THE FOURTH MOMENT THEOREM ON THE POISSON SPACE

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Abstract. We prove an exact fourth moment bound for the normal approximation of random variables belonging to the Wiener chaos of a general Poisson random measure. Such a result – that has been elusive for several years – shows that the so-called 'fourth moment phenomenon', first discovered by Nualart and Peccati (2005) in the context of Gaussian fields, also systematically emerges in a Poisson framework. Our main findings are based on Stein's method, Malliavin calculus and Mecke-type formulae, as well as on a methodological breakthrough, consisting in the use of carré-du-champ operators on the Poisson space for controlling residual terms associated with add-one cost operators. Our approach can be regarded as a successful application of Markov generator techniques to probabilistic approximations in a non-diffusive framework: as such, it represents a significant extension of the seminal contributions by Ledoux (2012) and Azmoodeh, Campese and Poly (2014). To demonstrate the flexibility of our results, we also provide some novel bounds for the Gamma approximation of non-linear functionals of a Poisson measure.

1. Introduction

1.1. **Overview.** The aim of this paper is to prove an exact *fourth moment bound* for the normal approximation of random variables belonging to the Wiener chaos of a general Poisson measure. Differently from previous fourth moment limit theorems on the Poisson space proved in the literature, our main findings do not require that the involved random variables have the form of multiple integrals with a kernel of constant sign (see [LRP13, Sch16, ET14]), nor that they are finite homogeneous sums (see [PZ14]) or that they belong to Wiener chaoses of lower orders (see [PT08, BP16b]). As discussed below, the methodological breakthrough yielding such an achievement, consists in the use of *carré-du-champ operators* on the Poisson space, that we shall systematically exploit in connection with Mecke-type formulae and Stein's method (see [CGS11,NP12]). We will see that using carré-du-champ operators instead of norms of Malliavin derivatives (as done in the already quoted references [LRP13, Sch16, ET14, PZ14, PT08, BP16b]) will allow us to bypass at once almost all combinatorial difficulties – in particular connected to multiplication formulae on configuration spaces – that have systematically marred previous attempts.

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We stress that the idea of using carré-du-champ operators, in order to deduce quantitative limit theorems by means of Stein's method, originates in the groundbreaking works [Led12, ACP14, AMMP16], where the authors apply the powerful techniques of *Gamma calculus* in the framework of eigenspaces of diffusive Markov generators (see [BGL14] for definitions, as well as [CP15] for an introduction to this approach). As demonstrated in Section 3, our results show that such an approach can be fruitfully applied and extended, in order to control residual terms arising from the application of Stein's method in a non-diffusive context.

1.2. **Further historical details.** The so-called *fourth moment phenomenon* was first discovered in [NP05], where the authors proved that a sequence of normalized random variables, belonging to a fixed Wiener chaos of a Gaussian field, converge in distribution to a Gaussian random variable if and only if their fourth cumulant converges to zero. Such a result constitutes a dramatic simplification of the method of moments and cumulants (see e.g. [NP12, p. 202]), and represents a rough infinite-dimensional counterpart of classical results by de Jong – see [dJ87, dJ89, dJ90], as well as [DP17, DP16] for recent advances. A particularly fruitful line of research was initiated in [NP09b], where it is proved that the results of [NP05] can be recovered from very general estimates, obtained by combining the Malliavin calculus of variations with Stein's method for normal approximation. Precisely, one remarkable achievement of this approach is the bound

$$d_{\mathrm{Kol}}(F,N) \le \sqrt{\frac{q-1}{3q}\big(\mathbb{E}[F^4]-3\big)}\,, \tag{1.1}$$

where d_{Kol} stands for the Kolmogorov distance between the laws of two random variables, F is a normalized *multiple Wiener-Itô integral* of order $q \ge 1$ on a Gaussian space and N denotes a standard normal random variable (see e.g. Theorem 5.2.6 in [NP12], where analogous bounds for other metrics are also stated). Such a discovery has been the seed of a fruitful stream of research, now consisting of several hundred papers, where the results of [NP05, NP09b] have been extended and applied to a variety of frameworks, ranging from free probability to stochastic geometry, compressed sensing and time-series analysis — see the webpage `https://sites.google.com/site/malliavinstein/home` for a constantly updated list, as well as the monograph [NP12] and the reference [LNP15] for recent developments related to functional inequalities.

The line of research pursued in the present work stems from the two papers [PSTU10, PZ10], where the authors adapted the techniques introduced in [NP09b] to the framework of non-linear Poisson functionals, in particular by combining Stein's method with a discrete version of Malliavin calculus on configuration spaces. As anticipated, the principal aim of this work is to positively answer the following question:

Can one prove a bound comparable to (1.1) *on the Poisson space?*

Such a question has stayed open since the publication of [PSTU10] and, so far, answers have only been found in very special cases — see Remark 1.5 below.

One should notice that the relevance of the techniques developed in [PSTU10, PZ10] has been greatly amplified by the pathbreaking reference [RS13], where it is shown that one can use Malliavin-Stein techniques on the Poisson space in order to study the fluctuation of random objects arising in the context of random geometric structures on configuration spaces, like e.g. random graphs or random tessellations. Such a connection with stochastic geometry has generated a remarkable body of work, that has recently culminated in the publication of the monograph [PR16]. The reader is referred to [LPS16, LRSY16] for recent developments connected to Mehler formulae, stabilization and second order Poincaré inequalities, and to [BP16a] for some related concentration estimates in a geometric context.

1.3. **Main results for normal approximations.** We fix an arbitrary measurable space $(\mathcal{Z}, \mathscr{Z})$ endowed with a σ-finite measure μ. Furthermore, we let

$$\mathscr{Z}_\mu := \{B \in \mathscr{Z} \,:\, \mu(B) < \infty\}$$

and denote by

$$\eta = \{\eta(B) \,:\, B \in \mathscr{Z}_\mu\}$$

a *Poisson measure* on $(\mathcal{Z}, \mathscr{Z})$ with *control* μ, defined on a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We recall that the distribution of η is completely determined by the following two facts: (i) for each finite sequence $B_1, \dots, B_m \in \mathscr{Z}$ of pairwise disjoint sets, the random variables $\eta(B_1), \dots, \eta(B_m)$ are independent, and (ii) that for every $B \in \mathscr{Z}$, the random variable $\eta(B)$ has the Poisson distribution with mean $\mu(B)$. Here, we have extended the family of Poisson distributions to the parameter region $[0, +\infty]$ in the usual way. For $B \in \mathscr{Z}_\mu$, we also write $\hat{\eta}(B) := \eta(B) - \mu(B)$ and denote by

$$\hat{\eta} = \{\hat{\eta}(B) \,:\, B \in \mathscr{Z}_\mu\}$$

the *compensated Poisson measure* associated with η. As discussed in Section 2.1, we require throughout the paper that η is *proper*, that is, that η can be a.s. represented as a (possibly infinite) random sum of Dirac masses. Without loss of generality, we may and will assume that $\mathcal{F} = \sigma(\eta)$. In order to state our main results, we introduce the following fundamental objects from stochastic analysis on the Poisson space. For precise definitions and further explanation we refer to [PR16], in particular to its first chapter [Las16], as well as to [LP17] and Section 2. For a nonnegative integer q and a square-integrable *kernel function* $f \in L^2(\mu^q)$ we denote by $I_q(f)$ the q-th order *multiple Wiener-Itô integral* of f with respect to $\widehat{\eta}$. If L denotes the generator of the *Ornstein-Uhlenbeck semigroup* with repect to η, then it is well-known that $-L$ has pure point spectrum given by the set of nonnegative integers and that, for $q \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$, F is an eigenfunction of $-L$ with eigenvalue q, if and only if $F = I_q(f)$ for some $f \in L^2(\mu^q)$. The corresponding eigenspace C_q is called the *q-th Wiener chaos* associated with η.

Next, we introduce the probabilistic distances in which our bounds are expressed. For $m \in \mathbb{N}$, denote by $\mathcal{H}_m$ the class of those $(m-1)$-times differentiable test functions h on $\mathbb{R}$ such that $h^{(m-1)}$ is Lipschitz-continuous and we have

$$\|h^{(l)}\|_\infty \le 1 \quad \text{for} \quad l = 1, \dots, m\,.$$

Here and elsewhere, for an arbitrary function g on $\mathbb{R}$, we use the notation

$$\|g'\|_\infty := \sup_{x \neq y} \frac{|g(y) - g(x)|}{|y - x|} \in [0, +\infty]$$

for the *minimum Lipschitz-constant* of g. This does not cause any confusion because this quantity coincides with the supremum norm of the derivative g' of g

when g happens to be differentiable. For real random variables X and Y such that $\mathbb{E}|X|, \mathbb{E}|Y| < \infty$ we denote by

$$d_m(X,Y) := d_m\big(\mathcal{L}(X), \mathcal{L}(Y)\big) := \sup_{h\in\mathcal{H}_m} \big|\mathbb{E}[h(X)] - \mathbb{E}[h(Y)]\big|$$

the distance between the distributions of X and Y induced by the class $\mathcal{H}_m$; observe that d_1 coincides with the classical 1-*Wasserstein distance*, see e.g. [NP12, Appendix C] and the references therein. We will also study the *Kolmogorov distance* between the laws of X and Y, given by

$$d_{\mathrm{Kol}}(X,Y) := \sup_{x\in\mathbb{R}} \big|\mathbb{P}[X \le x] - \mathbb{P}[Y \le x]\big|.$$

It is a well known fact (see e.g. [NP12, Appendix C] and the references therein) that if X is a generic random variable and Y has a density bounded by $c \in (0, \infty)$, then

$$d_{\mathrm{Kol}}(X,Y) \le \sqrt{2c\, d_1(X,Y)}. \tag{1.2}$$

The assumptions in our main results will be expressed in terms of the *add-one cost* operator D^+, that is defined as follows: if $F = \mathfrak{f}(\eta)$ is a functional of η, then

$$D_z^+ F := \mathfrak{f}(\eta + \delta_z) - \mathfrak{f}(\eta), \quad z \in \mathcal{Z},$$

in such a way that D^+F can be regarded as a random function with domain equal to $\mathcal{Z}$. See Section 2.1 for a formal discussion of such an object.

DEFINITION 1.1. Let F be an $\mathcal{F}(= \sigma(\eta))$-measurable random variable.

(i) We say that F satisfies **Assumption A** if $F \in L^4(\mathbb{P})$ and if the four random functions D^+F, FD^+F, $(D^+F)^4$ and F^3D^+F are all elements of the space $L^1(\Omega \times Z, \mathbb{P} \otimes \mu) := L^1(\mathbb{P} \otimes \mu)$.

(ii) We say that F satisfies **Assumption A**$^{(\mathrm{loc})}$ if there exists a set $Z_0 \in \mathscr{Z}$ with the following properties: (a) $\mu(\mathcal{Z}\backslash Z_0) = 0$, and (b) for every fixed $z \in Z_0$, the random variable D_z^+F verifies **Assumption A**.

REMARK 1.2. Requiring that a given functionals satisfies **Assumption A** or **Assumption A**$^{(\mathrm{loc})}$ is an unavoidable (minimal) restriction, ensuring that one can apply Mecke identities (see (2.3)–(2.4) below), as well as exploit several almost sure representations of Malliavin and carré-du-champ operators. Using e.g. the multiplication formula stated in [Las16, Proposition 5], one can easily prove that both **Assumption A** and **Assumption A**$^{(\mathrm{loc})}$ are verified, whenever F has the form

$$F = \sum_{q=0}^{M} I_q(f_q),$$

where $M < \infty$ and each f_q is bounded and such that its support is contained in a rectangle of the type $C \times \cdots \times C$, where $C \in \mathscr{Z}$ verifies $\mu(C) < \infty$. Such a class of random variables contains most U-statistics that are relevant for geometric applications (see the surveys [LRR16, ST16] and the references therein), as well as non-linear functionals of Volterra Lévy processes [PSTU10, PZ10], and the finite homogeneous sums in independent Poisson random variables considered in [PZ14]. A similar remark applies to the assumptions appearing in the statement of our main abstract bounds in Proposition 4.1 and Proposition 4.3.

The next result is the main finding of the paper: it provides quantitative fourth moment estimates with completely explicit constants, both in the 1-Wasserstein and Kolmogorov distances, for random variables living in the Wiener chaos of a Poisson measure. Remarkably, the order of the bound (as a function of the fourth cumulant $\mathbb{E}[F^4]-3$) is the same for the two distances, thus significantly improving the estimate on d_{Kol} that one could deduce from (1.2).

THEOREM **1.3** (Fourth moment bounds on the Poisson space)**.** *Fix an integer $q \geq 1$ and let $F = I_q(f)$ be a multiple Wiener-Itô integral with respect to $\hat{\eta}$. Assume that F verifies* **Assumption A** *and that $\mathbb{E}[F^2] = 1$; denote by $N \sim \mathscr{N}(0,1)$ a standard normal random variable. Then,*

$$d_1(F,N) \leq \left(\sqrt{\frac{2}{\pi}}\frac{2q-1}{2q} + \frac{\sqrt{4q-1}}{\sqrt{q}}\right)\sqrt{\mathbb{E}\big[F^4\big]-3} \tag{1.3}$$

$$\leq \left(\sqrt{\frac{2}{\pi}} + 2\right)\sqrt{\mathbb{E}\big[F^4\big]-3} \tag{1.4}$$

(in the above situation one automatically has that $\mathbb{E}[F^4] \geq 3$). Moreover, if F satisfies **Assumption A**$^{(\mathrm{loc})}$*, then*

$$d_{\mathrm{Kol}}(F,N) \leq \Big(11 + 2^{3/2}\big(\mathbb{E}[F^4]^{1/2} + \mathbb{E}[F^4]^{1/4}\big)\Big)\sqrt{\mathbb{E}\big[F^4\big]-3}\,. \tag{1.5}$$

The following result is an immediate consequence of the bound (1.4).

