] \to \mathbf{R}; \lambda)$. The series, which defines B,

$$B = \sum_{m=0}^{\infty} X_m I^1(\dot{h}_m),$$

converges with $\mu_{1/2}$-probability 1, in any $W_{\alpha,p}$, provided that $0 < \alpha - 1/p < 1/2$. By continuity of $\mathcal{K}_H$,

$$\mathcal{K}_H \left(\sum_{m=0}^{\infty} X_m I^1(\dot{h}_m) \right) = \sum_{m=0}^{\infty} X_m K_H(\dot{h}_m) \stackrel{\text{dist}}{=} B_H$$

converges on the same set of full measure in $I_{\alpha+H-1/2,p}$. Note that when $\alpha - 1/p$ runs through $(0, 1/2)$, $\alpha + H - 1/2 - 1/p$ varies along $(0, H)$. Hence, we retrieve the desired regularity of the sample-paths of B_H.

The same proof holds for the second identity.

Since the operator involved in the previous relation are all lower triangular,

4.4 Gradient and divergence

The gradient is defined as for the usual Brownian motion. The only modification is the Cameron-Martin space.

Definition 4.3. A function F is said to be cylindrical if there exists an integer n, $f \in \text{Schwartz}(\mathbf{R}^n)$, the Schwartz space on $\mathbf{R}^n$, $(h_1, \cdots, h_n) \in {\mathcal{H}_H}^n$ such that

$$F(\omega) = f(\delta_H h_1, \cdots, \delta_H h_n).$$

The set of such functionals is denoted by $\mathcal{S}_{\mathcal{H}_H}$.

Definition 4.4. Let $F \in \mathcal{S}$, $h \in \mathcal{H}_H$, with $F(\omega) = f(\delta_H h_1, \cdots, \delta_H h_n)$. Set

$$\nabla F = \sum_{j=1}^{n} \partial_j F(\delta_H h_1, \cdots, \delta_H h_n) \; h_j,$$

so that

$$\langle \nabla F, h \rangle_{\mathcal{H}_H} = \sum_{j=1}^{n} \partial_j F(\delta_H h_1, \cdots, \delta_H h_n) \; \langle h_j, h \rangle_{\mathcal{H}_H}.$$

Example 4.2. This means that

$$\nabla f(B_H(t)) = f'(B_H(t)) R_H(t, .)$$

and if we denote $\dot{\nabla} = K_H^{-1} \nabla$ (which corresponds for $H = 1/2$ to take the time derivative of the gradient), we get

$$\dot{\nabla}_s f(B_H(t)) = f'(B_H(t))K_H(t,s).$$

We can now improve Theorem 4.11.

Theorem 4.12. *Let*

$$B_H(t) = \delta_H\big(R_H(t,.)\big) \text{ and } B(t) = \delta_H\big(K_H(\mathbf{1}_{[0,t]})\big).$$

For any H, we have

$$\mu_H\Big(B = \mathcal{K}_H^{-1}(B_H)\Big) = 1. \tag{4.20}$$

Proof. To prove such an identity, it is necessary and sufficient to check that

$$\mathbf{E}\left[\psi \int_0^1 B(t)g(t)\ \mathrm{d}t\right] = \mathbf{E}\left[\psi \int_0^1 \mathcal{K}_H^{-1}(B_H)g(t)\ \mathrm{d}t\right] \tag{4.21}$$

for any $g \in L^2([0,1] \to \mathbf{R};\ \lambda)$ and any $\psi \in \mathcal{S}_H$. Indeed, $L^2([0,1] \to \mathbf{R};\ \lambda)\otimes\mathcal{S}_H$ is a dense subset of $L^2([0,1] \to \mathbf{R};\ \lambda) \otimes L^2(\mathcal{W} \to \mathbf{R};\ \mu_H) \simeq L^2([0,1] \otimes \mathcal{W} \to \mathbf{R};\ \lambda \otimes \mu_H)$ and (4.21) entails that $B = \mathcal{K}_H^{-1}(B_H)$ $\lambda\otimes\mu_H$-almost-surely. This means that for there exists $A \subset [0,1] \times W$ such that

$$\int_{[0,1]\times W} \mathbf{1}_A(s,\omega)\ \mathrm{d}s\ \mathrm{d}\mu_H(\omega) = 0,$$

and

$$B(\omega,s) = \mathcal{K}_H^{-1}(B_H)(\omega,s) \text{ for } (s,\omega) \notin A.$$

Hence, for any $s \in [0,1]$, the section of A at s fixed, i.e. $A_s = \{\omega,\ (s,\omega) \in A\}$, is a μ_H-negligeable set.

The sample-paths of B are known to be continuous and that of B_H belong to $W_{H-\varepsilon,p}$ for any $p \geq 1$ and ε sufficently small. Hence, according to Theorem 4.10, $\mathcal{K}_H^{-1}(B_H)$ almost-surely belongs to $I_{1/2-\varepsilon,p}$ for any $p \geq 1$. Choose $p > 2$ so that $I_{1/2-\varepsilon,p} \subset \mathcal{C}_0([0,1],\mathbf{R})$ to conclude that $\mathcal{K}_H^{-1}(B_H)$ has μ_H-a.s. continuous sample-paths. Consider

$$A_{\mathbf{Q}} = \bigcup_{t\in[0,1]\cap\mathbf{Q}} A_t.$$

It is a μ_H-negligeable set and for $\omega \in A_{\mathbf{Q}}^c$, for $t \in [0,1] \cap \mathbf{Q}$, $B(\omega,s) = \mathcal{K}_H^{-1}(B_H)(\omega,s)$. Thus, by continuity, this identity still holds for any $t \in [0,1]$ and any $\omega \in A_{\mathbf{Q}}^c$. This means that Eqn. (4.20) holds.

