Regularity properties of some stochastic Volterra integrals with singular kernel

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Abstract. We prove the Hölder continuity of some stochastic Volterra integrals, with singular kernels, under integrability assumptions on the integrand. Some applications to processes arising in the analysis of the fractional Brownian motion are given. The main tool is the embedding of some Besov spaces into some sets of Hölder continuous functions.

Keywords: Besov spaces, fractional Brownian motion, sample–path continuity, stochastic Volterra integral

Subj. Classification: Primary 60H07; Secondary 60H10, 60H20

1. Introduction

The goal of this paper is to prove, that under mild integrability assumptions on a progressively measurable process u, the sample-paths of the process :

$$t \mapsto \tilde{M}_t^V(u) \stackrel{def}{=} \int_0^t V(t,s) u_s \, dB_s$$

are a.s. Hölder-continuous even for some singular deterministic kernel V. The motivation comes from the analysis of the fractional Brownian motion (see below Section 4) where the corresponding kernel is of the form :

$$K_H(t,s) = l_H(t,s)(t-s)^{H-1/2} s^{-|H-1/2|} \mathbf{1}_{[0,t)}(s), \tag{1}$$

for some $H \in (0,1)$ where l_H is a continuous function on $[0,1]^2$. Several papers do exist on the sample-paths regularity properties of stochastic Volterra integrals but always with regular or convolutional kernels (see for instance [1, 2]), two hypothesis which are clearly not satisfied by the present kernel.

Formally, the strategy is similar to that of [6] : we prove that there exists a process, denoted by $M^{V_\gamma}(u)$, which is integrable and satisfies

$$I_{0+}^{\gamma}(M^{V_\gamma}(u)) = M^V(u), \tag{2}$$

where $M^V(u)$ is a measurable version of $\tilde{M}^V(u)$ and I_{0+}^{γ} is the fractional integral of order γ – see preliminaries below. Once we have proved that $M^{V_\gamma}(u)$ is sufficiently integrable, the embedding of Besov spaces in some space of Hölder continuous functions will give the regularity of M^V and a maximal inequality as well– for another application of these embeddings to Stratonovitch stochastic differential equations, see [5].

The problems we have to solve are mainly technical and due to the singularity of the kernels we want to work with in our applications. Actually, for a given deterministic and borelean kernel $\{V(t,s),\, t,s \in [0,1]\}$, we have to find conditions on u ensuring both the existence of $\tilde{M}_t^V(u)$ for each t and the existence of a measurable version, designated $M^V(u)$, of the whole process $\{\tilde{M}_t^V(u),\, t \in [0,1]\}$. Moreover, we need to be able to compute the fractional integral of such a process (see Eqn. (2)). For instance, for the kernel given in Eqn. (1), the very existence of $\tilde{M}_t^{K_H}(u)$ requires u to be r-times integrable with r strictly greater than $(1-H)^{-1}$. This turns to be a very restrictive condition as H goes to 1. This matter of fact is all the more disapointing that as H goes to 1, the regularity of the linear map canonically associated to K_H (i.e., the map which sends a square integrable function f on $K_H f \stackrel{def}{=} \int_0^1 K_H(.,s) f(s)\, ds$) is increasing.

It is thus more fruitful to work with the properties of the map V than with the expression of the kernel V. Unfortunately, the definition of $\tilde{M}^V(u)$ as a stochastic integral with the kernel V inside, is not well suited to such an approach. The key point is thus to view the process $\tilde{M}^V(u)$ as a random kernel on $\mathcal{L}^2$ and perform an integration by parts in order to be able to use the properties of the map V – see Proposition [3.2]. With this point of view, the proof of the existence of a measurable version and the computation of the fractional integral become almost straightforward. It should be noticed that the subsequent integration by parts are greatly simplified by considering stochastic integrals as divergences in the sense of the calculus of variation.

The paper is organized as follows : in Section 2, we recall the basic notions of deterministic fractional calculus and Malliavin calculus needed later, in Section 3, we give the general result and in Section 4, we treat the specific case of the fractional Brownian motion.