COROLLARY **1.4** (Fourth moment theorem on the Poisson space)**.** *For each $n \in \mathbb{N}$ let $q_n \geq 1$ be an integer and let $F_n = I_{q_n}(f_n)$ be a multiple Wiener-Itô integral of some symmetric kernel $f_n \in L^2(\mu^{q_n})$ such that*

$$\lim_{n\to\infty} \mathbb{E}\big[F_n^2\big] = \lim_{n\to\infty} q_n!\|f_n\|_2^2 = 1 \quad \textit{and} \quad \lim_{n\to\infty} \mathbb{E}\big[F_n^4\big] = 3\,.$$

Then, if each F_n satisfies **Assumption A***, the sequence $(F_n)_{n\in\mathbb{N}}$ converges in distribution to a standard normal random variable N in the sense of the* 1*-Wasserstein distance.*

REMARK 1.5. As mentioned before, so far, the fourth moment theorem on the Poisson space has only been known in very special cases: For double integrals, i.e. $q = 2$, the qualitative fourth moment theorem was proved in [PT08]. Under different assumptions, this result is also proved in [BP16b] where also a qualitative fourth moment theorem for $q = 3$ is derived. We would like to mention that the method of proof applied in [BP16b] is rather ad hoc and cannot be generalized to higher orders of q. We also stress that all existing quantitative fourth moment theorems on the Poisson space make the restrictive assumption that the kernel function f has a constant sign (see e.g. [LRP13, Sch16, ET14]). Furthermore, the multiplicative constants in these results depend on the order q in a non-explicit way such that e.g. a statement in the spirit of Corollary 1.4 cannot be inferred from them. We finally mention [PZ14], where one can find a fourth moment theorem for sequences of chaotic elements having the form of homogeneous sums in independent Poisson random variables whose variance is bounded away from zero, as well as [BP14], where the authors prove a fourth moment theorem for multiple integrals with respect of a non-commutative Poisson measure (in the framework of free probability theory), under an additional tameness assumption.

We also notice the following negative result.

PROPOSITION **1.6.** *For each $q \in \mathbb{N}$, there exists no Gaussian random variable with positive variance in the q-th Wiener chaos C_q associated with η.*

1.4. **Main results on Gamma approximations.** For $\nu > 0$, we denote by $\bar{\Gamma}(\nu)$ the so-called *centered Gamma distribution* with parameter ν which by definition is the distribution of

$$Z_\nu := 2X_{\nu/2,1} - \nu\,,$$

where, $X_{\nu/2,1}$ has the usual Gamma distribution on $[0, +\infty)$ with shape parameter $\nu/2$ and rate 1. In particular, one has

$$\mathbb{E}[Z_\nu] = 0 \quad \text{and} \quad \mathrm{Var}(Z_\nu) = \mathbb{E}[Z_\nu^2] = 2\nu\,.$$

Moreover, the following moment identity (already exploited in [NP09a]) will play an important role in what follows:

$$\mathbb{E}[Z_\nu^4] - 12\mathbb{E}[Z_\nu^3] - 12\nu^2 + 48\nu = 0\,. \tag{1.6}$$

The next result is the counterpart of Theorem 1.3 for centered Gamma approximation.

THEOREM **1.7** (Fourth moment bound for Gamma approximation)**.** *Fix $\nu > 0$ as well as an integer $q \geq 1$ and let $F = I_q(f)$ be a multiple Wiener-Itô integral with respect to $\hat{\eta}$, verifying* **Assumption A**. *Assume that $F \in L^4(\mathbb{P})$ and that $\mathbb{E}[F^2] = 2\nu$. Also, let $Z_\nu \sim \bar{\Gamma}(\nu)$ have the centered Gamma distribution with parameter ν. Then, we have the following bound:*

$$d_2(F, Z_\nu) \leq C_1(\nu)\sqrt{\Big|\mathbb{E}\big[F^4\big] - 12\mathbb{E}\big[F^3\big] - 12\nu^2 + 48\nu\Big|} + C_2(\nu)\Big(\frac{1}{q}\int_{\mathcal{Z}} \mathbb{E}\big[|D_z^+ F|^4\big]\mu(dz)\Big)^{1/2}\,, \tag{1.7}$$

where D^+ denotes the add-one-cost operator associated with η (see Section 2) and where we can let

$$C_1(\nu) := \frac{1}{\sqrt{3}}\max\Big(1, \frac{2}{\nu}\Big) \quad \textit{and}$$

$$C_2(\nu) := \frac{1}{\sqrt{6}}\max\Big(1, \frac{2}{\nu}\Big) + \max\Big(\sqrt{2\nu}, \sqrt{\frac{2}{\nu}} + \sqrt{\frac{\nu}{2}}\Big)\,.$$

REMARK 1.8. (a) The bound (1.7) displays an additional term, not directly connected to moments, that is not present in the estimate (1.3) for normal approximations. For the time being, it is a challenging open problem to determine whether such a term can be removed.

(b) The previous result implies that, if, for $n \in \mathbb{N}$, $F_n \in \mathrm{Ker}(L + q_n I_n)$ ($q_n \geq 1$) is a sequence of random variables verifying **Assumption A** and such that

$$\lim_{n\to\infty} \mathbb{E}[F_n^2] = 2\nu, \quad \lim_{n\to\infty} \big(\mathbb{E}\big[F_n^4\big] - 12\mathbb{E}\big[F_n^3\big]\big) = 12\nu^2 - 48\nu,$$

and

$$\lim_{n\to\infty} \int_{\mathcal{Z}} \mathbb{E}\big[|D_z^+ F_n|^4\big]\mu(dz) = 0, \tag{1.8}$$

then F_n converges in distribution to Z_ν. In the case where $q_n \equiv q$, and F_n has the form $I_q(f_n)$ for some sufficiently regular kernel f_n, then one sufficient condition in order to have (1.8) is that all contractions of the type $f_n \star_b^a f_n$ converge to zero in L^2, where the definition of $f_n \star_b^a f_n$ can be found e.g. in [Las16, Section 6]; see the computations contained in [PSTU10, p. 465-466]. A detailed discussion of the Gamma bound (1.7) via the use of contraction operators (in the sprit e.g. of [PT13, FT16]) seems to be outside the scope of the present work, and will be tackled elsewhere; see also [DP16].

1.5. **Plan.** The paper is organised as follows. Section 2 contains preliminary results concerning stochastic analysis on the Poisson space. Section 3 focusses on several new estimates for multiple integrals, whereas Section 4 and Section 5 deal with the proofs of our main results. Finally, Section 6 contains the proofs of some technical lemmas.

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2. Elements of stochastic analysis on the Poisson space

In this section, we describe our theoretical framework in more detail, by adopting the language of [Las16], corresponding to Chapter 1 in [PR16]. See also the monograph [LP17].

2.1. **Setup.** In what follows, we will view the Poisson process η as a random element taking values in the space $\mathbf{N}_\sigma = \mathbf{N}_\sigma(\mathcal{Z})$ of all σ-finite point measures χ on $(\mathcal{Z}, \mathscr{Z})$ that satisfy $\chi(B) \in \mathbb{N}_0 \cup \{+\infty\}$ for all $B \in \mathscr{Z}$. Such a space is equipped with the smallest σ-field $\mathscr{N}_\sigma := \mathscr{N}_\sigma(\mathcal{Z})$ such that, for each $B \in \mathscr{Z}$, the mapping $\mathbf{N}_\sigma \ni \chi \mapsto \chi(B) \in [0, +\infty]$ is measurable. As anticipated, throughout the paper we shall assume that the process η is *proper*, in the sense that η can be $\mathbb{P}$-a.s. represented in the form

$$\eta = \sum_{n=1}^{\eta(\mathcal{Z})} \delta_{X_n},$$

where $\{X_n : n \geq 1\}$ denotes a countable collection of random elements with values in $\mathcal{Z}$. A sufficient condition for η to be proper is e.g. that $(\mathcal{Z}, \mathscr{Z})$ is a Polish space endowed with its Borel σ-field, with μ being σ-finite as above. See [LP17, Section 6.1] and [Las16, p. 2-3] for more details. Furthermore, Corollary 3.7 in [LP17] states that for each Poisson process η, there exists (maybe on a different probability space) a proper Poisson process η^* which has the same distribution as η. Since all our results depend uniquely on the distribution of η, it is no restriction of generality to assume that η is proper.

Now denote by $\mathbf{F}(\mathbf{N}_\sigma)$ the class of all measurable functions $\mathfrak{f} : \mathbf{N}_\sigma \to \mathbb{R}$ and by $\mathcal{L}^0(\Omega) := \mathcal{L}^0(\Omega, \mathcal{F})$ the class of real-valued, measurable functions F on Ω. Note that, as $\mathcal{F} = \sigma(\eta)$, each $F \in \mathcal{L}^0(\Omega)$ can be written as $F = \mathfrak{f}(\eta)$ for some measurable function $\mathfrak{f}$. This $\mathfrak{f}$, called a *representative* of F, is $\mathbb{P}_\eta$-a.s. uniquely defined, where $\mathbb{P}_\eta = \mathbb{P} \circ \eta^{-1}$ is the image measure of $\mathbb{P}$ under η on the space $(\mathbf{N}_\sigma, \mathscr{N}_\sigma)$.

Using a representative $\mathfrak{f}$ of F, we can define the so-called *add-one cost operator* $D^+ = (D_z^+)_{z\in\mathcal{Z}}$ on $\mathcal{L}^0(\Omega)$ (recall that we assume $\mathcal{F} = \sigma(\eta)$) by

$$D_z^+ F := \mathfrak{f}(\eta + \delta_z) - \mathfrak{f}(\eta)\,, \quad z \in \mathcal{Z}; \tag{2.1}$$

similarly, we define D^- on $\mathcal{L}^0(\Omega)$ via

$$D_z^- F := \mathfrak{f}(\eta) - \mathfrak{f}(\eta - \delta_z)\,, \text{ if } z \in \operatorname{supp}(\eta)\,, \text{ and } D_z^- F := 0, \text{ otherwise}, \tag{2.2}$$

where $\operatorname{supp}(\eta)$ stands for the support of the measure η; note that, since η is proper, if $z \in \operatorname{supp}(\eta)$, then $\eta - \delta_z \in \mathbf{N}_\sigma$. Intuitively, $-D^-$ is a *remove-one cost operator*. We stress that the definitions of D^+F and D^-F are, respectively, $\mathbb{P}\otimes\mu$-a.e. and $\mathbb{P}$-a.s. independent of the choice of the representative $\mathfrak{f}$ — see e.g. [LP11, Lemma 2.4] for the case of D^+, whereas the case of D^- can be dealt with by using the Mecke formula (2.4) below. Similarly, the conditions stated in **Assumption A** and **Assumption** $\mathbf{A}^{(\mathrm{loc})}$ do not depend on the choice of the representative $\mathfrak{f}$.

We conclude the section by observing that the operator D^+ can be canonically iterated by setting $D^{(1)} := D^+$ and, for $n \geq 2$ and $z_1, \ldots, z_n \in \mathcal{Z}$ and $F \in \mathcal{L}^0(\Omega)$, by recursively defining

$$D^{(n)}_{z_1,\ldots,z_n} F := D^+_{z_1}\big(D^{(n-1)}_{z_2,\ldots,z_n} F\big).$$

2.2. L^1 **theory: Mecke formula and** Γ_0**.** A central formula in the theory of Poisson processes is the so-called *Mecke formula* from [Mec67] which says that for each measurable function $h : \mathbf{N}_\sigma \times \mathcal{Z} \to [0, +\infty]$ the identity

$$\mathbb{E}\Big[\int_{\mathcal{Z}} h(\eta + \delta_z, z)\mu(dz)\Big] = \mathbb{E}\Big[\int_{\mathcal{Z}} h(\eta, z)\eta(dz)\Big] \tag{2.3}$$

holds true; see [LP17, Chapter 4] for a modern discussion of this fundamental result. We will pervasively use the following consequence of (2.3):

LEMMA **2.1.** *For some integer $d \geq 1$, let $\mathfrak{f}_1, \ldots, \mathfrak{f}_d$ be measurable mappings from $\mathbf{N}_\sigma$ into $[0, +\infty]$, and let $V : [0, +\infty]^{2d} \to [0, +\infty]$ be measurable. Then,*

$$\begin{aligned}\mathbb{E}\Big[\int_{\mathcal{Z}} V(z)\mu(dz)\Big] &:= \mathbb{E}\Big[\int_{\mathcal{Z}} V\big(\mathfrak{f}_1(\eta), \mathfrak{f}_1(\eta + \delta_z), \ldots, \mathfrak{f}_d(\eta), \mathfrak{f}_d(\eta + \delta_z)\big)\mu(dz)\Big] \\ &= \mathbb{E}\Big[\int_{\mathcal{Z}} V\big(\mathfrak{f}_1(\eta - \delta_z), \mathfrak{f}_1(\eta), \ldots, \mathfrak{f}_d(\eta - \delta_z), \mathfrak{f}_d(\eta)\big)\eta(dz)\Big]. \end{aligned} \tag{2.4}$$

Both sides of (2.4) do not change if any of the $\mathfrak{f}_i$, $i = 1, \ldots, d$ is replaced with another measurable mapping $\widehat{\mathfrak{f}}_i$ such that $\mathfrak{f}_i = \widehat{\mathfrak{f}}_i$, a.s.-$\mathbb{P}_\eta$.

Proof. Apply relation (2.3) to the random function

$$\begin{aligned} h(\eta + \delta_z, z) &:= V(z) = V\big(\mathfrak{f}_1(\eta), \mathfrak{f}_1(\eta + \delta_z), \ldots, \mathfrak{f}_d(\eta), \mathfrak{f}_d(\eta + \delta_z)\big) \\ &= V\big(\mathfrak{f}_1(\eta + \delta_z - \delta_z), \mathfrak{f}_1(\eta + \delta_z), \ldots, \mathfrak{f}_d(\eta + \delta_z - \delta_z), \mathfrak{f}_d(\eta + \delta_z)\big)\mathbf{1}_{\{(\eta+\delta_z)(\{z\})\geq 1\}}, \end{aligned}$$

in such a way that

$$h(\eta, z) = V\big(\mathfrak{f}_1(\eta - \delta_z), \mathfrak{f}_1(\eta), \ldots, \mathfrak{f}_d(\eta - \delta_z), \mathfrak{f}_d(\eta)\big)\mathbf{1}_{\{\eta(\{z\})\geq 1\}}.$$

The last sentence in the statement follows from [LP11, Lemma 2.4]. □

REMARK 2.2. Plainly, formulae (2.3) and (2.4) continue to hold when the functions $h(\eta+\delta_z, z)$ and $V\big(z\big)$ are in $L^1(\mathbb{P}\otimes\mu)$, without necessarily having a constant sign.