We now prove (4.21),

$$\begin{aligned}
\mathbf{E}\left[\psi \int_0^1 \mathcal{K}_H^{-1}(B_H)g(t)\,\mathrm{d}t\right] &= \int_0^1 \mathbf{E}\left[\psi\, B_H(t)\right](\mathcal{K}_H^{-1})^*(g)(t)\,\mathrm{d}t\\
&= \int_0^1 \mathbf{E}\left[\psi\,\delta_H(R_H(t,.))\right](\mathcal{K}_H^{-1})^*(g)(t)\,\mathrm{d}t\\
&= \mathbf{E}\left[\int_0^1 (\mathcal{K}_H^{-1})^*(g)(t)\int_0^1 \dot{\nabla}_s\psi\, K_H(t,s)\,\mathrm{d}s\,\mathrm{d}t\right]\\
&= \mathbf{E}\left[\int_0^1 \dot{\nabla}_s\psi \int_0^1 K_H(t,s)(\mathcal{K}_H^{-1})^*(g)(t)\,\mathrm{d}t\,\mathrm{d}s\right]\\
&= \mathbf{E}\left[\int_0^1 \dot{\nabla}_s\psi K_H^*(\mathcal{K}_H^{-1})^*(g))(s)\,\mathrm{d}s\right]
\end{aligned}$$

By the very definition of $\mathcal{K}_H$,

$$K_H^* \circ (\mathcal{K}_H^{-1})^* = K_H^* \circ (K_H^{-1})^* \circ K_{1/2}^* = K_{1/2}^*.$$

Thus, we have

$$\begin{aligned}
\mathbf{E}\left[\psi \int_0^1 \mathcal{K}_H^{-1}(B_H)g(t)\,\mathrm{d}t\right] &= \mathbf{E}\left[\int_0^1 \dot{\nabla}_s\psi\; K_{1/2}^* g(s)\,\mathrm{d}s\right]\\
&= \mathbf{E}\left[\int_0^1 \dot{\nabla}_s\psi \int_s^1 g(t)\,\mathrm{d}t\,\mathrm{d}s\right]\\
&= \mathbf{E}\left[\int_0^1 \int_0^1 \dot{\nabla}_s\psi\; g(t)\,\mathbf{1}_{[s,1]}(t)\,\mathrm{d}t\,\mathrm{d}s\right]\\
&= \mathbf{E}\left[\int_0^1 \int_0^1 \dot{\nabla}_s\psi\; g(t)\,\mathbf{1}_{[0,t]}(s)\,\mathrm{d}t\,\mathrm{d}s\right]\\
&= \mathbf{E}\left[\int_0^1 g(t) \int_0^1 \dot{\nabla}_s\psi\,\mathbf{1}_{[0,t]}(s)\,\mathrm{d}s\,\mathrm{d}t\right].
\end{aligned}$$

On the other hand, $B(t) = \delta_H\big(K_H(\mathbf{1}_{[0,t]})\big)$ hence,

$$\begin{aligned}
\mathbf{E}\left[\psi \int_0^1 B(t)g(t)\,\mathrm{d}t\right] &= \mathbf{E}\left[\psi \int_0^1 \delta_H\big(K_H(\mathbf{1}_{[0,t]})\big)\, g(t)\,\mathrm{d}t\right]\\
&= \mathbf{E}\left[\int_0^1 g(t) \int_0^1 \dot{\nabla}_s\psi\; \mathbf{1}_{[0,t]}(s)\,\mathrm{d}s\,\mathrm{d}t\right].
\end{aligned}$$

Then, (4.21) follows.

We can even go further and show that B and B_H generate the same filtration.

Definition 4.5. Recall that $(\dot{\pi}_t, t \in [0,1])$ are the projections defined by

$$\begin{aligned}\dot{\pi}_t \, : \, L^2([0,1] \to \mathbf{R};\, \lambda) &\longrightarrow L^2([0,1] \to \mathbf{R};\, \lambda)\\ f &\longmapsto f\mathbf{1}_{[0,t)}.\end{aligned}$$

Let V be a closable map from $\operatorname{Dom} V \subset L^2([0,1] \to \mathbf{R};\, \lambda)$ into $L^2([0,1] \to \mathbf{R};\, \lambda)$.

Then, V is $\dot{\pi}$-causal if $\operatorname{Dom} V$ is $\dot{\pi}$-stable, i.e. $\dot{\pi}_t \operatorname{Dom} V \subset \operatorname{Dom} V$ for any $t \in [0,1]$ and if for any $t \in [0,1]$,

$$\dot{\pi}_t V \dot{\pi}_t = \dot{\pi}_t V.$$

Consider also π_t^H defined by

$$\begin{aligned}\pi_t^H \, : \, \mathcal{H}_H &\longrightarrow \mathcal{H}_H\\ h &\longmapsto K_H\big(\pi_t K_H^{-1}(h)\big) = K_H\big(\dot{h}\,\mathbf{1}_{[0,t]}\big).\end{aligned}$$

Remark 4.4. An integral operator, i.e.