2. Preliminaries

2.1. Deterministic fractional calculus

For $f \in \mathcal{L}^1([0,1];\ dt)$, (denoted by $\mathcal{L}^1$ for short) the left and right fractional integrals of f are defined by :

$$(I^\alpha_{0^+}f)(x) \overset{def}{=} \frac{1}{\Gamma(\alpha)}\int_0^x f(t)(x-t)^{\alpha-1}dt\ ,\ x \geq 0,$$
$$(I^\alpha_{1^-}f)(x) \overset{def}{=} \frac{1}{\Gamma(\alpha)}\int_x^1 f(t)(t-x)^{\alpha-1}dt\ ,\ x \leq 1,$$

where $\alpha > 0$ and $I^0_{0^+} = I^0_{1^-} = \mathrm{Id}$. For any $\alpha \geq 0$, any $f \in \mathcal{L}^p$ and $g \in \mathcal{L}^q$ where $p^{-1} + q^{-1} \leq \alpha$, we have :

$$\int_0^1 f(s)(I^\alpha_{0^+}g)(s)\ ds = \int_0^1 (I^\alpha_{1^-}f)(s)g(s)\ ds. \tag{3}$$

The Besov space $I^\alpha_{0^+}(\mathcal{L}^p) \overset{not}{=} \mathcal{I}_{\alpha,p}$ is usually equipped with the norm :

$$\|f\|_{\mathcal{I}_{\alpha,p}} = \|I^{-\alpha}_{0^+}f\|_{\mathcal{L}^p}. \tag{4}$$

We then have the following continuity results (see [6, 11]) :

Proposition 2.1. *i. If $0 < \alpha < 1$, $1 < p < 1/\alpha$, then $I^\alpha_{0^+}$ is a bounded operator from $\mathcal{L}^p$ into $\mathcal{L}^q$ with $q = p(1-\alpha p)^{-1}$.*

ii. For any $0 < \alpha < 1$ and any $p \geq 1$, $\mathcal{I}_{\alpha,p}$ is continuously embedded in $\mathrm{Hol}(\alpha - 1/p)$ provided that $\alpha - 1/p > 0$. $\mathrm{Hol}(\nu)$ denotes the space of Hölder-continuous functions, null at time 0, equipped with the usual norm.

iii. For any $0 < \alpha < \beta < 1$, $\mathrm{Hol}(\beta)$ is compactly embedded in $\mathcal{I}_{\alpha,\infty}$.

iv. Let $0 < \alpha < 1$, $\varphi \in \mathcal{L}^p$ for some $p > 1$ and $\mu > -1 + 1/p$. There exists $\phi \in \mathcal{L}^p$ such that

$$I^\alpha_{0^+}(s^\mu\varphi)(t) = t^\mu I^\alpha_{0^+}(\phi)(t) \text{ and } \|\phi\|_{\mathcal{L}^p} \leq c\|\varphi\|_{\mathcal{L}^p}.$$

v. By $I^{-\alpha}_{0^+}$, respectively $I^{-\alpha}_{1^-}$, we mean the inverse map of $I^\alpha_{0^+}$, respectively $I^\alpha_{1^-}$. The relation $I^\alpha_{0^+}I^\beta_{0^+}f = I^{\alpha+\beta}_{0^+}f$ holds whenever $\beta > 0$, $\alpha + \beta > 0$ and $f \in \mathcal{L}^1$.

2.2. Probabilistic setting

We work on the standard Wiener space $(\Omega, \mathbb{H}, I\!\!P)$ where Ω is the Banach space of continuous functions from $[0,1]$ into $I\!\!R$, null at time 0, equipped with the sup-norm. $\mathbb{H}$ is the Hilbert space of absolutely continuous functions vanishing at time 0, with the norm $\|h\|_{\mathbb{H}} = \|\dot{h}\|_{\mathcal{L}^2}$, where $\dot{h}$ is the (time) derivative of h. A mapping ϕ from Ω into some separable Hilbert space X is called cylindrical if it is of the form $\phi(w) = f(\langle v_1, w\rangle, \cdots, \langle v_n, w\rangle)$ where $f \in \mathcal{C}_0^\infty(I\!\!R^n, X)$ and $(v_i,\, i = 1 \ldots n)$ is sequence of Ω^* such that $(\tilde{v}_i,\, i = 1 \ldots n)$ (where $\tilde{v}_i$ is the image of v_i under the injection $\Omega^\star \hookrightarrow \mathbb{H}$) is an orthonormal system of $\mathbb{H}$. For such a function we define $\nabla\phi$ as

$$\nabla\phi(w) = \sum_{i=1}^{n} \partial_i f(\langle v_1, w\rangle, \cdots, \langle v_n, w\rangle)\tilde{v}_i.$$