For random variables $F, G \in \mathcal{L}^0(\Omega)$ such that $D^+F\,D^+G \in L^1(\mathbb{P}\otimes\mu)$, we define

$$\Gamma_0(F,G) := \frac{1}{2}\left\{\int_{\mathcal{Z}} (D_z^+ F D_z^+ G)\,\mu(dz) + \int_{\mathcal{Z}} (D_z^- F D_z^- G)\,\eta(dz)\right\} \tag{2.5}$$

which verifies $\mathbb{E}[|\Gamma_0(F,G)|] < \infty$, and $\mathbb{E}[\Gamma_0(F,G)] = \mathbb{E}[\int_{\mathcal{Z}} (D_z^+ F D_z^+ G)\,\mu(dz)]$, in view of the Mecke formula (2.4). The following statement will play a fundamental role in our work.

LEMMA **2.3** (L^1 integration by parts)**.** *Let $G, H \in \mathcal{L}^0(\Omega)$ be such that*

$$GD^+H,\ D^+G\,D^+H \in L^1(\mathbb{P}\otimes\mu).$$

Then,

$$\mathbb{E}\left[G\left(\int_{\mathcal{Z}} D_z^+ H\,\mu(dz) - \int_{\mathcal{Z}} D_z^- H\,\eta(dz)\right)\right] = -\mathbb{E}[\Gamma_0(G,H)]. \tag{2.6}$$

Proof. The assumptions in the statement imply that $(G + D^+G)D^+H \in L^1(\mathbb{P}\otimes\mu)$. Applying (2.4) and Remark 2.2 to

$$V(z) = \mathfrak{g}(\eta+\delta_z)\{\mathfrak{h}(\eta+\delta_z) - \mathfrak{h}(\eta)\},$$

where $\mathfrak{g}$ and $\mathfrak{h}$ are representatives of G and H, respectively, yields that

$$\mathbb{E}\left[G\int_{\mathcal{Z}} D_z^- H\,\eta(dz)\right] = \mathbb{E}\left[\int_{\mathcal{Z}} (G + D_z^+ G) D_z^+ H\,\mu(dz)\right],$$

which gives immediately the desired conclusion. □

2.3. L^2 **theory, part 1: multiple integrals.** For an integer $p \geq 1$ we denote by $L^2(\mu^p)$ the Hilbert space of all square-integrable and real-valued functions on $\mathcal{Z}^p$ and we write $L_s^2(\mu^p)$ for the subspace of those functions in $L^2(\mu^p)$ which are μ^p-a.e. symmetric. Moreover, for ease of notation, we denote by $\|\cdot\|_2$ and $\langle\cdot,\cdot\rangle_2$ the usual norm and scalar product on $L^2(\mu^p)$ for whatever value of p. We further define $L^2(\mu^0) := \mathbb{R}$. For $f \in L^2(\mu^p)$, we denote by $I_p(f)$ the *multiple Wiener-Itô integral* of f with respect to $\hat{\eta}$. If $p = 0$, then, by convention, $I_0(c) := c$ for each $c \in \mathbb{R}$. We refer to Section 3 of [Las16] for a precise definition and the following basic properties of these integrals in the general framework of a σ-finite measure space $(\mathcal{Z}, \mathscr{Z}, \mu)$. Let $p, q \geq 0$ be integers:

1) $I_p(f) = I_p(\tilde{f})$, where $\tilde{f}$ denotes the *canonical symmetrization* of $f \in L^2(\mu^p)$, i.e. with $\mathbb{S}_p$ the symmetric group acting on $\{1,\dots,p\}$ we have

$$\tilde{f}(z_1,\dots,z_p) = \frac{1}{p!}\sum_{\pi\in\mathbb{S}_p} f(z_{\pi(1)},\dots,z_{\pi(p)}).$$

2) $I_p(f) \in L^2(\mathbb{P})$, and $\mathbb{E}\big[I_p(f)I_q(g)\big] = \delta_{p,q}\,p!\,\langle\tilde{f},\tilde{g}\rangle_2$, where $\delta_{p,q}$ denotes *Kronecker's delta symbol*.

For $p \geq 0$, the Hilbert space consisting of all random variables $I_p(f)$, $f \in L^2(\mu^p)$, is called the *p-th Wiener chaos* associated with η, and is customarily denoted by C_p. It is a crucial fact that every $F \in L^2(\mathbb{P})$ admits a unique representation

$$F = \mathbb{E}[F] + \sum_{p=1}^{\infty} I_p(f_p)\,, \tag{2.7}$$

where $f_p \in L_s^2(\mu^p)$, $p \geq 1$, are suitable symmetric kernel functions, and the series converges in $L^2(\mathbb{P})$. Identity (2.7) is referred to as the *chaotic decomposition* of the functional $F \in L^2(\mathbb{P})$.

From Theorem 2 in [Las16] (which is Theorem 1.3 from the article [LP11]) it is known that, for all $F \in L^2(\mathbb{P})$ and all $p \geq 1$, the kernel f_p in (2.7) is explicitly given by

$$f_p(z_1, \dots, z_p) = \frac{1}{p!}\mathbb{E}\big[D^{(p)}_{z_1,\dots,z_p} F\big]\,, \quad z_1, \dots, z_p \in \mathcal{Z}\,. \tag{2.8}$$

The following new lemma, which relies on (2.8) and whose proof is deferred to Section 6, will be essential for the proof of Theorem 1.3.

Lemma **2.4.** *Let $p, q \geq 1$ be integers and let the multiple Wiener-Itô integrals $F = I_p(f)$ and $G = I_q(g)$ be in $L^4(\mathbb{P})$ and given by symmetric kernels $f \in L^2(\mu^p)$ and $g \in L^2(\mu^q)$, respectively.*

(a) *The product FG has a finite chaotic decomposition of the form $FG = \sum_{r=0}^{p+q} \mathrm{proj}\big\{FG \,\big|\, C_r\big\} = \sum_{r=0}^{p+q} I_r(h_r)$ with symmetric kernels $h_r \in L_s^2(\mu^r)$.*

(b) *The kernel h_{p+q} in* (a) *is explicitly given by $h_{p+q} = f\tilde{\otimes}g$, where $f \otimes g \in L^2(\mu^{p+q})$ denotes the tensor product of f and g defined by*

$$f \otimes g(z_1, \dots, z_{p+q}) = f(z_1, \dots, z_p)g(z_{p+1}, \dots, z_{p+q})$$

and $f\tilde{\otimes}g$ denotes its canonical symmetrization.

Remark 2.5. We stress that the statement of Lemma 2.4 *is not* a direct consequence of the product formula for multiple Wiener-Itô integrals on the Poisson space (see e.g. Proposition 5 in [Las16] and the discussion therein), since such a result assumes the square-integrability of the so-called 'star contractions kernels' $f \star_r^l g$ associated with f and g. It is easily seen that such an integrability property cannot be directly deduced from the minimal assumptions of Lemma 2.4.

2.4. **L^2 theory, part 2: Malliavin operators and carré-du-champ.** We now briefly discuss Malliavin operators on the Poisson space.

(i) The *domain* $\mathrm{dom}\, D$ of the *Malliavin derivative operator* D is the set of all $F \in L^2(\mathbb{P})$ such that the chaotic decomposition (2.7) of F satisfies $\sum_{p=1}^{\infty} p\, p! \|f_p\|_2^2 < \infty$. For such an F, the random function $\mathcal{Z} \ni z \mapsto D_z F \in L^2(\mathbb{P})$ is defined via

$$D_z F = \sum_{p=1}^{\infty} p I_{p-1}\big(f_p(z, \cdot)\big)\,, \tag{2.9}$$

whenever z is such that the series is converging in $L^2(\mathbb{P})$ (this happens a.e.-μ), and set to zero otherwise; note that $f_p(z, \cdot)$ is an a.e. symmetric function on $\mathcal{Z}^{p-1}$. Hence, $DF = (D_z F)_{z \in \mathcal{Z}}$ is indeed an element of $L^2\big(\mathbb{P} \otimes \mu\big)$. It is well-known (see e.g. [PT13, Lemma 3.1]) that, $F \in \operatorname{dom} D$ if and only if $D^+F \in L^2(\mathbb{P} \otimes \mu)$, and in this case

$$D_z F = D_z^+ F, \quad \mathbb{P} \otimes \mu\text{–a.e..} \tag{2.10}$$

(ii) The domain $\operatorname{dom} L$ of the *Ornstein-Uhlenbeck generator* L is the set of those $F \in L^2(\mathbb{P})$ whose chaotic decomposition (2.7) verifies $\sum_{p=1}^{\infty} p^2\, p! \|f_p\|_2^2 < \infty$ (so that $\operatorname{dom} L \subset \operatorname{dom} D$) and, for $F \in \operatorname{dom} L$, one defines

$$LF = -\sum_{p=1}^{\infty} p I_p(f_p)\,. \tag{2.11}$$

By definition, $\mathbb{E}[LF] = 0$; also, from (2.11) it is easy to see that L is *symmetric* in the sense that

$$\mathbb{E}\big[(LF)G\big] = \mathbb{E}\big[F(LG)\big]$$

for all $F, G \in \operatorname{dom} L$ for which these expectations are well-defined. Note that, from (2.11), we conclude that the spectrum of $-L$ is given by the nonnegative integers and that $F \in \operatorname{dom} L$ is an eigenfunction of $-L$ with corresponding eigenvalue p if and only if $F = I_p(f_p)$ for some $f_p \in L_s^2(\mu^p)$, that is:

$$C_p = \operatorname{Ker}(L + pI).$$

For $F \in L^2(\mathbb{P})$ given by (2.7) and $p \in \mathbb{N}_0$ we write

$$\operatorname{proj}\{F \,|\, C_p\} = I_p(f_p)$$

for the projection of F onto C_p, with $f_0 := \mathbb{E}[F]$. The following identity, which corresponds to formula (65) in [Las16], will play an important role in the sequel: if $F \in \operatorname{dom} L$ is such that $D^+F \in L^1(\mathbb{P} \otimes \mu)$, then

$$LF = \int_{\mathcal{Z}} \big(D_z^+ F\big)\mu(dz) - \int_{\mathcal{Z}} \big(D_z^- F\big)\eta(dz)\,. \tag{2.12}$$

(iii) In order to deal with bounds in the Kolmogorov distance, we will also exploit the properties of the *Skohorod integral operator* δ associated with η, which is characterised by the following *duality relation*:

$$\mathbb{E}\big[G\delta(u)\big] = \mathbb{E}\big[\langle DG, u\rangle_{L^2(\mu)}\big] \quad \text{for all } G \in \operatorname{dom} D,\ u \in \operatorname{dom} \delta, \tag{2.13}$$

where $\operatorname{dom} \delta$ stands for its domain (see [Las16, p.14-15]). Recall that the operator δ satisfies the classical identity

$$L = -\delta D, \tag{2.14}$$

that has to be understood in the following sense: $F \in \operatorname{dom} L$ if and only if $F \in \operatorname{dom} D$ and $DF \in \operatorname{dom} \delta$, and in this case $\delta DF = -LF$. Also, if $u(\eta, \cdot) \in L^1(\mathbb{P} \otimes \mu) \cap \operatorname{dom} \delta$, then

$$\delta(u) = \int_{\mathbb{Z}} u(\eta - \delta_z, z)\eta(dz) - \int_{\mathbb{Z}} u(\eta, z)\mu(dz), \quad \text{a.s.-}\mathbb{P}; \tag{2.15}$$

see [Las16, Theorem 6] for a proof of this fact.

(iv) As it is customary in the theory of Markov generators, see e.g. [BGL14], for suitable random variables $F, G \in \operatorname{dom} L$ such that $FG \in \operatorname{dom} L$, we introduce the *carré-du-champ operator* Γ associated with L by

$$\Gamma(F,G) := \frac{1}{2}\big(L(FG) - FLG - GLF\big)\,. \tag{2.16}$$

The symmetry of L implies immediately the crucial *integration by parts formula*

$$\mathbb{E}\big[(LF)G\big] = \mathbb{E}\big[F(LG)\big] = -\mathbb{E}\big[\Gamma(F,G)\big]. \tag{2.17}$$

The connection between (2.17) and (2.6) will be clarified in the discussion to follow.

(v) The domain $\operatorname{dom} L^{-1}$ of the *pseudo-inverse* L^{-1} of L is the class of mean zero elements F of $L^2(\mathbb{P})$. If $F = \sum_{p=1}^\infty I_p(f_p)$ is the chaotic decomposition of F, then L^{-1}F is given by

$$L^{-1}F = -\sum_{p=1}^\infty \frac{1}{p} I_p(f)\,.$$

Note that these definitions imply that $L^{-1}F \in \operatorname{dom} L$ (and therefore $L^{-1}F \in \operatorname{dom} D$), for every $F \in \operatorname{dom} L^{-1}$, and moreover

$$\begin{aligned} &LL^{-1}F = F \quad \text{for all } F \in \operatorname{dom} L^{-1} \quad \text{and} \\ &L^{-1}LF = F - \mathbb{E}[F] \quad \text{for all } F \in \operatorname{dom} L\,. \end{aligned}$$

Using the first of these identities as well as (2.17) we obtain that, for F, G such that G, $G\,L^{-1}(F - \mathbb{E}(F)) \in \operatorname{dom} L$,

$$\begin{aligned} \operatorname{Cov}(F,G) &= \mathbb{E}\big[G\big(F - \mathbb{E}[F]\big)\big] = \mathbb{E}\big[G \cdot LL^{-1}\big(F - \mathbb{E}[F]\big)\big] \\ &= -\mathbb{E}\big[\Gamma\big(G, L^{-1}\big(F - \mathbb{E}[F]\big)\big] \end{aligned} \tag{2.18}$$

In particular, if $F = I_q(f)$ is a multiple integral of order $q \geq 1$ such that $F^2 \in \operatorname{dom}\, L$, then $\mathbb{E}[F] = 0$, $L^{-1}F = -q^{-1}F$ and

$$\operatorname{Var}(F) = \frac{1}{q}\mathbb{E}\big[\Gamma(F,F)\big]\,. \tag{2.19}$$

Note that Lemma 2.4 immediately implies that $F^2 = I_q(f)^2 \in \operatorname{dom} L$ if and only if $F \in L^4(\mathbb{P})$. On the other hand, if $G \in \operatorname{dom} D$ and $GD^+(L^{-1}F)$, $D^+(L^{-1}F) \in L^1(\mathbb{P} \otimes \mu)$, then combining (in order) (2.12), (2.6) and (2.10) yields

$$\operatorname{Cov}(F,G) = \mathbb{E}\big[G \cdot LL^{-1}\big(F - \mathbb{E}[F]\big)\big] = -\mathbb{E}[\Gamma_0(G, L^{-1}\big(F - \mathbb{E}[F]\big))]. \tag{2.20}$$

2.5. **Combining L^1 and L^2 techniques.** The following result provides an explicit representation of the carré-du-champ operator Γ in terms of Γ_0, as introduced in (2.5). Although such a characterization follows quite straightforwardly from the (classical) results and definitions provided above, we were not able to locate it in the existing literature, and we will therefore provide a full proof. It is one of the staples of our approach.