$$Vf(t) = \int_0^1 V(t,s) f(s) \, \mathrm{d}s$$

is $\dot{\pi}$-causal if and only if $V(t,s) = 0$ for $s > t$. For V_1, V_2 two causal operators, their composition $V_1 V_2$ is still causal:

$$\begin{aligned}\pi_t V_1 V_2 \pi_t = (\pi_t V_1 \pi_t) V_2 \pi_t &= \pi_t V_1 (\pi_t V_2 \pi_t)\\ &= \pi_t V_1 (\pi_t V_2) = (\pi_t V_1 \pi_t) V_2 = \pi_t V_1 V_2.\end{aligned}$$

Corollary 4.1. *The filtrations generated by B_H and B do coincide.*

Proof. From the representation

$$B_H(t) = \int_0^t K_H(t,s) \, \mathrm{d}B(s),$$

we deduce that

$$\sigma\left\{B_H(s),\ s \le t\right\} \subset \sigma\left\{B(s),\ s \le t\right\}.$$

We have $\mathcal{K}_H^{-1} = K_{1/2} K_H^{-1}$. From Theorem 4.5, K_H^{-1} appears as the composition of fractional derivatives and multiplication operators:

$$f \mapsto x^\alpha f.$$

Time derivatives of any order (as in Definition 4.10) are local operators and as such are causal. It is straightforward that multiplication operators are also causal. Thus, $\mathcal{K}_H^{-1}$ appears as the composition of causal operators hence it is causal. This means that

$$B(t) = \int_0^t V(t,s) B_H(s)\, \mathrm{d}s$$

for some lower trianguler kernel V. Hence,

$$\sigma\left\{B_H(s),\ s \le t\right\} \supset \sigma\left\{B(s),\ s \le t\right\},$$

and the equality of filtrations is proved.

We can now reap the fruits of our not so usual presentation of the Malliavin calculus for the Brownian motion, in which we cautiously sidestepped chaos decomposition. The Theorem 4.9 entails the integration by parts formula, pending of (2.3): For any F and G in $\mathcal{S}_H$, for any $h \in \mathcal{H}_H$,

$$\mathbf{E}\left[G\left\langle \nabla F, h\right\rangle_{\mathcal{H}_H}\right] = -\mathbf{E}\left[F\left\langle \nabla G, h\right\rangle_{\mathcal{H}_H}\right] + \mathbf{E}\left[FG\,\delta_H h\right]. \tag{4.22}$$

Definition 4.4 is formally the very same as Definition 2.1 so that the definition of the Sobolev spaces are identical.

Definition 4.6. The space $\mathbb{D}^H_{p,1}$ is the closure of $\mathcal{S}_H$ for the norm

$$\|F\|_{p,1,H} = \mathbf{E}\left[|F|^p\right]^{1/p} + \mathbf{E}\left[\|\nabla F\|^p_{\mathcal{H}_H}\right]^{1/p}.$$

The iterated gradient are defined likewise and so do the Sobolev of higher order, $\mathbb{D}_{p,k,H}$. We sill clearly have

$$\begin{aligned}\nabla(FG) &= F\nabla G + G\nabla F\\ \nabla\phi(F) &= \phi'(F)\nabla F\end{aligned}$$

for $F \in \mathbb{D}_{p,1,H}$, $G \in \mathbb{D}_{q,1,H}$ and ϕ Lipschitz continuous. As long as we do not use the temporal scale, there is no difference between the identities established for the usual Brownian motion and that relative to the fractional Brownian motion.

Theorem 4.13. *For any F in $L^2(\mathcal{W} \to \mathbf{R};\, \mu_H)$,*

$$\Gamma(\pi^H_t)F = \mathbf{E}\left[F \,|\, \mathcal{F}^H_t\right],$$

in particular,

$$\begin{aligned}\mathbf{E}\left[W_t \,|\, \mathcal{F}^H_r\right] &= \int_0^t K_H(t,s)\mathbf{1}_{[0,r]}(s)\,\delta B(s), \text{ and}\\ \mathbf{E}\left[\exp(\delta_H u - 1/2\|u\|^2_{\mathcal{H}_H}) \,|\, \mathcal{F}^H_t\right] &= \exp(\delta_H \pi^H_t u - 1/2\|\pi^H_t u\|^2_{\mathcal{H}_H}),\end{aligned}$$

for any $u \in \mathcal{H}_H$.

Proof. Let $\{h_n,\, n \ge 0\}$ be a denumerable family of elements of $\mathcal{H}_H$ and let $V_n = \sigma\{\delta_H h_k,\, 1 \le k \le n\}$. Denote by p_n the orthogonal projection on span$\{h_1, \ldots, h_n\}$. For any f bounded, for any $u \in \mathcal{H}_H$, by the Cameron–Martin theorem we have

$$\begin{aligned}\mathbf{E}\left[\Lambda_1^u f(\delta_H h_1, \ldots, \delta_H h_n)\right] \\ &= \mathbf{E}\left[f(\delta_H h_1(w+u), \ldots, \delta_H h_n(w+u))\right] \\ &= \mathbf{E}\left[f(\delta_H h_1 + (h_1, u)_{\mathcal{H}_H}, \ldots, \delta_H h_n + (h_n, u)_{\mathcal{H}_H})\right] \\ &= \mathbf{E}\left[f(\delta_H h_1(w+p_n u), \ldots, \delta_H h_n(w+p_n u))\right] \\ &= \mathbf{E}\left[\Lambda_1^{p_n u} f(\delta_H h_1, \ldots, \delta_H h_n)\right],\end{aligned}$$

hence

$$\mathbf{E}\left[\Lambda_1^u \,|\, V_n\right] = \Lambda_1^{p_n u}. \tag{4.23}$$

Choose h_n of the form $\pi_t^H(e_n)$ where $\{e_n,\, n \ge 0\}$ is an orthonormal basis of $\mathcal{H}_H$, i.e., $\{h_n,\, n \ge 0\}$ is an orthonormal basis of $\pi_t^H(\mathcal{H}_H)$. By the previous theorem, $\bigvee_n V_n = \mathcal{F}_t^H$ and it is clear that p_n tends pointwise to π_t^H, hence from (4.23) and martingale convergence theorem, we can conclude that

$$\mathbf{E}\left[\Lambda_1^u \,|\, \mathcal{F}_t^H\right] = \Lambda_1^{\pi_t^H u} = \Lambda_t^u.$$