From the quasi-invariance of the Wiener measure, it follows that ∇ is a closable operator on $L^p(\Omega; X)$, $p \geq 1$, and we will denote its closure with the same notation. The powers of ∇ are defined by iterating this procedure. For $p > 1$, $k \in I\!\!N$, we denote by $\mathbb{D}_{p,k}(X)$ the completion of X-valued cylindrical functions under the following norm

$$\|\phi\|_{p,k} = \sum_{i=0}^{k} \|\nabla^i \phi\|_{L^p(\Omega; X \otimes \mathbb{H}^{\otimes i})} \,.$$

Let us denote by ∇^* the formal adjoint of ∇ with respect to Wiener measure. Since it is often more appealing to work with $\mathcal{L}^2$-valued integrand, we introduce the map δ which is such that for any $\mathbb{H}$ -valued process v, belonging to $\operatorname{Dom}\nabla^*$, we have $\delta(\dot{v}) = \nabla^*(v)$. A classical result stands that δ is an extension of the Itô integral thus we have

$$\mathbf{E}\left[\int_0^t u_s\, dB_s\ \varphi\right] = \mathbf{E}\left[\int_0^t u_s \dot{\nabla}_s \varphi\, ds\right] \tag{5}$$

for any u adapted in $L^2(\Omega; \mathcal{L}^2)$ and any $\varphi \in \mathbb{D}_{2,1}$, where $\{B_t \stackrel{def}{=} \delta(\mathbf{1}_{[0,t]}),\, t \in [0,1]\}$ is a standard Brownian motion on $(\Omega, I\!\!P)$. We denote by $\mathcal{F}$ the complete σ-field generated by the sample-paths of B up to time 1 and by $\mathcal{B}([0,1])$ the set of Borel sets of $[0,1]$.

3. General result

In this section, we deal with a fixed deterministic borelean function $V(t,s)$ defined on $[0,1] \times [0,1]$. We also denote by V the linear map canonically associated to $V(t,s)$ by $Vf(t) = \int_0^1 V(t,s) f(s)\, ds$. We set :

$$\theta(x) \stackrel{def}{=} \frac{2x}{2-x} \text{ for } x \leq 2 \text{ and } \psi(x) \stackrel{def}{=} \frac{2x}{2+x} \text{ so that } \theta \circ \psi = \mathrm{Id}\,.$$

Hypothesis I. *We assume that there exists* $\alpha > 0$ *such that* V *is continuous from* $\mathcal{L}^2$ *into* $\mathcal{I}_{\alpha+1/2,2}$.

Hypothesis II. *We assume that for the same* α, *there exists* $\eta \leq 2$ *such that* V *is continuous from* $\mathcal{L}^\eta$ *into* $\mathcal{I}_{\alpha,\theta(\eta)}$.

Proposition 3.1. *Under hypothesis I, for any* $0 \leq \gamma < \alpha$, *the maps* V *and* $V_\gamma \stackrel{def}{=} I_{0^+}^{-\gamma} \circ V$ *are Hilbert-Schmidt from* $\mathcal{L}^2$ *into itself.*

Proof. Since $\alpha - \gamma > 0$, V_γ is continuous from $\mathcal{L}^2$ into $\mathcal{I}_{1/2+\alpha-\gamma,2}$ and the embedding of $\mathcal{I}_{1/2+\alpha-\gamma,2}$ in $\mathcal{L}^2$ is Hilbert-Schmidt (see [12])and so is V_γ from $\mathcal{L}^2$ into itself. The same holds for $V \equiv V_0$.

Remark 3.1 (Comments on the hypothesis). *Since the map* V_γ *is Hilbert-Schmidt, there exists a borelean kernel* $\{V_\gamma(t,s),\, t,s \in [0,1]\}$ *such that* $V_\gamma f = \int_0^1 V_\gamma(.,s) f(s)\, ds$. *Since* $\epsilon \stackrel{def}{=} \alpha - \gamma > 0$, *the space* $\mathcal{I}_{\epsilon+1/2,2}$ *is continuously embedded (see Proposition 2.1) in the space of continuous functions on* $[0,1]$, *null at time* 0, *hereafter designated* $\mathcal{C}_0$. *Accordingly, the adjoint* V_γ^* *of* V_γ *is continuous from* $\mathcal{C}_0^*$ *into* $\mathcal{L}^2$, *hence, for any* $t \in [0,1]$, $\int_0^1 V_\gamma(t,s)^2\, ds$ *is finite. It follows that for* u *progressively measurable and bounded, the stochastic integral* $\int_0^1 V_\gamma(t,s) u_s\, dBs$ *is well defined. The same holds* a fortiori *for* $\int_0^1 V(t,s) u_s\, dB_s$ *since* $V = V_0$.