PROPOSITION **2.6.** *For all $F, G \in \operatorname{dom} L$ such that $FG \in \operatorname{dom} L$ and*

$$DF,\ DG,\ FDG,\ GDF \in L^1(\mathbb{P} \otimes \mu),$$

we have that $DF = D^+F$, $DG = D^+G$, in such a way that $DF\, DG = D^+F\, D^+G \in L^1(\mathbb{P} \otimes \mu)$, and

$$\Gamma(F, G) = \Gamma_0(F, G), \tag{2.21}$$

where Γ_0 is defined in (2.5).

In order to prove Proposition 2.6, we state the following lemma which will be exploited in several occasions.

LEMMA **2.7.** (a) *For $F \in \mathcal{L}^0(\Omega)$ and $z \in \mathcal{Z}$ we have the identities*

$$D_z^+ F^2 = \big(D_z^+ F\big)^2 + 2F D_z^+ F \tag{2.22}$$

$$D_z^+ F^3 = \big(D_z^+ F\big)^3 + 3F^2 D_z^+ F + 3F\big(D_z^+ F\big)^2 \tag{2.23}$$

$$D_z^- F^2 = -\big(D_z^- F\big)^2 + 2F D_z^- F \tag{2.24}$$

$$D_z^- F^3 = \big(D_z^- F\big)^3 + 3F^2 D_z^- F - 3F\big(D_z^- F\big)^2 . \tag{2.25}$$

(b) *Let $\psi \in C^1(\mathbb{R})$ be such that ψ' is Lipschitz with minimum Lipschitz-constant $\|\psi''\|_\infty$. Then, for $F \in \mathcal{L}^0(\Omega)$ and $z \in \mathcal{Z}$, there are random quantities $R_\psi^+(F, z)$ and $R_\psi^-(F, z)$ such that*

$$\big|R_\psi^+(F, z)\big| \le \frac{\|\psi''\|_\infty}{2}, \quad \big|R_\psi^-(F, z)\big| \le \frac{\|\psi''\|_\infty}{2}$$

and

$$D_z^+ \psi(F) = \psi'(F) D_z^+ F + R_\psi^+(F, z)\big(D_z^+ F\big)^2 \quad \textit{and}$$
$$D_z^- \psi(F) = \psi'(F) D_z^- F + R_\psi^-(F, z)\big(D_z^- F\big)^2 .$$

Proof. The proof of this result is deferred to Section 6.

□

REMARK 2.8. Note that, by virtue of (2.22) and polarization, for $F, G \in \mathcal{L}^0(\Omega)$ and $z \in \mathcal{Z}$ we also deduce the product rules

$$D_z^+\big(FG\big) = G D_z^+ F + F D_z^+ G + \big(D_z^+ F\big)\big(D_z^+ G\big) \tag{2.26}$$

$$D_z^-\big(FG\big) = G D_z^- F + F D_z^- G - \big(D_z^- F\big)\big(D_z^- G\big) \tag{2.27}$$

If, furthermore, $F, G, FG \in \operatorname{dom} D$, then, from (2.10) we conclude that

$$D_z(FG) = G D_z F + F D_z G + (D_z F)(D_z G)\,, \quad z \in \mathcal{Z}\,, \tag{2.28}$$

for the Malliavin derivative D. Relations (2.26)–(2.27) combined with (2.21) imply that Γ is not a derivation, and confirm the well-known fact that L is not a diffusion operator (see e.g. [BGL14, Definition 1.11.1] for definitions).

Proof of Proposition 2.6. We need only prove (2.21) — as the rest of the assertions in the statement follows from elementary considerations. Since our assumptions imply that $D(FG) \in L^1(\mathbb{P} \otimes \mu)$, we can apply (2.12) in order to deduce that

$$\begin{aligned}2\Gamma(F,G) = LFG - GLF - FLG = &\int_{\mathcal{Z}} D_z^+(FG)\mu(dz) - \int_{\mathcal{Z}} D_z^-(FG)\eta(dz)\\ &- G\int_{\mathcal{Z}} D_z^+F\,\mu(dz) + G\int_{\mathcal{Z}} D_z^-F\,\eta(dz)\\ &- F\int_{\mathcal{Z}} D_z^+G\,\mu(dz) + F\int_{\mathcal{Z}} D_z^-G\,\eta(dz).\end{aligned}$$

Using (2.26) and (2.27) yields immediately the desired formula. □

3. Identities and estimates for multiple integrals

We will now prove several important relations involving multiple stochastic integrals of a fixed order $q \geq 1$. They constitute the backbone of the forthcoming proof of Theorem 1.3.

Lemma **3.1.** *Let $q \geq 1$, and consider a random variable F such that $F = I_q(f) \in C_q = \operatorname{Ker}(L + qI)$ and $\mathbb{E}[F^4] < \infty$. Then, $F, F^2 \in \operatorname{dom} L$, and*

$$\begin{aligned}\operatorname{Var}\big(q^{-1}\Gamma(F,F)\big) &= \sum_{p=1}^{2q-1}\Big(1-\frac{p}{2q}\Big)^2 \operatorname{Var}\big(\operatorname{proj}\{F^2 \,|\, C_p\}\big)\\ &\leq \frac{(2q-1)^2}{4q^2}\big(\mathbb{E}\big[F^4\big] - 3\mathbb{E}[F^2]^2\big). \qquad (3.1)\end{aligned}$$

Moreover, one has also that

$$\frac{1}{q^2}\mathbb{E}[\Gamma(F,F)^2] \leq \mathbb{E}[F^4] \qquad (3.2)$$

$$\frac{1}{q}\mathbb{E}[F^2\Gamma(F,F)] \leq \mathbb{E}[F^4] \qquad (3.3)$$

Proof. From Lemma 2.4, we know that $F^2 = I_q(f)^2$ has a chaos decomposition of the form

$$F^2 = \sum_{p=0}^{2q} \operatorname{proj}\{F^2 \,|\, C_p\} = \mathbb{E}[F^2] + \sum_{p=1}^{2q-1} \operatorname{proj}\{F^2 \,|\, C_p\} + I_{2q}(g_{2g}) \qquad (3.4)$$

with $g_{2q} = f\tilde{\otimes}f$, thus ensuring that F^2 is in the domain of L. By homogeneity, without loss of generality we can assume for the rest of the proof that $\mathbb{E}[F^2] = 1$. As $LF = -qF$, by the definitions of Γ and L we have

$$\begin{aligned}2\Gamma(F,F) = LF^2 - 2FLF &= \sum_{p=1}^{2q} -p\operatorname{proj}\{F^2 \,|\, C_p\} + 2q\sum_{p=0}^{2q}\operatorname{proj}\{F^2 \,|\, C_p\}\\ &= \sum_{p=0}^{2q}(2q-p)\operatorname{proj}\{F^2 \,|\, C_p\}. \qquad (3.5)\end{aligned}$$

By orthogonality, one has that

$$\begin{aligned}\operatorname{Var}\big(q^{-1}\Gamma(F,F)\big) &= \frac{1}{4q^2}\sum_{p=1}^{2q}(2q-p)^2\operatorname{Var}\big(\operatorname{proj}\{F^2 \,|\, C_p\}\big)\\ &= \frac{1}{4q^2}\sum_{p=1}^{2q-1}(2q-p)^2\operatorname{Var}\big(\operatorname{proj}\{F^2 \,|\, C_p\}\big),\end{aligned}$$

proving the first equality in (3.1). For the inequality, first note that from (3.4) and the isometry property of multiple integrals we have

$$\begin{aligned}\mathbb{E}\big[F^4\big]-1=\operatorname{Var}\big(F^2\big) &= \sum_{p=1}^{2q}\operatorname{Var}\big(\operatorname{proj}\{F^2 \,|\, C_p\}\big)\\ &= \sum_{p=1}^{2q-1}\operatorname{Var}\big(\operatorname{proj}\{F^2 \,|\, C_p\}\big)+(2q)!\|f\tilde{\otimes}f\|_2^2\,. \end{aligned}\tag{3.6}$$

Now, identity (5.2.12) in the book [NP12] yields that

$$(2q!)\|f\tilde{\otimes}f\|_2^2 = 2(q!)^2\|f\|_2^4 + D_q\,, \tag{3.7}$$

where $D_q \geq 0$ is a finite non-negative quantity that can be expressed in terms of the contraction kernels associated with F, and whose explicit form is immaterial for the present proof. Also,

$$2(q!)^2\|f\|_2^4 = 2\Big(\mathbb{E}\big[F^2\big]\Big)^2 = 2,$$

and we deduce from (3.6) and (3.7) that

$$\begin{aligned}\frac{(2q-1)^2}{4q^2}\Big(\mathbb{E}\big[F^4\big]-3\Big) &= \frac{(2q-1)^2}{4q^2}\sum_{p=1}^{2q-1}\operatorname{Var}\big(\operatorname{proj}\{F^2 \,|\, C_p\}\big)+\frac{(2q-1)^2}{4q^2}D_q\\ &\geq \frac{(2q-1)^2}{4q^2}\sum_{p=1}^{2q-1}\operatorname{Var}\big(\operatorname{proj}\{F^2 \,|\, C_p\}\big)\\ &\geq \frac{1}{4q^2}\sum_{p=1}^{2q-1}(2q-p)^2\operatorname{Var}\big(\operatorname{proj}\{F^2 \,|\, C_p\}\big)\\ &= \operatorname{Var}\big(q^{-1}\Gamma(F,F)\big)\,,\end{aligned}$$

which is exactly the second estimate in (3.1). Relations (3.2) and (3.3) are immediate consequences of (3.4) and (3.5). □

The following result will allow us to effectively control residual quantities arising from the applicaton of Stein's method on the Poisson space.

LEMMA **3.2.** *Let $q \geq 1$ be an integer and let $F \in L^4(\mathbb{P})$ be an element of the q-th Wiener chaos C_q, such that F verifies* **Assumption A**. *Then,*

$$\frac{1}{2q}\int_{\mathcal{Z}}\mathbb{E}\big[|D_z^+F|^4\big]\mu(dz) = \frac{3}{q}\mathbb{E}\big[F^2\Gamma(F,F)\big]-\mathbb{E}\big[F^4\big] \leq \frac{4q-3}{2q}\Big(\mathbb{E}\big[F^4\big]-3\mathbb{E}[F^2]^2\Big).$$

Proof. Again by homogeneity, we can assume without loss of generality that F has unit variance. Observe that $F \in \operatorname{dom} D$, and therefore $DF = D^+F$ (up to a $\mathbb{P} \otimes \mu$-negligible set), and also, by virtue of Proposition 2.6, one has that $\Gamma(F,F) = \Gamma_0(F,F)$, a.s.-$\mathbb{P}$. It follows that

$$\mathbb{E}\left[F^2 \int_{\mathcal{Z}} (D_z^+ F)^2 \mu(dz)\right] \le 2\mathbb{E}\left[F^2 \Gamma_0(F,F)\right] = 2\mathbb{E}\left[F^2\Gamma(F,F)\right] \le 2q\mathbb{E}[F^4] < \infty,$$

where we have used (3.3), and moreover, by Cauchy-Schwarz,

$$\mathbb{E}\left[|F| \int_{\mathcal{Z}} |D_z^+ F|^3 \mu(dz)\right] \le \mathbb{E}\left[F^2 \int_{\mathcal{Z}} (D_z^+ F)^2 \mu(dz)\right]^{1/2} \mathbb{E}\left[\int_{\mathcal{Z}} (D_z^+ F)^4 \mu(dz)\right]^{1/2} < \infty,$$

so that $F^2(D^+F)^2$, $F(D^+F)^3 \in L^1(\mathbb{P} \otimes \mu)$. Since $LF = -qF$ and $DF \in L^1(\mathbb{P} \otimes \mu)$, one infers from (2.12) that

$$F = -\frac{1}{q}\left(\int_{\mathcal{Z}} (D_z^+ F)\mu(dz) - \int_{\mathcal{Z}} (D_z^- F)\eta(dz)\right).$$

Since the above discussion also implies that $F^3 D^+ F$, $D^+(F^3)D^+F \in L^1(\mathbb{P} \otimes \mu)$ (via (2.23)), we can now exploit the integration by parts relation stated in Lemma 2.3 to deduce that

$$\mathbb{E}\big[F^4\big] = -\frac{1}{q}\mathbb{E}\left[F^3\left(\int_{\mathcal{Z}} (D_z^+ F)\mu(dz) - \int_{\mathcal{Z}} (D_z^- F)\eta(dz)\right)\right] = \frac{1}{q}\mathbb{E}\big[\Gamma_0(F, F^3)\big].$$

Now, using (2.23) and (2.25) we obtain

$$\begin{aligned}
\Gamma_0(F,F^3) &= \frac{1}{2}\bigg(\int_{\mathcal{Z}} D_z^+F\Big((D_z^+F)^3 + 3F^2 D_z^+F + 3F(D_z^+F)^2\Big)\mu(dz) \\
&\quad + \int_{\mathcal{Z}} D_z^-F\Big((D_z^-F)^3 + 3F^2 D_z^-F - 3F(D_z^-F)^2\Big)\eta(dz)\bigg) \\
&= \frac{1}{2}\bigg(\int_{\mathcal{Z}} \Big((D_z^+F)^4 + 3F^2(D_z^+F)^2 + 3F(D_z^+F)^3\Big)\mu(dz) \\
&\quad + \int_{\mathcal{Z}} \Big((D_z^-F)^4 + 3F^2(D_z^-F)^2 - 3F(D_z^-F)^3\Big)\eta(dz)\bigg),
\end{aligned}$$

and we also have

$$3F^2\Gamma_0(F,F) = 3F^2\Gamma(F,F) = \frac{1}{2}\left(\int_{\mathcal{Z}} 3F^2(D_z^+F)^2\mu(dz) + \int_{\mathcal{Z}} 3F^2(D_z^-F)^2\eta(dz)\right).$$