Moreover, for $u \in \mathcal{H}_H$,

$$\Gamma(\pi_t^H)(\Lambda_1^u) = \Lambda_1^{\pi_t^H u},$$

hence by density of linear combinations of Wick exponentials, for any $F \in L^2(\mu_H)$,

$$\Gamma(\pi_t^H)F = \mathbf{E}\left[F \,|\, \mathcal{F}_t^H\right],$$

and the proof is completed.

Definition 4.7. For the sake of notations, we set, for $\dot{u}$ such that $K_H\dot{u}$ belongs to $\operatorname{Dom}_p \delta_H$ for some $p > 1$,

$$\int_0^1 \dot{u}(s)\delta B(s) = \delta_H(K_H\dot{u}) \text{ and } \int_0^t \dot{u}(s)\delta B(s) = \delta_H(\pi_t^H K_H\dot{u}). \tag{4.24}$$

Note that, for any $\psi \in \mathbb{D}_{p/(p-1),1}$

$$\mathbf{E}\left[\psi \int_0^1 \dot{u}(s)\delta B(s)\right] = \mathbf{E}\left[\int_0^1 \dot{\nabla}_s\psi\ \dot{u}(s)\ \mathrm{d}s\right].$$

The next result is the Clark formula. It reads formally as (3.11) but we should take care that the $\dot{\nabla}$ does not represent the same object. Here it is defined as $\dot{\nabla} = K_H^{-1}\nabla$.

Corollary 4.2. *For any $F \in L^2(\mathcal{W} \to \mathbf{R};\, \mu_H)$,*

$$F = \mathbf{E}\left[F\right] + \int_0^1 \mathbf{E}\left[\dot{\nabla}_s F \,|\, \mathcal{F}_s\right]\ \delta B(s).$$

Proof. With the notations at hand, Theorem 4.13 implies that

$$\begin{aligned}\mathbf{E}\left[\Lambda_1^h \,|\, \mathcal{F}_t\right] &= \exp\left(\delta_H(\pi_t^H h) - \frac{1}{2}\|\pi_t^H h\|^2_{\mathcal{H}_H}\right)\\ &= \exp\left(\int_0^t \dot{h}(s)\ \delta B(s) - \frac{1}{2}\int_0^t \dot{h}^2(s)\ \mathrm{d}s\right).\end{aligned}$$

This means that we have the usual relation

$$\Lambda_t^h = 1 + \int_0^t \Lambda_s \dot{h}(s)\ \delta B(s) = \mathbf{E}\left[\Lambda_1^h\right] + \int_0^1 \mathbf{E}\left[\dot{\nabla}_s \Lambda_1^h \,|\, \mathcal{F}_s\right]\ \delta B(s).$$

By density of the Doléans exponentials, we obtain the result.

Should we want to obfuscate everything, we could write

$$F = \mathbf{E}\left[F\right] + \delta_H\left(K_H\big(\mathbf{E}\left[(K_H^{-1}\nabla)_. F \,|\, \mathcal{F}_.\right]\big)\right).$$

4.5 Itô formula

Definition 4.8. Consider the operator $\mathcal{K}$ defined by $\mathcal{K} = I_{0^+}^{-1} \circ K_H$.

For $H > 1/2$, it is a continuous map from $L^p([0,1] \to \mathbf{R};\, \lambda)$ into $I_{H-1/2,p}$, for any $p \geq 1$. Let $\mathcal{K}_t^*$ be its adjoint in $L^p([0,t] \to \mathbf{R};\, \lambda)$, i.e. for any $f \in L^p([0,1] \to \mathbf{R};\, \lambda)$, any g sufficently regular,

$$\int_0^t \mathcal{K}f(s)\, g(s)\ \mathrm{d}s = \int_0^t f(s)\, \mathcal{K}_t^* g(s)\ \mathrm{d}s.$$

The map $\mathcal{K}_t^*$ is continuous from $(I_{H-1/2,p})^*$ into $L^q([0,t] \to \mathbf{R};\, \lambda)$, where $q = p/(p-1)$.

Theorem 4.14. *Assume $H > 1/2$. For $f \in \mathcal{C}_b^2$,*

$$f(B_H(t)) = f(0) + \int_0^t \mathcal{K}_t^*\big(f' \circ B_H\big)(s)\ \delta B(s) + H\, V_H \int_0^1 f''\big(B_H(s)\big) s^{2H-1}\ ds.$$

Proof. Introduce the function g as

$$g(x) = f(\frac{a+b}{2} + x) - f(\frac{a+b}{2} - x).$$

This function is even, satisfies

$$g^{(2j+1)}(0) = 2f^{(2j+1)}((a+b)/2) \text{ and } g(\frac{b-a}{2}) = f(b) - f(a).$$