Under hypothesis II, for any $0 \leq \gamma \leq \alpha$, *the map* V_γ *is continuous from* $\mathcal{L}^\eta$ *into* $\mathcal{L}^{\theta(\eta)}$ *and we set*

$$c_{\gamma,\eta} \stackrel{def}{=} \sup_{g : \|g\|_{\mathcal{L}^\eta} = 1} \|V_\gamma g\|_{\mathcal{L}^{\theta(\eta)}}.$$

Proposition 3.2. *For* V *a Hilbert-Schmidt map from* $\mathcal{L}^2$ *into itself, for any bounded and progressively measurable process* u, *there exists a* $\mathcal{F} \otimes \mathcal{B}([0,1])$*-measurable process, denoted by* $M_t^V(u)$, *such that for any* $f \in \mathcal{L}^2$,

$$\delta(V^* f.u) = \int_0^1 M_t^V(u) f(t)\, dt \quad I\!\!P \otimes dt \text{ a.e..} \tag{6}$$

Moreover,

$$\mathbf{E}\left[\int_0^1 |M_t^V(u)|^2\, dt\right] \text{ is finite.} \tag{7}$$

Proof. Consider the map

$$\begin{aligned} \Theta_u \ :\ & \mathcal{L}^2 \to L^2(\Omega) \\ & f \mapsto \delta(V^* f.u). \end{aligned}$$

Θ_u is a Hilbert-Schmidt operator : Let $(\varphi_n,\, n \ge 1)$ be an CONS of $\mathcal{L}^2$, we have

$$\begin{aligned}\sum_{n\ge 1} \|\Theta_u \varphi_n\|^2_{L^2(\Omega)} &= \sum_{n\ge 1} \mathbf{E}\left[\int_0^1 V^*\varphi_n(s)^2 u_s^2\, ds\right]\\ &\le \|u\|^2_\infty \|V^*\|^2_{HS}.\end{aligned}$$

Hence there exists (see [4]) a $\mathcal{F}\otimes\mathcal{B}([0,1])$-measurable kernel $M_t^V(u)$ such that (6) and the integrability condition (7) hold.

Remark 3.2. *Note that the existence of $\int_0^1 V(t,s)u_s\, dB_s$ as a stochastic integral requires that $V^*(\epsilon_t).u$ belongs to $L^2(\Omega\times[0,1])$. On the other hand, for $f\in\mathcal{L}^2$, the existence of $\delta(V^*fu)$ requires $V^*(f).u$ to belong to $L^2(\Omega\times[0,1])$. This latter condition is likely to be weaker than the former because V^*f is a priori more regular (with respect to the time variable) than $V^*\epsilon_t$. This means that for a given u, $M^V(u)$ may exist whereas $\tilde{M}^V(u)$ may not.*

Proposition 3.3. *Under assumptions I and II; for any bounded and progressively measurable process u, $M^{V_\gamma}(u)$ belongs to $L^r(\Omega;\mathcal{L}^r)$ where $r=\theta(\eta)$ and*

$$\|M^{V_\gamma}(u)\|_{L^r(\Omega;\mathcal{L}^r)} \le c_{\gamma,\eta}\|u\|_{L^r(\Omega;\mathcal{L}^r)}. \tag{8}$$

Note that since η is assumed to be less than 2, r is greater than 2 (in particular not equal to 1) so that $L^r(\Omega\times[0,1])$ can be viewed as the strong dual of $L^{r^*}(\Omega\times[0,1])$.