Hence, using the Mecke formula (2.4) (as well as the content of Remark 2.2) in the case

$$V(z) = -\big(\mathfrak{f}(\eta+\delta_z) - \mathfrak{f}(\eta)\big)^4 - 3\mathfrak{f}(\eta)\big(\mathfrak{f}(\eta+\delta_z) - \mathfrak{f}(\eta)\big)^3,$$

where $\mathfrak{f}$ is some representative of F, we can conclude that

$$
\begin{aligned}
\frac{3}{q}\mathbb{E}\big[F^2\Gamma(F,F)\big]-\mathbb{E}\big[F^4\big] &= \frac{1}{2q}\mathbb{E}\bigg[\int_{\mathcal{Z}}\Big(-(D_z^+F)^4-3F(D_z^+F)^3\Big)\mu(dz)\\
&\qquad+\int_{\mathcal{Z}}\Big(-(D_z^-F)^4+3F(D_z^-F)^3\Big)\eta(dz)\bigg]\\
&=\frac{1}{2q}\mathbb{E}\bigg[-2\int_{\mathcal{Z}}(D_z^+F)^4\mu(dz)+3\int_{\mathcal{Z}}(D_z^+F)^3\big(\mathfrak{f}(\eta+\delta_z)-\mathfrak{f}(\eta)\big)\mu(dz)\bigg]\\
&=\frac{1}{2q}\mathbb{E}\bigg[\int_{\mathcal{Z}}(D_z^+F)^4\mu(dz)\bigg].
\end{aligned}
$$

Finally, using relations (3.4) and (3.5) from the proof of Lemma 3.1, we obtain

$$
\begin{aligned}
\frac{1}{q}\int_{\mathcal{Z}}\mathbb{E}\big[|D_z^+F|^4\big]\mu(dz) &= 2\Big(\frac{3}{q}\mathbb{E}\big[F^2\Gamma(F,F)\big]-\mathbb{E}\big[F^4\big]\Big)\\
&=2\bigg(\frac{3}{q}q\big(\mathbb{E}[F^2]\big)^2-\mathbb{E}\big[F^4\big]+\frac{3}{2q}\sum_{p=1}^{2q-1}(2q-p)\operatorname{Var}\big(\operatorname{proj}\{F^2\,|\,C_p\}\big)\bigg)\\
&\le 2\Big(3-\mathbb{E}\big[F^4\big]\Big)+\frac{3(2q-1)}{q}\sum_{p=1}^{2q-1}\operatorname{Var}\big(\operatorname{proj}\{F^2\,|\,C_p\}\big)\\
&\le 2\Big(3-\mathbb{E}\big[F^4\big]\Big)+\frac{3(2q-1)}{q}\Big(\mathbb{E}\big[F^4\big]-3\Big)\\
&=\frac{4q-3}{q}\Big(\mathbb{E}\big[F^4\big]-3\Big),
\end{aligned}
\tag{3.8}
$$

where the last inequality is again a consequence of (3.6) and (3.7). □

We eventually prove an estimate that will be crucial in order to deal with bounds in the Kolmogorov distance.

Lemma **3.3.** *For some fixed $q\ge 1$, let $F\in\operatorname{Ker}(L+qI)$ satisfy both* **Assumption A** *and* **Assumption** $\mathbf{A}^{(\mathrm{loc})}$. *Then,*

$$
0\le\frac{1}{q}\sup_{x\in\mathbb{R}}\mathbb{E}\bigg[\int_{\mathcal{Z}}(D_z^+\mathbf{1}_{\{F>x\}}|D_z^+F|D_z^+F\,\mu(dz)\bigg]\le 10\sqrt{\mathbb{E}[F^4]-3\mathbb{E}[F^2]^2}. \tag{3.9}
$$

Proof. One checks immediately that $D_z^+\mathbf{1}_{\{F>x\}}D_z^+F\ge 0$, so that we need only prove the second inequality in the statement; also, without loss of generality and by homogeneity, we can once again assume that F has unit variance. According to (2.9)–(2.10), we can choose a version of D^+F such that, for μ–almost every $z\in\mathcal{Z}$, the random variable $D_z^+F=D_zF$ is an element of the $(q-1)$th Wiener chaos C_{q-1}. Applying Lemma 3.1 and Lemma 3.2 to every D_zF such that z lies outside the exceptional set, one therefore infers that

$$\begin{aligned} A &:= \int_{\mathcal{Z}} \mathbb{E}\left[\int_{\mathcal{Z}} \big(D_{z_2}D_{z_1}F\big)^4 \mu(dz_2)\right]\mu(dz_1) \\ &= \int_{\mathcal{Z}} \mathbb{E}\left[\int_{\mathcal{Z}} \big(D^+_{z_2}(D^+_{z_1}F)\big)^4 \mu(dz_2)\right]\mu(dz_1) \\ &\le 4(q-1)\mathbb{E}\left[\int_{\mathcal{Z}} (D^+_z F)^4\mu(dz)\right] \le 16q(q-1)\Big(\mathbb{E}\big[F^4\big]-3\Big), \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} B &:= \int_{\mathcal{Z}} \mathbb{E}\left[(D_{z_1}F)^2\int_{\mathcal{Z}} (D_{z_2}D_{z_1}F)^2 \mu(dz_2)\right]\mu(dz_1) \\ &= \int_{\mathcal{Z}} \mathbb{E}\left[(D^+_{z_1}F)^2\int_{\mathcal{Z}} (D^+_{z_2}D^+_{z_1}F)^2 \mu(dz_2)\right]\mu(dz_1) \\ &\le 2(q-1)\int_{\mathcal{Z}} \mathbb{E}\left[(D^+_{z_1}F)^2\Gamma_0(D^+_{z_1}F, D^+_{z_1}F)\right]\mu(dz_1) \\ &= 2(q-1)\int_{\mathcal{Z}} \mathbb{E}\left[(D^+_{z_1}F)^2\Gamma(D^+_{z_1}F, D^+_{z_1}F)\right]\mu(dz_1) \\ &\le 2(q-1)\mathbb{E}\left[\int_{\mathcal{Z}} (D^+_{z_1}F)^4\mu(dz_1)\right] \le 8q(q-1)\Big(\mathbb{E}\big[F^4\big]-3\Big), \end{aligned} \tag{3.11}$$

where we have used twice the fact that, by virtue of Lemma 3.2,

$$C := \int_{\mathcal{Z}} \mathbb{E}\big[|D^+_z F|^4\big]\mu(dz) \le 4q\Big(\mathbb{E}\big[F^4\big]-3\Big). \tag{3.12}$$

Now write $\Phi(a) := a|a|$, $a \in \mathbb{R}$. In view of the inequality (proved e.g. in [PT13, Section 4.2])

$$[D^+_{z_2}\Phi(D^+_{z_1}F)]^2 \le 8(D^+_{z_1}F)^2(D^+_{z_2}D^+_{z_1}F)^2 + 2(D^+_{z_2}D^+_{z_1}F)^4, \tag{3.13}$$

valid μ^2–almost everywhere, we deduce immediately that the process $z \mapsto v(z) := \Phi(D^+_z F)$ is such that $v(z) \in \operatorname{dom} D$ for μ-almost every z, and $v \in \operatorname{dom} \delta$ — this last fact being a a consequence of the classical criterion stated in [Las16, Theorem 5] and of the estimates (3.10)–(3.12), together with the fact that $\mathbb{E}[F^4] < \infty$ by assumption. Also, in view of the fact that $v \in L^1(\mathbb{P}\otimes\mu)$ by assumption, equation (2.15) yields that

$$\delta(v) = \int_{\mathcal{Z}} \Phi(D^-_z F)\eta(dz) - \int_{\mathcal{Z}} \Phi(D^+_z F)\mu(dz).$$

We now fix $x \in \mathbb{R}$. Relation (2.4) applied to the mapping

$$V(z) = \mathbf{1}_{\{\mathfrak{f}(\eta+\delta_z)>x\}}\Phi\big(\mathfrak{f}(\eta+\delta_z)-\mathfrak{f}(\eta)\big),$$

where $\mathfrak{f}$ is a representative of F, yields that

$$\begin{aligned} &\frac{1}{q}\mathbb{E}\left[\int_{\mathcal{Z}} D^+_z\mathbf{1}_{\{F>x\}}|D^+_z F|D^+_z F\,\mu(dz)\right] \\ &= \frac{1}{q}\mathbb{E}\left[\mathbf{1}_{\{F>x\}}\left(\int_{\mathcal{Z}} \Phi(D^-_z F)\eta(dz) - \int_{\mathcal{Z}} \Phi(D^+_z F)\mu(dz)\right)\right] \\ &= \frac{1}{q}\mathbb{E}\left[\mathbf{1}_{\{F>x\}}\delta(v)\right] \le \frac{1}{q}\mathbb{E}\left[\delta(v)^2\right]^{1/2}. \end{aligned}$$

To conclude, we use [Las16, formula (56)] as well as (3.13) to deduce that

$$\begin{aligned}\mathbb{E}\left[\delta(v)^2\right] &\leq \mathbb{E}\left[\int_{\mathcal{Z}} v(z)^2\mu(dz)\right] + \mathbb{E}\left[\int_{\mathcal{Z}}\int_{\mathcal{Z}} (D_y^+ v(z))^2 \mu(dz)\mu(dy)\right] \\ &\leq C + 8B + 2A \leq \big(4q + 64q(q-1) + 32q(q-1)\big)\Big(\mathbb{E}\big[F^4\big] - 3\Big) \\ &\leq 100q^2\Big(\mathbb{E}\big[F^4\big] - 3\Big),\end{aligned}$$

which in turn implies that

$$\frac{1}{q}\mathbb{E}\left[\delta(v)^2\right]^{1/2} \leq 10\sqrt{\mathbb{E}\big[F^4\big] - 3}\,,$$

where A, B, C have been defined above, and where we have used the estimates (3.10)–(3.12). □

4. Proof of Theorem 1.3

In order to prove Theorem 1.3 we have to establish new abstract bounds on the normal approximation of functionals on the Poisson space in the Wasserstein and Kolmogorov distances, respectively. Recall the definition of Γ_0 given in (2.5).

Proposition **4.1.** *Let $F \in \operatorname{dom} D$ be such that $\mathbb{E}[F] = 0$ and let $N \sim \mathscr{N}(0,1)$ be a standard normal random variable. Assume that*

$$D^+(L^{-1}F),\ FD^+(L^{-1}F) \in L^1(\mathbb{P}\otimes\mu). \tag{4.1}$$

Then, we have the bounds

$$d_1(F,N) \leq \sqrt{\frac{2}{\pi}}\mathbb{E}\Big|1 - \Gamma_0\big(F, -L^{-1}F\big)\Big| + \int_{\mathcal{Z}} \mathbb{E}\Big[\big|D_z^+F\big|^2\big|D_z^+L^{-1}F\big|\Big]\mu(dz) \tag{4.2}$$

$$\begin{aligned}&\leq \sqrt{\frac{2}{\pi}}\big|1 - \mathbb{E}[F^2]\big| + \sqrt{\frac{2}{\pi}}\sqrt{\operatorname{Var}\big(\Gamma_0(F,-L^{-1}F)\big)} \\ &\quad + \int_{\mathcal{Z}} \mathbb{E}\Big[\big|D_z^+F\big|^2\big|D_z^+L^{-1}F\big|\Big]\mu(dz)\,.\end{aligned} \tag{4.3}$$

If, furthermore, $F = I_q(f)$ for some $q \geq 1$ and some square-integrable, symmetric kernel f on $\mathcal{Z}^q$ and $\mathbb{E}[F^2] = q!\|f\|_2^2 = 1$, then $-L^{-1}F = q^{-1}F$,

$$\mathbb{E}\big[\Gamma_0(F,-L^{-1}F)\big] = q^{-1}\mathbb{E}\big[\Gamma_0(F,F)\big] = 1 \quad \textit{and}$$

$$\begin{aligned}\int_{\mathcal{Z}} \mathbb{E}\big[|D_z^+F|^2|D_z^+L^{-1}F|\big]\mu(dz) &= q^{-1}\int_{\mathcal{Z}} \mathbb{E}\big[|D_z^+F|^3\big]\mu(dz) \\ &\leq \Big(q^{-1}\int_{\mathcal{Z}} \mathbb{E}\big[|D_z^+F|^4\big]\mu(dz)\Big)^{1/2}\end{aligned}$$

so that the previous estimate (4.3) gives

$$d_1(F,N) \leq \sqrt{\frac{2}{\pi}}\sqrt{\operatorname{Var}\big(q^{-1}\Gamma_0(F,F)\big)} + \frac{1}{\sqrt{q}}\Big(\int_{\mathcal{Z}} \mathbb{E}\big[|D_z^+F|^4\big]\mu(dz)\Big)^{1/2}. \tag{4.4}$$

Remark 4.2. Under the assumptions of Theorem 4.1, we have that $F, L^{-1}F \in \operatorname{dom} D$, in such a way that $\Gamma_0(F, -L^{-1}F)$ is an element ot $L^1(\mathbb{P})$. It follows that the variance $\operatorname{Var}\big(\Gamma_0(F,-L^{-1}F)\big)$ is always well-defined, albeit possibly infinite.