Apply the Taylor formula to g between the points 0 and $(b-a)/2$ to get

$$f(b) - f(a) = \sum_{j=0}^{n} \frac{2^{-2j}}{(2j+1)!}(b-a)^{2j+1} f^{(2j+1)}(\frac{a+b}{2})$$
$$+ \frac{(b-a)^{2(n+1)}}{2} \int_0^1 \lambda^{2n+1} g^{(2(n+1))}(\lambda a + (1-\lambda)b) \mathrm{d}\lambda.$$

For any $\psi \in \mathcal{E}$ of the form $\psi = \exp(\delta_H h - \frac{1}{2}\|h\|^2_{\mathcal{H}_H})$ with $h \in \mathcal{C}_b^1 \subset \mathcal{H}_H$. Note that ψ satisfies $\nabla\psi = \psi\, h \in L^2(\mathcal{W} \to \mathbf{R};\, \mu_H)$. Since $\mathcal{C}_b^1$ is dense into $\mathcal{H}_H$, these functionals are dense in $L^2(\mathcal{W})$. We thus have

$$\begin{aligned}
&\mathbf{E}\left[\big(f\big(B_H(t+\varepsilon)\big) - f\big(B_H(t)\big)\big)\,\psi\right] \\
&= \mathbf{E}\left[\big(B_H(t+\varepsilon) - B_H(t)\big)\ f'\left(\frac{B_H(t) + B_H(t+\varepsilon)}{2}\right)\ \psi\right] \\
&+\frac{1}{2}\mathbf{E}\left[\big(B_H(t+\varepsilon) - B_H(t)\big)^2 \int_0^1 r\, g^{(2)}(rB_H(t) + (1-r)B_H(t+\varepsilon))\mathrm{d}r\ \psi\right] \\
&= A_0 + \frac{1}{2}A_1. \quad (4.25)
\end{aligned}$$

For A_0, we have

$$\begin{aligned}
A_0 &= \mathbf{E}\left[\big(B_H(t+\varepsilon) - B_H(t)\big)\ f'\left(\frac{B_H(t) + B_H(t+\varepsilon)}{2}\right)\ \psi\right] \\
&= \mathbf{E}\left[\int_0^1 \big(K_H(t+\varepsilon, s) - K_H(t,s)\big)\delta B(s)\ f'\left(\frac{B_H(t) + B_H(t+\varepsilon)}{2}\right)\ \psi\right] \\
&= \mathbf{E}\left[\int_0^1 \big(K_H(t+\varepsilon, s) - K_H(t,s)\big)\ \dot{\nabla}_s \left(f'\left(\frac{B_H(t) + B_H(t+\varepsilon)}{2}\right)\ \psi\right)\ \mathrm{d}s\right].
\end{aligned}$$

Since $\dot{\nabla}$ is a true derivation operator

$$\begin{aligned}
\dot{\nabla}_s \left(f'\left(\frac{B_H(t) + B_H(t+\varepsilon)}{2}\right)\ \psi\right) &= f'\left(\frac{B_H(t) + B_H(t+\varepsilon)}{2}\right)\dot{\nabla}_s\psi \\
&+ f''\left(\frac{B_H(t) + B_H(t+\varepsilon)}{2}\right)\big(K_H(t+\varepsilon, s) + K_H(t,s)\big).
\end{aligned}$$

Thus,

$$A_0 = \mathbf{E}\left[f'\left(\frac{B_H(t)+B_H(t+\varepsilon)}{2}\right)\int_0^1 \big(K_H(t+\varepsilon,s)-K_H(t,s)\big)\dot{\nabla}_s\psi \; \mathrm{d}s\right]$$
$$+\,\mathbf{E}\left[\psi\, f''\left(\frac{B_H(t)+B_H(t+\varepsilon)}{2}\right)\right.$$
$$\left.\times\int_0^1 \big(K_H(t+\varepsilon,s)-K_H(t,s)\big)\big(K_H(t+\varepsilon,s)+K_H(t,s)\big)\;\mathrm{d}s\right]$$
$$= B_1 + B_2.$$

By the very definition of $\dot{\nabla}$,

$$\frac{1}{\varepsilon}\int_0^1 \big(K_H(t+\varepsilon,s)-K_H(t,s)\big)\dot{\nabla}_s\psi\;\mathrm{d}s = \frac{1}{\varepsilon}\big(\nabla\psi(t+\varepsilon)-\nabla\psi(t)\big)$$
$$\xrightarrow{\varepsilon\to 0}\frac{d}{dt}\nabla\psi(t) = I_{0^+}^{-1}\circ K_H(\dot{\nabla}\psi)(t) = \mathcal{K}(\dot{\nabla}\psi)(t).$$

Moreover, since $\nabla\psi$ belongs to $L^2(W;I_{H+1/2,2})$,

$$\mathbf{E}\left[|\nabla\psi(t+\varepsilon)-\nabla\psi(t)|^2\right] \le c\,\|\mathcal{K}\dot{\nabla}\psi\|_{L^2(W;I_{H-1/2,2})}\;|\varepsilon|.$$

Hence,

$$\varepsilon^{-1}\,B_1 \xrightarrow{\varepsilon\to 0}\mathbf{E}\left[f'(B_H(t))\,\mathcal{K}\dot{\nabla}\psi(t)\right].$$

Simple calculations give that

$$B_2 = \mathbf{E}\left[\psi\, f''\left(\frac{B_H(t)+B_H(t+\varepsilon)}{2}\right)\;\Big(R_H(t+\varepsilon,t+\varepsilon)-R_H(t,t)\Big)\right]$$

and that

$$\varepsilon^{-1}\Big(R_H(t+\varepsilon,t+\varepsilon)-R_H(t,t)\Big) = V_H\,\frac{(t+\varepsilon)^{2H}-t^{2H}}{\varepsilon}\xrightarrow{\varepsilon\to 0} 2H\,V_H\,t^{2H-1}.$$