Proof. For any $g\in\mathcal{L}^2\subset\mathcal{L}^{r^*}$ and any $\varphi\in L^{r^*}(\Omega)$, using (6), Hölder inequality, the isometry property of stochastic integrals and assumption II, we have :

$$\begin{aligned}\left|\mathbf{E}\left[\int_0^1 M_t^{V_\gamma}(u)g(t)\,dt\ \varphi\right]\right| &= \left|\mathbf{E}\left[\delta(V_\gamma^*(g).u)\varphi\right]\right|\\ &\le \|\varphi\|_{L^{r^*}}\|\delta(V_\gamma^*(g).u)\|_{L^r}\\ &= \|\varphi\|_{L^{r^*}}\|V_\gamma^* g.u\|_{L^r(\Omega;\mathcal{L}^2)}\\ &\le \|\varphi\|_{L^{r^*}}\|V_\gamma^* g\|_{\mathcal{L}^{2(r/2)^*}}\|u\|_{\mathcal{L}^r(\Omega;\mathcal{L}^r)} \qquad \text{(a)}\\ &\le c_{\gamma,\eta}\|\varphi\|_{L^{r^*}}\|g\|_{\mathcal{L}^{r^*}}\|u\|_{\mathcal{L}^r(\Omega;\mathcal{L}^r)}. \qquad \text{(b)}\end{aligned}$$

The key points from line (a) to line (b) are that

$$2\big(\frac{r}{2}\big)^* = \frac{2r}{r-2} = \psi(r)^* = \eta^*,$$

and that a map and its adjoint have the same operator norm. By the density of $\mathcal{L}^2$ in $\mathcal{L}^{r^*}$, it follows that $M^{V_\gamma}(u)$ belongs to $L^r(\Omega;\mathcal{L}^r)$ and (8) follows.

Proposition 3.4. *Under assumption I, for any bounded, progressively measurable, process u, we have*

$$I^\gamma_{0+}(M^{V_\gamma}_{.}(u))(t) = M^V_t(u) \quad I\!\!P\otimes dt - a.e.. \tag{9}$$

Proof. For $f \in \mathcal{L}^\infty$, by Proposition [2.1], $I^\gamma_{1-}(f)$ belongs to $\mathcal{L}^2$. Moreover, since V_γ is Hilbert-Schmidt, according to (7), $M^{V_\gamma}(u)$ belongs $I\!\!P$-a.s. to $\mathcal{L}^2$ hence to the domain of I^γ_{0+}, so that we have :

$$\begin{aligned}\int_0^1 I^\gamma_{0+}(M^{V_\gamma}(u))(t)f(t)\,dt = &\int_0^1 M^{V_\gamma}_t(u) I^\gamma_{1-}(f)(t)\,dt\\ =&\delta(V^*_\gamma\circ I^\gamma_{1-}(f).u)\\ =&\delta(V^* f.u) = \int_0^1 M^V(u)(t)f(t)\,dt,\end{aligned}$$

where we have twice used (6).

Theorem 3.1. *Assume that hypothesis I and II hold. Let $r=\theta(\eta)$ where η is given by hypothesis II. Let u be a progressively measurable process belonging to $L^r(\Omega;\mathcal{L}^r)$ and satisfying*

$$\mathbf{E}\left[\int_0^1 V(t,s)^2 u_s^2\,ds\right] < \infty, \text{ for all } t\in[0,1]. \tag{10}$$

Then $\{\tilde{M}^V_t(u),\, t\in[0,1]\}$ has a version which belongs to $\bigcap_{\gamma<\alpha}\mathcal{I}_{\gamma,r}$ Moreover, we have, for any $\gamma<\alpha$:

$$\|\tilde{M}^V(u)\|_{L^r(\Omega;\mathcal{I}_{\gamma,r})} \le c_{\gamma,\eta}\,\|u\|_{L^r(\Omega;\mathcal{L}^r)} \tag{11}$$

Proof. Let $0\le\gamma<\alpha$ be fixed. In a first step, assume that u is bounded; according to remark 3.1, condition (10) is satisfied thus $\tilde{M}^V_t(u)$ exists for all t. By Proposition [3.3], $M^{V_\gamma}(u)$ belongs almost surely to $\mathcal{L}^r$ hence, according to (9), this means that $M^V(u)$ belongs almost surely to $\mathcal{I}_{\gamma,r}$. By Proposition [2.1], part (ii), $s\mapsto M^V_s(u)$ is thus an almost surely continuous function.