Proof of Proposition 4.1. We apply Stein's method for normal approximation. Define the class $\mathscr{F}_1$ of all continuously differentiable functions ψ on $\mathbb{R}$ such that both ψ and ψ' are Lipschitz-continuous with minimal Lipschitz constants

$$\|\psi'\|_\infty \leq \sqrt{\frac{2}{\pi}} \quad \text{and} \quad \|\psi''\|_\infty \leq 2\,. \tag{4.5}$$

Then, it is well-known (see e.g. Theorem 3 of [BP16b], and the references therein) that

$$d_1(F,N) \leq \sup_{\psi\in\mathscr{F}_1} \bigl|\mathbb{E}\bigl[\psi'(F) - F\psi(F)\bigr]\bigr|\,. \tag{4.6}$$

Let us thus fix $\psi \in \mathscr{F}_1$. The Lipschitz property of ψ implies that $\psi(F) \in \mathrm{dom} D$, whereas the trivial estimate

$$|\psi(F)D^+(L^{-1}F)| \leq \bigl(|\psi(0)| + \sqrt{2/\pi}\,|F|\bigr) \times |D^+(L^{-1}F)|$$

implies that $\psi(F)D^+(L^{-1}F) \in L^1(\mathbb{P}\otimes\mu)$. Using that $\mathbb{E}[F]=0$ we therefore deduce from (2.20) that

$$\mathbb{E}\bigl[F\psi(F)\bigr] = \mathbb{E}\bigl[\psi(F)\cdot LL^{-1}F\bigr] = -\mathbb{E}\bigl[\Gamma_0\bigl(\psi(F),L^{-1}F\bigr)\bigr] \tag{4.7}$$

Now, by the definition of Γ_0 and Lemma 2.7 (b) we obtain that

$$\begin{aligned}
&2\Gamma_0\bigl(\psi(F),L^{-1}F\bigr) = \int_{\mathcal{Z}}\bigl(D_z^+\psi(F)\bigr)\bigl(D_z^+L^{-1}F\bigr)\mu(dz) + \int_{\mathcal{Z}}\bigl(D_z^-\psi(F)\bigr)\bigl(D_z^-L^{-1}F\bigr)\eta(dz)\\
&= \psi'(F)\int_{\mathcal{Z}}\bigl(D_z^+F\bigr)\bigl(D_z^+L^{-1}F\bigr)\mu(dz) + \int_{\mathcal{Z}} R_\psi^+(F,z)\bigl(D_z^+F\bigr)^2\bigl(D_z^+L^{-1}F\bigr)\mu(dz)\\
&+ \psi'(F)\int_{\mathcal{Z}}\bigl(D_z^-F\bigr)\bigl(D_z^-L^{-1}F\bigr)\eta(dz) + \int_{\mathcal{Z}} R_\psi^-(F,z)\bigl(D_z^-F\bigr)^2\bigl(D_z^-L^{-1}F\bigr)\eta(dz)\\
&=: \psi'(F)\int_{\mathcal{Z}}\bigl(D_z^+F\bigr)\bigl(D_z^+L^{-1}F\bigr)\mu(dz) + R_+\\
&+ \psi'(F)\int_{\mathcal{Z}}\bigl(D_z^-F\bigr)\bigl(D_z^-L^{-1}F\bigr)\eta(dz) + R_-\\
&= 2\psi'(F)\Gamma_0(F,L^{-1}F) + R_+ + R_-
\end{aligned} \tag{4.8}$$

with

$$\begin{aligned}
\mathbb{E}|R_+| &\leq \frac{\|\psi''\|_\infty}{2}\mathbb{E}\Bigl[\int_{\mathcal{Z}}\bigl|D_z^+F\bigr|^2\bigl|D_z^+L^{-1}F\bigr|\mu(dz)\Bigr]\\
&\leq \mathbb{E}\Bigl[\int_{\mathcal{Z}}\bigl|D_z^+F\bigr|^2\bigl|D_z^+L^{-1}F\bigr|\mu(dz)\Bigr]
\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}
\mathbb{E}|R_-| &\leq \frac{\|\psi''\|_\infty}{2}\mathbb{E}\Bigl[\int_{\mathcal{Z}}\bigl|D_z^-F\bigr|^2\bigl|D_z^-L^{-1}F\bigr|\eta(dz)\Bigr]\\
&\leq \mathbb{E}\Bigl[\int_{\mathcal{Z}}\bigl|D_z^-F\bigr|^2\bigl|D_z^-L^{-1}F\bigr|\eta(dz)\Bigr]\\
&= \mathbb{E}\Bigl[\int_{\mathcal{Z}}\bigl|D_z^+F\bigr|^2\bigl|D_z^+L^{-1}F\bigr|\mu(dz)\Bigr]\,,
\end{aligned} \tag{4.10}$$

where the last identity holds by virtue of (2.4), as applied to

$$V(z) = \big(\mathfrak{f}(\eta+\delta_z) - \mathfrak{f}(\eta)\big)^2 \big|\mathfrak{f}^*(\eta+\delta_z) - \mathfrak{f}^*(\eta)\big|,$$

where $\mathfrak{f}$ is a representative of F and $\mathfrak{f}^*$ is a representative of $L^{-1}F$. Thus, from (4.7) and (4.8) we infer

$$\mathbb{E}\big[\psi'(F) - F\psi(F)\big] = \mathbb{E}\big[\psi'(F)\big(1 - \Gamma_0\big(F, -L^{-1}F\big)\big] + \frac{1}{2}(\mathbb{E}|R_+| + \mathbb{E}|R_-|)\,, \quad (4.11)$$

and from (4.5), (4.9), (4.10) and (4.11) we conclude that

$$\big|\mathbb{E}\big[\psi'(F) - F\psi(F)\big]\big| \le \sqrt{\frac{2}{\pi}}\mathbb{E}\big|1 - \Gamma_0\big(F, -L^{-1}F\big)\big| + \mathbb{E}\bigg[\int_{\mathcal{Z}} \big|D_z^+F\big|^2 \big|D_z^+L^{-1}F\big|\mu(dz)\bigg].$$

Plugging such an estimate into (4.6) yields (4.2). By (2.20) we know that

$$\mathbb{E}\big[\Gamma_0(F, -L^{-1}F)\big] = \mathrm{Var}(F) = \mathbb{E}[F^2]$$

and, hence, (4.3) follows from (4.2) by using the triangle and Cauchy-Schwarz inequalities. To prove (4.4) we first apply the Cauchy-Schwarz inequality to obtain

$$\int_{\mathcal{Z}} \mathbb{E}\big[|D_z^+F|^3\big]\mu(dz) \le \Big(\int_{\mathcal{Z}} \mathbb{E}\big[|D_z^+F|^4\big]\mu(dz)\Big)^{1/2}\Big(\int_{\mathcal{Z}} \mathbb{E}\big[|D_z^+F|^2\big]\mu(dz)\Big)^{1/2}$$

But, by using the isometry properties of multiple integrals we have

$$\begin{aligned}\int_{\mathcal{Z}} \mathbb{E}\big[|D_z^+F|^2\big]\mu(dz) &= q^2 \int_{\mathcal{Z}} \mathbb{E}\big[I_{q-1}\big(f(z,\cdot)\big)^2\big]\mu(dz) \\ &= q^2(q-1)! \int_{\mathcal{Z}} \|f(z,\cdot)\|_2^2 \mu(dz) = qq!\|f\|_2^2 = q\mathbb{E}[F^2] = q\,. \quad (4.12)\end{aligned}$$

Hence, we obtain

$$q^{-1}\int_{\mathcal{Z}} \mathbb{E}\big[|D_z^+F|^3\big]\mu(dz) \le \frac{1}{\sqrt{q}}\Big(\int_{\mathcal{Z}} \mathbb{E}\big[|D_z^+F|^4\big]\mu(dz)\Big)^{1/2}$$

proving (4.4).

□

The next result provides a similar estimate in the Kolmogorov distance.

Proposition **4.3.** *Under the same assumptions as in Proposition 4.1, one has the bounds*

$$
\begin{aligned}
d_{\mathrm{Kol}}(F,N) \;&\leq\; \mathbb{E}\Big|1-\Gamma_0\big(F,-L^{-1}F\big)\Big| && (4.13)\\
&\quad+\mathbb{E}\left[\big(|F|+\sqrt{2\pi}/4\big)\int_{\mathcal{Z}}(D_z^+F)^2|D_z^+L^{-1}F|\mu(dz)\right]\\
&\quad+\sup_{x\in\mathbb{R}}\mathbb{E}\left[\int_{\mathcal{Z}}(D_z^+F)\,|D_z^+(L^{-1}F)|\,D_z^+\mathbf{1}_{\{F>x\}}\mu(dz)\right]\\
&\leq\; \big|1-\mathbb{E}[F^2]\big|+\sqrt{\mathrm{Var}\big(\Gamma_0(F,-L^{-1}F)\big)} && (4.14)\\
&\quad+\mathbb{E}\left[\left(\int_{\mathcal{Z}}(D_z^+F)^2\mu(dz)\right)^2\right]^{1/4}\big(1+\mathbb{E}[F^4]^{1/4}\big)\\
&\quad\times\sqrt{\mathbb{E}\left[\int_{\mathcal{Z}}(D_z^+F)^2(D_z^+(L^{-1}F))^2\mu(dz)\right]}\\
&\quad+\sup_{x\in\mathbb{R}}\mathbb{E}\left[\int_{\mathcal{Z}}(D_z^+F)\,|D_z^+(L^{-1}F)|\,D^+\mathbf{1}_{\{F>x\}}\mu(dz)\right].
\end{aligned}
$$

If $F=I_q(f)$ for some $q\geq 1$ and some square-integrable, symmetric kernel f on $\mathcal{Z}^q$ and $\mathbb{E}[F^2]=q!\|f\|_2^2=1$, then (4.14) *becomes*

$$
\begin{aligned}
d_{\mathrm{Kol}}(F,N)&\leq\sqrt{\mathrm{Var}\big(q^{-1}\Gamma_0(F,F)\big)}\\
&\quad+\frac{1}{q}\big(1+\mathbb{E}[F^4]^{1/4}\big)\mathbb{E}\left[\left(\int_{\mathcal{Z}}(D_z^+F)^2\mu(dz)\right)^2\right]^{1/4}\sqrt{\mathbb{E}\left[\int_{\mathcal{Z}}(D_z^+F)^4\mu(dz)\right]}\\
&\quad+\frac{1}{q}\sup_{x\in\mathbb{R}}\mathbb{E}\left[\int_{\mathcal{Z}}(D_z^+F)\,|D_z^+F|\,D_z^+\mathbf{1}_{\{F>x\}}\mu(dz)\right]. \qquad (4.15)
\end{aligned}
$$

Proof. Fix $x\in\mathbb{R}$. According to Propositon 6.1, we can write

$$
|\mathbb{P}(F\leq x)-\mathbb{P}(N\leq x)|=|\mathbb{E}[g_x'(F)-Fg_x(F)]|\,,
$$

where g_x is the solution of the Stein equation (6.3) associated with x, whose properties are stated in Proposition 6.1. Using Proposition 6.1 and reasoning as in the proof of Proposition 4.1, one deduces that

$$
\begin{aligned}
&|\mathbb{E}[g_x'(F)-Fg_x(F)]|\\
&\leq\mathbb{E}\Big[|g_x'(F)||1-\Gamma_0(F,-L^{-1}F)|\Big]\\
&\quad+\frac{1}{4}\mathbb{E}\left[\big(|F|+\sqrt{2\pi}/4\big)\int_{\mathcal{Z}}(D_z^+F)^2|D_z^+(L^{-1}F)|\mu(dz)\right]\\
&\quad+\frac{1}{2}\mathbb{E}\left[\int_{\mathcal{Z}}(D_z^+F)|D_z^+(L^{-1}F)|\,D_z^+\mathbf{1}_{\{F>x\}}\mu(dz)\right]\\
&\quad+\frac{3}{4}\mathbb{E}\left[\int_{\mathcal{Z}}\big(|F-D_z^-F|+\sqrt{2\pi}/4\big)\,(D_z^-F)^2|D_z^-(L^{-1}F)|\eta(dz)\right]\\
&\quad+\frac{1}{2}\mathbb{E}\left[\int_{\mathcal{Z}}(D_z^-F)|D_z^-(L^{-1}F)|\,D_z^-\mathbf{1}_{\{F>x\}}\eta(dz)\right].
\end{aligned}
$$

Note that, in order to obtain the previous estimate, one has to use Point (f) and Point (g) in Proposition 6.1, respectively, in order to control the quantities $|D_z^+ g_x(F) - g_x'(F) D_z^+ F|$ and $|D_z^- g_x(F) - g_x'(F) D_z^- F|$. Bound (4.13) can now be deduced by applying (2.4) to the mappings

$$V(z) = \left(|\mathfrak{f}(\eta)| + \sqrt{2\pi}/4\right) \left(\mathfrak{f}(\eta+\delta_z) - \mathfrak{f}(\eta)\right)^2 |\mathfrak{f}^*(\eta+\delta_z) - \mathfrak{f}^*(\eta)|,$$

and

$$V(z) = \mathbf{1}_{\{\mathfrak{f}(\eta+\delta_z)>x\}} \left(\mathfrak{f}(\eta+\delta_z) - \mathfrak{f}(\eta)\right) |\mathfrak{f}^*(\eta+\delta_z) - \mathfrak{f}^*(\eta)|,$$

where $\mathfrak{f}$ and $\mathfrak{f}^*$ are representatives of F and $L^{-1}F$, respectively. The estimate (4.14) can be deduced by applying the Cauchy-Schwarz and triangle inequalities to the middle term of (4.13). The second part of the statement immediately follows from (4.14) and from the fact that, if $F = I_q(f)$, then $-L^{-1}F = q^{-1}F$. □

End of the proof of Theorem 1.3. Since, under **Assumption A**, one has that

$$\Gamma(F,F) = \Gamma_0(F,F), \quad \text{a.s.–}\mathbb{P},$$

the estimate (1.3) is a direct consequence of (4.4), Lemma 3.1 and Lemma 3.2, as well as of elementary simplifications. Similarly, (1.5) follows from (4.15), Lemma 3.1, Lemma 3.2 and Lemma 3.3, combined with the estimate

$$\mathbb{E}\left[\left(\int_{\mathcal{Z}} (D_z^+ F)^2 \mu(dz)\right)^2\right]^{1/4} \leq 4^{1/4} \mathbb{E}[(\Gamma_0(F,F))^2]^{1/4} \leq \sqrt{2q}\, \mathbb{E}[F^4]^{1/4},$$

where we have used (3.2). □

Proof of Proposition 1.6. Fix $q \geq 2$. Reasoning as in [NP05, Corollary 2], if a Gaussian random variable $F := I_q(f) \in C_q$ such that $\mathbb{E}[I_q(f)^2] := c > 0$ existed, then $\mathbb{E}[F^4] - 3c^2 = 0$. Formulae (3.6)–(3.7), together with the explcit form of D_q would therefore imply that $f \otimes_r f = 0$ for every $r = 1, ..., q-1$, where q is the rth contraction of f with itself, as defined in [NP12, Appendix B]. This conclusion contradicts the fact that $c = q!\|f\|_2^2 > 0$. The case $q = 1$ follows immediately from the relation $\mathbb{E}[I_1(f)^4] = 3\|f\|_2^4 + \int_{\mathcal{Z}} f^4 d\mu$. □

5. Proof of Theorem 1.7

We begin by giving the analog of Proposition 4.1 for Gamma approximation.