The dominated convergence theorem then yields

$$\varepsilon^{-1}B_2 \xrightarrow{\varepsilon\longrightarrow 0} H\,V_H\mathbf{E}\left[\psi f''(B_H(t))\,t^{2H-1}\right].$$

If $H>1/2$, $\varepsilon^{-1}A_1$ does vanish. Actually, recall that $B_H(t+\varepsilon)-B_H(t)$ is a centered Gaussian random variable of variance proportional to ε^{2H}, hence

$$\varepsilon^{-1}|A_1| \le c\mathbf{E}\left[|B_H(t+\varepsilon)-B_H(t)|^2\right]\|f^{(2)}\|_{L^\infty} \le c\,\varepsilon^{2H-1}\,\|f^{(2)}\|_{L^\infty}\xrightarrow{\varepsilon\to 0} 0,$$

since $2H-1>0$.

We have proved so far that

$$\frac{d}{dt}\mathbf{E}\left[\psi\, f'\big(B_H(t)\big)\right] = \mathbf{E}\left[f\big(B_H(t)\big)\,\nabla\psi(t)\right] + H\,V_H\mathbf{E}\left[\psi\;f''\big(B_H(t)\big)\,t^{2H-1}\right]. \tag{4.26}$$

It is straightforward that the right-hand-side of (4.26) is continuous as a function of t on any interval $[0,T]$. Hence we can integrate the previous relation and we get

$$\begin{aligned}\mathbf{E}\left[\psi\, f\big(B_H(t)\big)\right] - \mathbf{E}\left[\psi\, f\big(B_H(0)\big)\right] &= \mathbf{E}\left[\int_0^t f'\big(B_H(s)\big)\,\mathcal{K}\dot{\nabla}\psi(s)\;\mathrm{d}s\right]\\ &+ H\,V_H\;\mathbf{E}\left[\psi\int_0^t\; f''\big(B_H(s)\big)\,s^{2H-1}\;\mathrm{d}s\right].\end{aligned}$$

Remark now that

$$\begin{aligned}\mathbf{E}\left[\int_0^t f'\big(B_H(s)\big)\,\mathcal{K}\dot{\nabla}\psi(s)\;\mathrm{d}s\right] &= \mathbf{E}\left[\int_0^1 f'\big(B_H(s)\big)\mathbf{1}_{[0,t]}(s)\,\mathcal{K}\dot{\nabla}\psi(s)\;\mathrm{d}s\right]\\ = \mathbf{E}\left[\int_0^1 \mathcal{K}_1^*\big(f'\circ B_H\,\mathbf{1}_{[0,t]}\big)\,\dot{\nabla}_s\psi\;\mathrm{d}s\right] &= \mathbf{E}\left[\psi\int_0^1 \mathcal{K}_1^*\big(f'\circ B_H\,\mathbf{1}_{[0,t]}\big)(s)\;\delta B(s)\right].\end{aligned}$$

Note that

$$\mathcal{K}_1^*(f'\mathbf{1}_{[0,t]})(s) = \frac{d}{ds}\int_s^1 K(r,s)f'(r)\mathbf{1}_{[0,t]}(r)\;\mathrm{d}r = 0 \text{ if } s>t.$$

This means that

$$\pi_t^H\left(\mathcal{K}_t^*(f'\mathbf{1}_{[0,t]})\right) = \mathcal{K}_t^*(f'\mathbf{1}_{[0,t]})$$

and by the definition (4.24),

$$\int_0^1 \mathcal{K}_t^*\big(f\circ B_H\,\mathbf{1}_{[0,t]}\big)(s)\;\delta B(s) = \int_0^t \mathcal{K}_t^*\big(f\circ B_H\big)(s)\;\delta B(s).$$

Consequently, we have

$$\begin{aligned}\mathbf{E}\left[\psi\, f\big(B_H(t)\big)\right] - \mathbf{E}\left[\psi\, f\big(B_H(0)\big)\right] &= \mathbf{E}\left[\psi\int_0^t \mathcal{K}_t^*(f\circ B_H)(s)\delta B(s)\right]\\ &+ H\,V_H\;\mathbf{E}\left[\psi\int_0^t\; f''\big(B_H(s)\big)\,s^{2H-1}\;\mathrm{d}s\right].\end{aligned}$$

Since the functionals ψ we considered form a dense subset in L^2, we have

$$\begin{aligned}f\big(B_H(t)\big) - f\big(B_H(0)\big) &= \int_0^t \mathcal{K}_t^*(f\circ B_H)(s)\delta B(s)\\ &+ H\,V_H\int_0^t\; f''\big(B_H(s)\big)\,s^{2H-1}\;\mathrm{d}s,\quad \mathrm{d}t\otimes\mu_H\text{-a.s.}\end{aligned} \tag{4.27}$$

Admit for a while that

$$t \longrightarrow \int_0^t \mathcal{K}_t^*(f' \circ B_H)(s)\delta B(s)$$

has almost-surely continuous sample-paths. It is clear that the other terms of (4.27) have also continuous trajectories. Let A be the negligeable set of $W \times [0,1]$ where (4.27) does not hold. According to the Fubini theorem, for any $t \in [0,1]$, the set

$$A_t = \{\omega \in W, \, (\omega, t) \in\} \in A\}$$

is negligeable and so does $A_{\mathbf{Q}} = \cup_{t \in [0,1] \cap \mathbf{Q}} A_t$. For any $t \in \mathbf{Q} \cap [0,1]$, Eqn. (4.27) holds on $A_{\mathbf{Q}}^c$, i.e. holds μ_H-almost surely. By continuity, this is still true for any $t \in [0,1]$.