We now consider the following sequence of embeddings :

$$\mathcal{C}_0 \underset{id}{\subseteq} \mathcal{L}^2 \approx \mathcal{L}^{2*} \underset{id^*}{\subseteq} \mathcal{C}_0^*,$$

where $\approx$ denotes the canonical isomorphism between $\mathcal{L}^2$ and its dual. For $(f_n,\, n \geq 1)$ a $\mathcal{L}^{2*}$-sequence weakly-$*$ converging in $\mathcal{C}_0$, to ϵ_t, we have :

$$\begin{aligned} M_t^V(u) = \langle \epsilon_t, M^V(u) \rangle_{\mathcal{C}_0^*,\mathcal{C}_0} &= \lim_{n\to\infty} < id^*(f_n),\, M^V(u) >_{\mathcal{C}_0^*,\mathcal{C}_0} \\ &= \lim_{n\to\infty} < f_n,\, id(M^V(u)) >_{\mathcal{L}^2,\mathcal{L}^2} = \lim_{n\to\infty} \delta(V^* f_n.u), \end{aligned}$$

according to the definition of $M^V(u)$. Since V is continuous from $\mathcal{L}^2$ in $\mathcal{C}_0$, the sequence $(V^* f_n,\ n \geq 1)$ is weakly convergent in $\mathcal{L}^2$ and thus is bounded. For u bounded and progressively measurable,

$$\sup_n \mathbf{E}\left[\delta(V^* f_n.u)^2\right] \leq \|u\|_\infty^2 \sup_n \|V^* f_n\|_{\mathcal{L}^2}^2 \ < \infty.$$

This entails that the sequence $(\delta(V^* f_n.u),\, n \geq 1)$ is uniformly integrable and hence that for φ bounded, we have :

$$\mathbf{E}\left[M_t^V(u)\varphi\right] = \lim_{n\to\infty} \mathbf{E}\left[\delta(V^* f_n.u)\varphi\right] \tag{12}$$

It remains to prove that $M^V(u)$ is a version of $\tilde{M}^V(u)$. For, consider a cylindrical functional φ with bounded Gross-Sobolev derivative, we have on one hand :

$$\mathbf{E}\left[\tilde{M}_t^V(u)\varphi\right] = \mathbf{E}\left[\int_0^1 V(t,s) u_s \dot{\nabla}_s \varphi \, ds\right], \text{ by (5)}$$

and on the other hand :

$$\begin{aligned} \mathbf{E}\left[M_t^V(u)\varphi\right] &= \lim_{n\to\infty} \mathbf{E}\left[\delta(V^* f_n.u)\varphi\right], \text{ by (12)} \\ &= \lim_{n\to\infty} \mathbf{E}\left[\int_0^1 V^* f_n(s) u_s \dot{\nabla}_s \varphi \, ds\right], \text{ by (5)} \\ &= \mathbf{E}\left[\int_0^1 V(t,s) u_s \dot{\nabla}_s \varphi \, ds\right], \end{aligned}$$

according to hypothesis I and the dominated convergence theorem. Thus the theorem is proved provided that u is bounded.

Now, let u be not bounded but belong to $L^r(\Omega; \mathcal{L}^r)$ and satisfy (10). Consider the sequence $(u_n,\, n \geq 1)$ defined by $u_n(s) = u(s)\mathbf{1}_{\{|u(s)| \leq n\}}$; u_n converges clearly to u in $L^r(\Omega; \mathcal{L}^r)$ by dominated convergence. Applying (11), $(Z_n \stackrel{def}{=} M^V(u_n),\, n \geq 1)$ is a Cauchy sequence in $L^r(\Omega; \mathcal{I}_{\gamma,r})$, thus converges to some process Z, the sample-paths of which belonging to $\mathcal{I}_{\gamma,r}$ and satisfying

$$\|Z\|_{L^r(\Omega;\mathcal{I}_{\gamma,r})} \leq c_\gamma \|u\|_{L^r(\Omega;\mathcal{L}^r)}.$$

Furthermore, for fixed t, by the first part of the proof, $Z_n(t) = \tilde{M}_t^V(u_n)$ for any n, almost surely and by (10) and dominated convergence, $\tilde{M}_t^V(u_n)$ converges in $L^2(\Omega)$ to $\tilde{M}_t^V(u)$, hence Z is a $\mathcal{I}_{\gamma,r}$-valued version of $\tilde{M}^V(u)$.