Proposition **5.1.** *Let $F \in \operatorname{dom} D$ satisfy the same assumptions as in the statement of Proposition 4.1, and let $Z_\nu \sim \bar{\Gamma}(\nu)$ have the centered Gamma distribution with parameter $\nu > 0$. Then, we have the bounds*

$$\begin{aligned} d_2(F, Z_\nu) &\leq \max\Big(1, \frac{2}{\nu}\Big) \mathbb{E}\Big|2(F+\nu) - \Gamma_0\big(F, -L^{-1}F\big)\Big| \\ &\quad + \max\Big(1, \frac{1}{\nu} + \frac{1}{2}\Big) \int_{\mathcal{Z}} \mathbb{E}\Big[\big|D_z^+ F\big|^2 \big|D_z^+ L^{-1} F\big|\Big] \mu(dz) \qquad (5.1) \\ &\leq \max\Big(1, \frac{2}{\nu}\Big) \big|2\nu - \mathbb{E}[F^2]\big| + \max\Big(1, \frac{2}{\nu}\Big) \sqrt{\operatorname{Var}\Big(2F - \Gamma_0\big(F, -L^{-1}F\big)\Big)} \\ &\quad + \max\Big(1, \frac{1}{\nu} + \frac{1}{2}\Big) \int_{\mathcal{Z}} \mathbb{E}\Big[\big|D_z^+ F\big|^2 \big|D_z^+ L^{-1} F\big|\Big] \mu(dz)\,. \qquad (5.2) \end{aligned}$$

If, furthermore, $F = I_q(f)$ for some $q \geq 1$ and some square-integrable, symmetric kernel f on $\mathcal{Z}^q$ and $\mathbb{E}[F^2] = q!\|f\|_2^2 = 2\nu$, then $-L^{-1}F = q^{-1}F$,

$$\mathbb{E}\big[\Gamma(F, -L^{-1}F) - 2F\big] = q^{-1}\mathbb{E}\big[\Gamma_0(F,F)\big] = 2\nu \quad \text{and}$$
$$\int_{\mathcal{Z}} \mathbb{E}\big[|D_z^+F|^2|D_z^+L^{-1}F|\big]\mu(dz) = q^{-1}\int_{\mathcal{Z}} \mathbb{E}\big[|D_z^+F|^3\big]\mu(dz)$$
$$\leq \left(\frac{2\nu}{q}\int_{\mathcal{Z}} \mathbb{E}\big[|D_z^+F|^4\big]\mu(dz)\right)^{1/2}$$

so that the previous estimate (5.2) *can be further bounded to give*

$$d_2(F, Z_\nu) \leq \max\Big(1, \frac{2}{\nu}\Big)\sqrt{\operatorname{Var}\Big(2F - q^{-1}\Gamma_0\big(F,F\big)\Big)}$$
$$+ \max\left(\sqrt{2\nu}, \sqrt{\frac{2}{\nu}} + \sqrt{\frac{\nu}{2}}\right)\left(\frac{1}{q}\int_{\mathcal{Z}} \mathbb{E}\big[|D_z^+F|^4\big]\mu(dz)\right)^{1/2}. \tag{5.3}$$

Proof. Using the recently obtained bounds on the solution to the centered Gamma Stein equation from Theorem 2.3 of [DP16], it is easy to see that

$$d_2(F, Z_\nu) \leq \sup_{\psi \in \mathcal{F}_{2,\nu}} \Big|\mathbb{E}\big[2(F+\nu)\psi'(F) - F\psi(F)\big]\Big|,$$

where $\mathcal{F}_{2,\nu}$ denotes the class of all continuously differentiable functions ψ in $\mathbb{R}$ such that both ψ and ψ' are Lipschitz-continuous with minimum Lipschitz-constants

$$\|\psi'\|_\infty \leq \max\Big(1, \frac{2}{\nu}\Big) \quad \text{and} \quad \|\psi''\|_\infty \leq \max\Big(2, \frac{1}{\nu} + 1\Big).$$

The rest of the argument follows a route that is completely analogous to the one leading to the proof of Proposition 4.1; the details are omitted for the sake of conciseness.

□

Lemma **5.2.** *Let $q \geq 1$ be an integer and and consider a random variable F such that $F = I_q(f) \in C_q = \operatorname{Ker}(L + qI)$, $\mathbb{E}[F^2] = 2\nu$ and $\mathbb{E}[F^4] < \infty$. Then, $F, F^2 \in \operatorname{dom} L$, and*

$$\operatorname{Var}\Big(2F - q^{-1}\Gamma\big(F,F\big)\Big) = \sum_{\substack{1\leq p\leq 2q-1:\\ p\neq q}} \Big(1 - \frac{p}{2q}\Big)^2 \operatorname{Var}\big(\operatorname{proj}\{F^2 \,|\, C_p\}\big)$$
$$+ \frac{1}{4}\operatorname{Var}\Big(\operatorname{proj}\{F^2 \,|\, C_q\} - 4F\Big)$$
$$= \sum_{\substack{1\leq p\leq 2q-1:\\ p\neq q}} \Big(1 - \frac{p}{2q}\Big)^2 \operatorname{Var}\big(\operatorname{proj}\{F^2 \,|\, C_p\}\big)$$
$$+ \frac{1}{4}\operatorname{Var}\big(\operatorname{proj}\{F^2 \,|\, C_q\}\big) + 8\nu - 2\mathbb{E}\big[F^3\big] = V_1 + V_2,$$

where we define

$$V_1 := \sum_{\substack{1\le p\le 2q-1:\\ p\neq q}} \Big(1-\frac{p}{2q}\Big)^2 \operatorname{Var}\big(\operatorname{proj}\{F^2\,|\,C_p\}\big) \quad \text{and} \tag{5.4}$$

$$V_2 := \frac{1}{4}\operatorname{Var}\big(\operatorname{proj}\{F^2\,|\,C_q\}\big) + 8\nu - 2\mathbb{E}\big[F^3\big] = \frac{1}{4}\operatorname{Var}\Big(\operatorname{proj}\{F^2\,|\,C_q\} - 4F\Big). \tag{5.5}$$

Proof. The first identity easily follows from (3.5) and the orthogonality of the chaos decomposition. The second one follows from this and the formula

$$\operatorname{Var}(X+Y) = \operatorname{Var}(X) + \operatorname{Var}(Y) + 2\operatorname{Cov}(X,Y)$$

upon observing that

$$\operatorname{Cov}\Big(\operatorname{proj}\{F^2\,|\,C_q\}\,, -4F\Big) = -4\mathbb{E}\big[F^3\big]\,,$$

again by orthogonality.

□

Lemma **5.3.** *Let $q\geq 1$ be an integer and let $F\in L^4(\mathbb{P})$ be an element of the q-th Wiener chaos C_q, such that F verifies* **Assumption A** *and $\mathbb{E}[F^2]=2\nu$. The following relations are in order:*

$$\begin{aligned}
&\frac{1}{6q}\Big(\mathbb{E}\big[F^4\big] - 12\mathbb{E}\big[F^3\big] - 12\nu^2 + 48\nu\Big) + \frac{1}{12q^2}\int_{\mathcal{Z}} \mathbb{E}\big[|D_z^+F|^4\big]\mu(dz)\\
&\le \operatorname{Var}\Big(2F - q^{-1}\Gamma\big(F,F\big)\Big) = \operatorname{Var}\Big(2F - q^{-1}\Gamma_0\big(F,F\big)\Big)\\
&\le \frac{1}{3}\Big(\mathbb{E}\big[F^4\big] - 12\mathbb{E}\big[F^3\big] - 12\nu^2 + 48\nu\Big) + \frac{1}{6q}\int_{\mathcal{Z}} \mathbb{E}\big[|D_z^+F|^4\big]\mu(dz)
\end{aligned}$$

Proof. Recall that, under the assumptions in the statement, $\Gamma(F,F) = \Gamma_0(F,F)$. Using orthogonality, from Lemma 3.2 and (3.5) we obtain

$$\begin{aligned}
\mathbb{E}\big[F^4\big] &= \frac{3}{q}\mathbb{E}\big[F^2\Gamma(F,F)\big] - \frac{1}{2q}\int_{\mathcal{Z}} \mathbb{E}\big[|D_z^+F|^4\big]\mu(dz)\\
&= 3\big(\mathbb{E}[F^2]\big)^2 + 3\sum_{p=1}^{2q}\Big(1-\frac{p}{2q}\Big)\operatorname{Var}\big(\operatorname{proj}\{F^2\,|\,C_p\}\big) - \frac{1}{2q}\int_{\mathcal{Z}} \mathbb{E}\big[|D_z^+F|^4\big]\mu(dz)\\
&= 12\nu^2 + 3\sum_{\substack{1\le p\le 2q-1:\\ p\neq q}}\Big(1-\frac{p}{2q}\Big)\operatorname{Var}\big(\operatorname{proj}\{F^2\,|\,C_p\}\big) + \frac{3}{2}\operatorname{Var}\big(\operatorname{proj}\{F^2\,|\,C_q\}\big)\\
&\quad - \frac{1}{2q}\int_{\mathcal{Z}} \mathbb{E}\big[|D_z^+F|^4\big]\mu(dz)\,.
\end{aligned}$$

Hence, recalling the definition of V_2 in (5.5) we conclude from Lemma 5.2 that

$$\begin{aligned}
&\mathbb{E}\big[F^4\big] - 12\mathbb{E}\big[F^3\big] - 12\nu^2 + 48\nu \\
&= 3 \sum_{\substack{1\leq p\leq 2q-1:\\ p\neq q}} \Big(1-\frac{p}{2q}\Big) \operatorname{Var}\big(\operatorname{proj}\{F^2 \,|\, C_p\}\big) - \frac{1}{2q}\int_{\mathcal{Z}} \mathbb{E}\big[|D_z^+F|^4\big]\mu(dz) \\
&\quad + \frac{3}{2}\operatorname{Var}\big(\operatorname{proj}\{F^2 \,|\, C_q\}\big) - 12\mathbb{E}\big[F^3\big] + 48\nu \\
&= 3 \sum_{\substack{1\leq p\leq 2q-1:\\ p\neq q}} \Big(1-\frac{p}{2q}\Big) \operatorname{Var}\big(\operatorname{proj}\{F^2 \,|\, C_p\}\big) + 6V_2 - \frac{1}{2q}\int_{\mathcal{Z}} \mathbb{E}\big[|D_z^+F|^4\big]\mu(dz)\,.
\end{aligned}$$

Now, recalling also the definition (5.4) of V_1 and using the simple chain of inequalities

$$\Big(1-\frac{p}{2q}\Big)^2 \leq \Big(1-\frac{p}{2q}\Big) \leq 2q\Big(1-\frac{p}{2q}\Big)^2, \quad 1\leq p\leq 2q-1\,,$$

we obtain on the one hand that

$$\begin{aligned}
&\mathbb{E}\big[F^4\big] - 12\mathbb{E}\big[F^3\big] - 12\nu^2 + 48\nu \\
&\geq 3 \sum_{\substack{1\leq p\leq 2q-1:\\ p\neq q}} \Big(1-\frac{p}{2q}\Big)^2 \operatorname{Var}\big(\operatorname{proj}\{F^2 \,|\, C_p\}\big) + 6V_2 - \frac{1}{2q}\int_{\mathcal{Z}} \mathbb{E}\big[|D_z^+F|^4\big]\mu(dz) \\
&= 3V_1 + 6V_2 - \frac{1}{2q}\int_{\mathcal{Z}} \mathbb{E}\big[|D_z^+F|^4\big]\mu(dz) \\
&\geq 3\operatorname{Var}\Big(2F - q^{-1}\Gamma\big(F,F\big)\Big) - \frac{1}{2q}\int_{\mathcal{Z}} \mathbb{E}\big[|D_z^+F|^4\big]\mu(dz)\,, \qquad (5.6)
\end{aligned}$$

and, on the other hand,

$$\begin{aligned}
&\mathbb{E}\big[F^4\big] - 12\mathbb{E}\big[F^3\big] - 12\nu^2 + 48\nu \\
&\leq 6q \sum_{\substack{1\leq p\leq 2q-1:\\ p\neq q}} \Big(1-\frac{p}{2q}\Big)^2 \operatorname{Var}\big(\operatorname{proj}\{F^2 \,|\, C_p\}\big) + 6V_2 - \frac{1}{2q}\int_{\mathcal{Z}} \mathbb{E}\big[|D_z^+F|^4\big]\mu(dz) \\
&\leq 6q\operatorname{Var}\Big(2F - q^{-1}\Gamma\big(F,F\big)\Big) - \frac{1}{2q}\int_{\mathcal{Z}} \mathbb{E}\big[|D_z^+F|^4\big]\mu(dz)\,. \qquad (5.7)
\end{aligned}$$

The statement of the Lemma now follows from (5.6) and (5.7).

□

End of the proof of Theorem 1.7. The claim of Theorem 1.7 is now an immediate consequence of the bound (5.3) and of the upper bound given in Lemma 5.3. □

6. Proofs of technical lemmas

6.1. **Proof of Lemma 2.7.** We first prove part (a). We just prove (2.22) and (2.23), since the derivation of (2.24) and (2.25) is very similar. Let f be a representative for F, i.e. $F = f(\eta)$. Then, by the binomial identity, we have

$$\begin{aligned}
\big(D_z^+F\big)^2 &= \big(f(\eta+\delta_z) - f(\eta)\big)^2 = f(\eta+\delta_z)^2 - f(\eta)^2 - 2f(\eta+\delta_z)f(\eta) + 2f(\eta)^2 \\
&= D_z^+F^2 - 2f(\eta)\big(f(\eta+\delta_z) - f(\eta)\big) = D_z^+F^2 - 2FD_z^+F
\end{aligned}$$

such that (2.22) holds true. Similarly, using (2.22), we obtain

$$\begin{aligned}\big(D_z^+F\big)^3 &= \big(f(\eta+\delta_z)-f(\eta)\big)^3 = f(\eta+\delta_z)^3 - f(\eta)^3 - 3f(\eta+\delta_z)^2 f(\eta)\\ &\quad + 3f(\eta+\delta_z)f(\eta)^2\\ &= D_z^+F^3 + 3f(\eta)^2\big(f(\eta+\delta_z)-f(\eta)\big) - 3f(\eta)\big(f(\eta+\delta_z)^2 - f(\eta)^2\big)\\ &= D_z^+F^3 + 3F^2D_z^+F - 3FD_z^+F^2\\ &= D_z^+F^3 + 3F^2D_z^+F - 3F\big(D_z^+F\big)^2 - 6F^2D_z^+F\\ &= D_z^+F^3 - 3F^2D_z^+F - 3F\big(D_z^+F\big)^2\end{aligned}$$

which is equivalent to (2.23). Now we turn to the proof of (b). Again, we just prove the part involving D_z^+. By a suitable version of Taylor's formula, for $x, y \in \mathbb{R}$ we have

$$\psi(y) = \psi(x) + \psi'(x)(y-x) + R_\psi(x,y)(y-x)^2\,,$$

where

$$\big|R_\psi(x,y)\big| \le \frac{\|\psi''\|_\infty}{2}\,.$$

Now the result follows by letting $x = F = f(\eta)$, $y = f(\eta+\delta_z)$ and $R_\psi^+(F,z) = R_\psi(f(\eta), f(\eta+\delta_z))$.