Theorem 4.15. *For any $H \in [1/2,1)$. Let u belong to $\mathbb{D}_{p,1}(L^p)$ with $Hp > 1$. The process*

$$U(t) = \int_0^t \mathcal{K}_t^* u(s)\delta B(s), \; t \in [0,1]$$

admits a modification with $(H - 1/p)$-Hölder continuous paths and we have the maximal inequality :

$$\mathbf{E}\left[\sup_{r \neq t \in [0,1]^2} \frac{\left| \int_0^1 \left(\mathcal{K}_t^* u(s) - \mathcal{K}_r^* u(s) \right) \delta B(s) \right|^p}{|t-r|^{pH}} \right]^{1/p} \le c \|\mathcal{K}_1^*\|_{H,2} \|u\|_{\mathbb{D}_{p,1}}.$$

Proof. For $g \in \mathcal{C}^\infty$ and ψ a cylindric real-valued functional,

$$\mathbf{E}\left[\int_0^1 \int_0^t \mathcal{K}_t^* u(s)\delta B(s) \; g(t) \, \mathrm{d}t \, \psi \right] = \mathbf{E}\left[\iint_{[0,1]^2} \mathcal{K}_1^*(u \mathbf{1}_{[0,t]})(r) g(t) \dot{\nabla}_r \psi \, \mathrm{d}t \, \mathrm{d}r \right]$$
$$= \mathbf{E}\left[\int_0^1 \mathcal{K}_1^*(u I^1_{1-} g)(r) \dot{\nabla}_r \psi \, \mathrm{d}r \right] = \mathbf{E}\left[\delta(\mathcal{K}_1^*(u.I^1_{1-} g) \psi \right].$$

Thus,

$$\int_0^1 \int_0^t \mathcal{K}_t^* u(s) \, \delta B(s) \; g(t) \, \mathrm{d}t = \int_0^1 \mathcal{K}_1^*(u.I^1_{1-} g)(s) \, \delta B(s) \; \mu_H - \text{a.s.} \quad (4.28)$$

Since $H > 1/2$, it is clear that $\mathcal{K}$ is continuous from $L^2([0,1] \to \mathbf{R}; \, \lambda)$ into $I_{H-1/2,2}$ thus that $\mathcal{K}_1^*$ is continuous from $I^*_{H-1/2,2}$ in $L^2([0,1] \to \mathbf{R}; \, \lambda)$. Since $I_{H-1/2,2}$ is continuously embedded in $L^{(1-H)^{-1}}([0,1] \to \mathbf{R}; \, \lambda)$, it follows that $L^{1/H}([0,1] \to \mathbf{R}; \, \lambda) = (L^{(1-H)^{-1}}([0,1] \to \mathbf{R}; \, \lambda))^*$ is continuously embedded in $I_{1/2-H,2}$. Since u belongs to $\mathbb{D}_{p,1}(L^p)$, the generalized Hölder inequality implies that

$$\|uI^1_{1-}g\|_{L^{1/H}} \le \|u\|_{L^p}\|I^1_{1-}g\|_{L^{(H-1/p)^{-1}}}.$$

It follows that U belongs to $L^p(\mathcal{W} \to I^+_{1,(1-H+1/p)^{-1}};\, \mu_H)$ with

$$\|U\|_{L^p(\mathcal{W}\to I^+_{1,(1-H+1/p)^{-1}};\,\mu_H)} \le c\|\mathcal{K}^*_1\|_{H,2}\|u\|_{\mathbb{D}_{p,1}}.$$

The proof is completed remarking that $1 - 1/(1-H+1/p)^{-1} = H - 1/p$ so that $I^+_{1,(1-H+1/p)^{-1}}$ is embedded in $\mathrm{Hol}(H-1/p)$.

4.6 Exercises

Exercise 4.1. Let V be a causal operator from $L^2([0,1] \to \mathbf{R};\, \lambda)$ into itself. Let

$$\begin{aligned} V_t = \dot{\pi}_t \circ V \circ \dot{\pi}_t \, : \, L^2([0,1] \to \mathbf{R};\, \lambda) &\longrightarrow L^2([0,t] \to \mathbf{R};\, \lambda) \\ f &\longmapsto V(f\mathbf{1}_{[0,t]})\mathbf{1}_{[0,t]}. \end{aligned}$$

Let V_t^* be the adjoint of V_t.

1. Show that V_t^* is continuous from $L^2([0,t] \to \mathbf{R};\, \lambda)$ into $L^2([0,1] \to \mathbf{R};\, \lambda)$. (We here identify $L^2([0,1] \to \mathbf{R};\, \lambda)$ with its dual)

Consider the situation where

$$Vf(r) = \int_0^t V(r,s)f(s)\, \mathrm{d}s$$

with $V(r,s) = 0$ whenever $s > r$. *Note that this is the case of $\mathcal{K}_H$ for $H > 1/2$.*

2. Show that
$$V_t^* f = V_1^*(\dot{\pi}_t f).$$
3. Derive the same identity using solely the causality of V.

$V = \mathcal{K}_H$ for $H < 1/2$ corresponds to this last situation.