4. Application

We now apply the previous result to some integrals related to the fractional Brownian motion. In the same vein, note that the Besov regularity of stochastic integral with respect to the standard Brownian motion is studied in [6, 7, 10, 13]. Namely there are two processes which deserve the name of fractional Brownian motion. One is the process defined by :

$$B_t^H = \int_0^t J_H(t,s)\, dB_s \text{ with } J_H(t,s) = \frac{1}{\Gamma(H+1/2)}(t-s)_+^{H-1/2}, \quad (13)$$

where $x_+ = \max(x,0)$. The parameter H which belongs to $(0,1)$, is the so-called Hurst index. The other one is the process defined by $W_t^H = \int_0^t K_H(t,s)\, dB_s$ where

$$K_H(t,s) = \frac{(t-s)_+^{H-1/2}}{\Gamma(H+1/2)} F(H-1/2, 1/2-H, H+1/2, 1-t/s). \quad (14)$$

The Gauss hyper-geometric function $F(\alpha,\beta,\gamma,z)$ (see [9]) is the analytic continuation on $\mathbb{C}\times\mathbb{C}\times\mathbb{C}\backslash\{-1,-2,\ldots\}\times\{z\in\mathbb{C}, Arg|1-z|<\pi\}$ of the power series

$$\sum_{k=0}^{+\infty} \frac{(\alpha)_k(\beta)_k}{(\gamma)_k k!} z^k. \quad (15)$$

Here $(\alpha)_k$ is defined by

$$(a)_0 = 1 \text{ and } (a)_k \overset{def}{=} \frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1)\ldots(a+k-1).$$

Note that W^H is more widely used in applications than B^H because it has stationary increments (see [3, 8] for some other features of B^H and W^H), in fact the covariance kernel of W^H is given by :

$$\mathbf{E}\left[W_s^H W_t^H\right] \overset{def}{=} \frac{V_H}{2}(s^{2H} + t^{2H} - |t-s|^{2H}),$$

where,

$$V_H \stackrel{def}{=} \frac{\Gamma(2-2H)\cos(\pi H)}{\pi H(1-2H)}.$$

It has been shown in [3] that for any regular f, we have

$$f(W_t^H) - \mathbf{E}\left[f(W_t^H)\right] = \int_0^t K_H(t,s) u_s\, dB_s,$$

where u is a progressively measurable process. Since the same can be done for B^H, this justifies our interest in stochastic Volterra integrals with singular kernel.

Theorem 4.1 (fBm with non stationary increments)**.** *Assume that u belongs to $L^r(\Omega;\mathcal{L}^r)$ for some $r > 1/H$ and $r \geq 2$ then $\tilde{M}^{J_H}(u)$ has a modification which is $H_r^\varepsilon \stackrel{def}{=} (H - 1/r - \varepsilon)$-Hölder continuous for any $0 < \varepsilon \leq H - 1/r$ and*

$$\mathbf{E}\left[\|M^{J_H}(u)\|^r_{\mathrm{Hol}(H_r^\varepsilon)}\right] \leq c_\varepsilon^r\, \mathbf{E}\left[\int_0^1 |u_s|^r\, ds\right], \tag{16}$$

where c_ε is the operator norm of $I_{0^+}^{1/2+\varepsilon}$ from $\mathcal{L}^{\psi(r)}$ to $\mathcal{L}^r$.

Remark 4.1. *As a byproduct, choosing any $\varepsilon \in (0, H - 1/r]$ yields to*

$$\mathbf{E}\left[\sup_{t\leq 1}\left|\int_0^t J_H(t,s) u_s\, dB_s\right|^r\right] \leq c\, \mathbf{E}\left[\int_0^1 |u_s|^r\, ds\right]. \tag{17}$$

The value of c depends on ε but the other terms of (17) don't, so we have as many inequalities as many choices of ε.

Proof. Since, J_H is simply $I_{0^+}^{H+1/2}$, hypothesis I is clearly satisfied with $\alpha = H$. As to assumption II, it is sufficient to observe that in view of Proposition [2.1] point (i), we have :

$$\mathcal{I}_{H+1/2,\eta} \subset \mathcal{I}_{H,\theta(\eta)} \text{ for any } \eta \leq 2.$$

To conclude, it remains to show that the integrability condition on u entails condition (10) which reads here as :

$$\mathbf{E}\left[\int_0^t (t-s)^{2H-1} u_s^2\, ds\right] < \infty,$$

or equivalently,

$$I_{0^+}^{2H}(\mathbf{E}\left[u_.^2\right])(t) \text{ finite for any } t.$$

Actually, since u belongs to $L^r(\Omega;\mathcal{L}^r)$, $(s \mapsto \mathbf{E}\left[u_s^2\right])$ belongs to $\mathcal{L}^{r/2}$ and thus $I_{0^+}^{2H}(\mathbf{E}\left[u_.^2\right])$ belongs to $\mathrm{Hol}(2(H-1/r)) \subset \mathcal{L}^\infty$.

Theorem 4.2 (fBm with stationary increments)**.** *Assume that u is a progressively measurable process which belongs to $L^r(\Omega;\mathcal{L}^r)$ for some $r > 1/H$ and $r \geq 2$. Assume also that u satisfies* (10) *then $\widetilde{M}^{K_H}(u)$ has a modification which is H^ε_r-Hölder continuous for any $0 < \varepsilon \leq H - 1/r$ and*

$$\mathbf{E}\left[\|M^{K_H}(u)\|^r_{\mathrm{Hol}(H^\varepsilon_r)}\right] \leq c^r_\varepsilon \mathbf{E}\left[\int_0^1 |u_s|^r\, ds\right],$$

where c_ε is the operator norm of $I^{\varepsilon-H}_{0+} \circ K_H$ from $\mathcal{L}^{\psi(r)}$ to $\mathcal{L}^r$.

Proof. It is proved in [11, Table 10.2, page 188] that K_H is continuous from $\mathcal{L}^\eta$ into $\mathcal{I}_{H+1/2,\eta}$ so that the situation is very similar to what it is in the previous theorem. The proof follows.

Remark 4.2. *To examplify how necessary it is to work with the properties of the maps rather than the expression of the kernel, consider at which condition on r, the integrability condition $u \in L^r(\Omega;\mathcal{L}^r)$ entails* (10) *in the case of the kernel K_H. It is proved in [3] that :*

$$0 \leq K_H(t,s) \leq c(t-s)^{H-1/2} s^{-|H-1/2|}\mathbf{1}_{[0,t)}(s). \tag{18}$$

Let $H_0 = |H - 1/2|$. For any $p < H_0^{-1}$, since from its very definition (see (14)*) $K_H(t,s)$ is $(H-1/2)$–homogeneous,*

$$\int_0^t |K_H(t,s)|^p\, ds = t^{p(H-1/2)+1}\int_0^1 |K_H(1,s)|^p\, ds,$$

hence, by (18)*,*

$$\int_0^t |K_H(t,s)|^p\, ds \leq \ t^{p(H-1/2)+1} B(1-pH_0,\, 1-pH_0) \text{ if } H < 1/2,$$

where B is the usual Beta function, and

$$\int_0^t |K_H(t,s)|^p\, ds \leq \ t^{p(H-1/2)+1}\frac{1}{1-pH_0} \text{ if } H > 1/2.$$

Hence, applying the Hölder inequality, we see that $\int_0^\cdot K_H(.,s)^2 f(s)^2\, ds$ is bounded provided that f belongs to $\mathcal{L}^{2(1-2H_0)^{-1}}$. Note that $2(1-2H_0)^{-1}$ is equal to H^{-1}, respectively $(1-H)^{-1}$, when $H \leq 1/2$, respectively $H \geq 1/2$. Thus for $H \leq 1/2$, the conditions $r > 1/H$ and $u \in L^r(\Omega;\mathcal{L}^r)$ entail that the function u satisfies (10) *hence the result is very much same as it is for the fBm with non-stationary increments. Meanwhile, for $H > 1/2$ to ensure* (10) *with the only constraint $u \in L^r(\Omega;\mathcal{L}^r)$ requires r to be greater than $(1-H)^{-1}$. Unfortunately, for $H > 1/2$,*

$(1-H)^{-1}$ *is strictly greater than* $1/H$*, hence, all the more* H *is close to* 1*, all the more the condition on* u *becomes stringent. Meanwhile,* W^H *has more and more regular sample-paths when* H *increases : the sample-paths are* $(H-\varepsilon)$*-Hölder continuous for any* $\varepsilon > 0$ *(Theorem [4.2] for* u *bounded). One could thus expect that for a fixed regularity condition like* $u \in L^r(\Omega; \mathcal{L}^r)$*,* $M^{K_H}(u)$ *would become all the more regular as* H *increases. The reason for which this does not happen is that the kernel* $K_H(t, s)$ *is more and more singular (see Eqn.* (1)*), al though the map* K_H *is a more and more regularizing operator, and the existence of the stochastic integral* $\tilde{M}^V(u)$ *can only be expressed in terms of the kernel (see* (10)*). Hence, whenever* r *is only greater than* 2*,* $M^{K_H}(u)$ *is still well defined for* $u \in L^r(W; \mathcal{L}^r)$ *whereas* $\tilde{M}^{K_H}(u)$ *may not exist.*

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