Exercise 4.2. One approach to define a stochastic integral with respect to B_H for $H > 1/2$ is to look at Riemann like sums:

$$RS_n(U) = \sum_{i=0}^{n-1} U(i/n)\left(B_H\left(\frac{i+1}{n}\right) - B_H\left(\frac{i}{n}\right)\right)$$

Consider that $U(s) = \delta_H h\; u(s)$ where u is deterministic and continuous on $[0,1]$ and h is $\mathcal{C}^1$, hence belongs to $\mathcal{H}_H$.

1. Show that
$$\dot{\nabla}_r U(s) = u(s)\dot{h}(r).$$
where $\dot{h} = K_H^{-1}(h)$.
2. Derive
$$\big(K_{1/2}^{-1} \circ K_H \circ \dot{\nabla}\big)_r \dot{U}(s) = u(s)h'(r).$$
3. Show that
$$RS_n(U) = \int_0^1 \sum_{i=0}^{n-1} U(\frac{i}{n}) \, \Big(K_H(\frac{i+1}{n}, r) - K_H(\frac{i}{n}, r)\Big) \, \delta B(r) \\ + \sum_{i=0}^{n-1} u(\frac{i}{n}) \left(h(\frac{i+1}{n}) - h(\frac{i}{n})\right).$$
4. Assume for the next two questions only that K_H is a regular as it needs to be. Show that
$$\sum_{i=0}^{n-1} U(\frac{i}{n}) \, \Big(K_H(\frac{i+1}{n}, r) - K_H(\frac{i}{n}, r)\Big) \xrightarrow{n\to\infty} \int_0^1 U(s)\frac{d}{ds}K_H(\varepsilon_r)(s) \text{ d}s$$
where ε_r is the Dirac measure at r.
5. Derive the following identity:
$$\int_0^1 U(s)\frac{d}{ds}K_H(\varepsilon_r)(s) \text{ d}s = \widehat{\mathcal{K}_H}^* U(r),$$
where $\widehat{\mathcal{K}_H} = K_{1/2}^{-1} \circ K_H$.
6. Show that
$$\sum_{i=0}^{n-1} u(\frac{i}{n}) \left(h(\frac{i+1}{n}) - h(\frac{i}{n})\right) \xrightarrow{n\to\infty} \int_0^1 u(s)h'(s) \text{ d}s = \operatorname{trace}\big(\widehat{\mathcal{K}_H}\dot{\nabla}U\big).$$

The map $\widehat{\mathcal{K}_H} = K_{1/2}^{-1} \circ K_H$ is a continuous map from $L^2([0,1] \to \mathbf{R}; \lambda)$ into $I_{H-1/2,2}$ so that a possible definition of a stochastic integral (in the sense of Riemann integrals) could be
$$\delta_H(\widehat{\mathcal{K}_H}^* U) + \operatorname{trace}(\widehat{\mathcal{K}_H}\dot{\nabla}U)$$
provided that U has the necessary regularity for these terms to make sense.

For some other definitions of a stochastic integral with respect to B_H, see [Dec05] and references therein.

Deterministic fractional calculus

We now consider some basic aspects of the deterministic fractional calculus – the main reference for this subject is [SKM93].

Definition 4.9. Let $f \in L^1([a,b] \to \mathbf{R};\ \lambda)$, the integrals

$$(I^\alpha_{a+} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(t)(x-t)^{\alpha-1} dt\ ,\ x \geq a,$$
$$(I^\alpha_{b-} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b f(t)(x-t)^{\alpha-1} dt\ ,\ x \leq b,$$

where $\alpha > 0$, are respectively called right and left fractional integral of the order α.

For any $\alpha \geq 0$, any $f \in L^p([0,1] \to \mathbf{R};\ \lambda)$ and $g \in L^q([0,1] \to \mathbf{R};\ \lambda)$ where $p^{-1} + q^{-1} \leq \alpha$, we have :

$$\int_0^t f(s)(I^\alpha_{0+} g)(s)\ ds = \int_0^t (I^\alpha_{t-} f)(s) g(s)\ ds. \tag{4.29}$$

Moreover, the family of fractional integrals constitute a semi-group of transformations: For any $\alpha, \beta > 0$,

$$I^\alpha_{0+} \circ I^\beta_{0+} = I^{\alpha+\beta}_{0+}. \tag{4.30}$$

Definition 4.10. For f given in the interval $[a,b]$, each of the expressions

$$(\mathcal{D}^\alpha_{a+} f)(x) = \left(\frac{d}{dx}\right)^{[\alpha]+1} I^{1-\{\alpha\}}_{a^+} f(x),$$
$$(\mathcal{D}^\alpha_{b-} f)(x) = \left(-\frac{d}{dx}\right)^{[\alpha]+1} I^{1-\{\alpha\}}_{b^-} f(x),$$

are respectively called the right and left fractional derivative (proved they exist), where $[\alpha]$ denotes the integer part of α and $\{\alpha\} = \alpha - [\alpha]$.

Theorem 4.16. *We have the following embeddings and continuity results:*

1. *If* $0 < \gamma < 1$, $1 < p < 1/\gamma$, *then* I^γ_{0+} *is a bounded operator from* $L^p([0,1] \to \mathbf{R};\ \lambda)$ *into* $L^q([0,1] \to \mathbf{R};\ \lambda)$ *with* $q = p(1-\gamma p)^{-1}$.
2. *For any* $0 < \gamma < 1$ *and any* $p \geq 1$, $I^+_{\gamma,p}$ *is continuously embedded in* $\mathrm{Hol}(\gamma - 1/p)$ *provided that* $\gamma - 1/p > 0$.
3. *For any* $0 < \gamma < \beta < 1$, $\mathrm{Hol}(\beta)$ *is compactly embedded in* $I_{\gamma,\infty}$.