□

6.2. **Proof of Lemma 2.4.** The method of proof we apply is similar to the one used for the proof of Proposition 5 in [Las16], which gives the product formula for multiple Wiener-Itô integrals. Let

$$FG = \mathbb{E}[FG] + \sum_{m=1}^{\infty} I_m(h_m)$$

denote the chaos decomposition of FG. We prove (a) and (b) simultaneously by induction on $k := p+q \ge 2$. If $k=2$, then necessarily $p=q=1$ and, by (2.28), for all $y, z \in \mathcal{Z}$ we have

$$\begin{aligned}D_z(FG) &= f(z)I_1(g) + g(z)I_1(f) + f(z)g(z) \quad \text{and}\\ D^{(2)}_{y,z}(FG) &= f(z)g(y) + f(y)g(z) = 2f\tilde{\otimes}g(y,z)\,.\end{aligned}$$

This immediately implies that $D^{(m)}(FG) = 0$ for all $m > 2$. From (2.8) we thus infer that

$$\begin{aligned}h_2(z_1,z_2) &= \frac{1}{2}\mathbb{E}\big[D^{(2)}_{z_1,z_2}(FG)\big] = f\tilde{\otimes}g(z_1,z_2) \quad \text{and}\\ h_m(z_1,\dots,z_m) &= \frac{1}{m!}\mathbb{E}\big[D^{(m)}_{z_1,\dots,z_m}(FG)\big] = 0\end{aligned}$$

for all $m > 2$ and $z_1,\dots,z_m \in \mathcal{Z}$. Now assume that $k > 2$. Then, again from (2.28) we have that

$$\begin{aligned}D_{z_k}(FG) &= pI_q(g)I_{p-1}\big(f(z_k,\cdot)\big) + qI_p(f)I_{q-1}\big(g(z_k,\cdot)\big)\\ &\quad + pqI_{p-1}\big(f(z_k,\cdot)\big)I_{q-1}\big(g(z_k,\cdot)\big)\\ &=: p\tilde{F}_{z_k}G + q\tilde{G}_{z_k}F + pq\tilde{F}_{z_k}\tilde{G}_{z_k} \qquad (6.1)\end{aligned}$$

holds for all $z_k \in \mathcal{Z}$, where $\tilde{F}_{z_k}$ and $\tilde{G}_{z_k}$ are multiple integrals of orders $p-1$ and $q-1$, respectively. Hence, by the induction hypothesis for claim (b) we already conclude that

$$\mathbb{E}\big[D^{(k-1)}_{z_1,\dots,z_{k-1}}\big(\tilde{F}_{z_k}\tilde{G}_{z_k}\big)\big] = 0\,.$$

so that

$$\mathbb{E}\big[D^{(k)}_{z_1,\dots,z_k}(FG)\big] = p\mathbb{E}\Big[D^{(k-1)}_{z_1,\dots,z_{k-1}}\big(\tilde{F}_{z_k}G\big)\Big] + q\mathbb{E}\Big[D^{(k-1)}_{z_1,\dots,z_{k-1}}\big(F\tilde{G}_{z_k}\big)\Big]\,.$$

By the induction hypothesis for claim (a) we have

$$\begin{aligned}\mathbb{E}\Big[D^{(k-1)}_{z_1,\dots,z_{k-1}}\big(\tilde{F}_{z_k}G\big)\Big] &= (k-1)!\big(f(z_k,\cdot)\tilde{\otimes}g\big)(z_1,\dots,z_{k-1}) \quad \text{and}\\ \mathbb{E}\Big[D^{(k-1)}_{z_1,\dots,z_{k-1}}\big(F\tilde{G}_{z_k}\big)\Big] &= (k-1)!\big(f\tilde{\otimes}\big(g(z_k,\cdot)\big)(z_1,\dots,z_{k-1})\,.\end{aligned}$$

and, in order to prove (a), it remains to show that

$$\begin{aligned}k!(f\tilde{\otimes}g)(z_1,\dots,z_k) &= p(k-1)!\big(f(z_k,\cdot)\tilde{\otimes}g\big)(z_1,\dots,z_{k-1})\\ &\quad + q(k-1)!\big(f\tilde{\otimes}\big(g(z_k,\cdot)\big)(z_1,\dots,z_{k-1})\,.\end{aligned} \tag{6.2}$$

This, however, follows from

$$\begin{aligned}k!(f\tilde{\otimes}g)(z_1,\dots,z_k) &= \sum_{\pi\in\mathbb{S}_{p+q}} f\big(z_{\pi(1)},\dots,z_{\pi(p)}\big)g\big(z_{\pi(p+1)},\dots,z_{\pi(p+q)}\big)\\ &= \sum_{\pi:k\in\{\pi(1),\dots,\pi(p)\}} f\big(z_{\pi(1)},\dots,z_{\pi(p)}\big)g\big(z_{\pi(p+1)},\dots,z_{\pi(p+q)}\big)\\ &\quad + \sum_{\pi:k\notin\{\pi(1),\dots,\pi(p)\}} f\big(z_{\pi(1)},\dots,z_{\pi(p)}\big)g\big(z_{\pi(p+1)},\dots,z_{\pi(p+q)}\big)\\ &\overset{!}{=} p\sum_{\tau\in\mathbb{S}_{p+q-1}} f\big(z_k,z_{\tau(1)},\dots,z_{\tau(p-1)}\big)g\big(z_{\tau(p)},\dots,z_{\tau(p+q-1)}\big)\\ &\quad + q\sum_{\tau\in\mathbb{S}_{p+q-1}} f\big(z_{\tau(1)},\dots,z_{\tau(p)}\big)g\big(z_{\tau(p+1)},\dots,z_{\tau(p+q-1)},z_k\big)\\ &= p(k-1)!\big(f(z_k,\cdot)\tilde{\otimes}g\big)(z_1,\dots,z_{k-1})\\ &\quad + q(k-1)!\big(f\tilde{\otimes}\big(g(z_k,\cdot)\big)(z_1,\dots,z_{k-1})\,.\end{aligned}$$

We explain the identity involving ! in some more detail. Consider the first sum appearing there and note that

$$\begin{aligned}&\sum_{\pi:k\in\{\pi(1),\dots,\pi(p)\}} f\big(z_{\pi(1)},\dots,z_{\pi(p)}\big)g\big(z_{\pi(p+1)},\dots,z_{\pi(p+q)}\big)\\ &= \sum_{j=1}^{p}\sum_{\pi:\pi(j)=k} f\big(z_{\pi(1)},\dots,z_{\pi(j-1)},z_k,z_{\pi(j+1)},\dots,z_{\pi(p)}\big)g\big(z_{\pi(p+1)},\dots,z_{\pi(p+q)}\big)\\ &= p\sum_{\pi:\pi(1)=k} f\big(z_k,z_{\pi(2)},\dots,z_{\pi(p)}\big)g\big(z_{\pi(p+1)},\dots,z_{\pi(p+q)}\big)\end{aligned}$$

where we have used the symmetry of the kernel f to obtain the last identity. Now, since the mapping

$$\Psi : \mathbb{S}_{k-1} \to \{\pi \in \mathbb{S}_k \,:\, \pi(1) = k\}\,, \quad \Psi(\sigma)(j) := \begin{cases} k\,, & j = 1 \\ \sigma(j-1)\,, & j \in \{2, \dots, k\} \end{cases}$$

is a bijection, we obtain that

$$\sum_{\pi:\pi(1)=k} f\big(z_k, z_{\pi(2)}, \dots, z_{\pi(p)}\big) g\big(z_{\pi(p+1)}, \dots, z_{\pi(p+q)}\big)$$
$$= \sum_{\tau \in \mathbb{S}_{p+q-1}} f\big(z_k, z_{\tau(1)}, \dots, z_{\tau(p-1)}\big) g\big(z_{\tau(p)}, \dots, z_{\tau(p+q-1)}\big)$$

proving the claim. Thus, we have proved (a).

If $m > k$ and $z_1, \dots, z_m \in \mathcal{Z}$, then, by the induction hypothesis on (b) and from (6.1) we obtain

$$\begin{aligned} m! h(z_1, \dots, z_m) &= \mathbb{E}\big[D^{(m)}_{z_1,\dots,z_m}(FG)\big] \\ &= p\mathbb{E}\Big[D^{(m-1)}_{z_1,\dots,z_{m-1}}\big(\tilde{F}_{z_m} G\big)\Big] + q\mathbb{E}\Big[D^{(m-1)}_{z_1,\dots,z_{m-1}}\big(F \tilde{G}_{z_m}\big)\Big] + pq\mathbb{E}\Big[D^{(m-1)}_{z_1,\dots,z_{m-1}}\big(\tilde{F}_{z_m} \tilde{G}_{z_m}\big)\Big] \\ &= 0 \end{aligned}$$

for all $z_1, \dots, z_m \in \mathcal{Z}$, proving (b). $\square$

6.3. **Stein's equation in the Kolmogorov distance.** In order to deal with bounds in the Kolmogorov distance involving remove-one cost operators, we need the following result, containing several estimates on the solution of the Stein's equation associated with test functions having the form of indicators of half-lines. Points (a)-(f) are well-known. Point (g) is standard but not explicitly stated in the literature (to our knowledge) — a proof is provided for the sake of completeness.

PROPOSITION **6.1.** *Let $N \sim N(0,1)$ be a centred Gaussian random variable with unit variance and, for every $x \in \mathbb{R}$, introduce the Stein's equation*

$$g'(w) - wg(w) = \mathbf{1}_{\{w \leqslant x\}} - \mathbb{P}(N \leqslant x), \tag{6.3}$$

where $w \in \mathbb{R}$. Then, for every real x, there exists a function $g_x : \mathbb{R} \to \mathbb{R} : w \mapsto g_x(w)$ satisfying the following properties (a)-(g)*:*

(a) *g_x is continuous at every point $w \in \mathbb{R}$, and infinitely differentiable at every $w \neq x$;*
(b) *g_x satisfies the relation* (6.3)*, for every $w \neq x$;*
(c) $0 < g_x \leqslant \frac{\sqrt{2\pi}}{4}$*;*
(d) *for every $u, v, w \in \mathbb{R}$,*

$$|(w+u)g_x(w+u) - (w+v)g_x(w+v)| \leqslant \left(|w| + \frac{\sqrt{2\pi}}{4}\right)(|u| + |v|)\,; \tag{6.4}$$

(e) *adopting the convention*

$$g_x'(x) := xg_x(x) + 1 - \mathbb{P}(N \leqslant x), \tag{6.5}$$

one has that $|g_x'(w)| \leqslant 1$, for every real w ;

(f) *using again the convention* (6.5), *for all* $w, h \in \mathbb{R}$ *one has that*

$$|g_x(w+h) - g_x(w) - g_x'(w)h| \leqslant \frac{|h|^2}{2}\left(|w| + \frac{\sqrt{2\pi}}{4}\right) + |h|(\mathbf{1}_{[w,w+h)}(x) + \mathbf{1}_{[w+h,w)}(x)) \tag{6.6}$$

$$= \frac{|h|^2}{2}\left(|w| + \frac{\sqrt{2\pi}}{4}\right) + h\left(\mathbf{1}_{[w,w+h)}(x) - \mathbf{1}_{[w+h,w)}(x)\right); \tag{6.7}$$

(g) *under* (6.5), *for every* $w, h \in \mathbb{R}$ *one has that*

$$|g_x(w) - g_x(w-h) - g_x'(w)h| \leqslant \frac{3|h|^2}{2}\left(|w-h| + \frac{\sqrt{2\pi}}{4}\right) + |h|(\mathbf{1}_{[w-h,w)}(x) + \mathbf{1}_{[w,w-h)}(x)) \tag{6.8}$$

$$= \frac{3|h|^2}{2}\left(|w-h| + \frac{\sqrt{2\pi}}{4}\right) + h\left(\mathbf{1}_{[w-h,w)}(x) - \mathbf{1}_{[w,w-h)}(x)\right). \tag{6.9}$$

Proof. The content of Points (a)–(f) is well-known – see e.g. [BP16b, Section 2.2.2] and the references therein. To show (g), fix $t \in \mathbb{R}$, recall (6.5) and write, for every $w, h \in \mathbb{R}$,

$$g_x(w) - g_x(w-h) - hg_t'(w) = \int_0^h (g_x'(w-h+u) - g_x'(w))\, du.$$

Since g_x is a solution of (6.3) for every real w, we have that, for all $w, h \in \mathbb{R}$,

$$\begin{aligned} &g_x(w) - g_x(w-h) - hg_x'x(w) \\ &= \int_0^h ((w-h+u)g_x(w-h+u) - wg_x(w))\, du + \int_0^h \left(\mathbf{1}_{\{w-h+u \leqslant x\}} - \mathbf{1}_{\{w \leqslant x\}}\right) du \\ &:= J_1 + J_2. \end{aligned}$$

It follows that, by the triangle inequality,

$$|g_x(w) - g_x(w-h) - hg_x'(x)| \leqslant |J_1| + |J_2|. \tag{6.10}$$

Using (6.4), we have

$$|J_1| \leqslant \int_0^h \left(|w-h| + \frac{\sqrt{2\pi}}{4}\right)(|u| + |h|)du = \frac{3h^2}{2}\left(|w-h| + \frac{\sqrt{2\pi}}{4}\right). \tag{6.11}$$

On the other hand, we have that

$$\begin{aligned} |J_2| &= \mathbf{1}_{\{h<0\}}\left|\int_0^h \left(\mathbf{1}_{\{w-h+u \leqslant x\}} - \mathbf{1}_{\{w \leqslant x\}}\right) du\right| \\ &\quad + \mathbf{1}_{\{h \geq 0\}}\left|\int_0^h \left(\mathbf{1}_{\{w-h+u \leqslant x\}} - \mathbf{1}_{\{w \leqslant x\}}\right) du\right| \\ &= \mathbf{1}_{\{h<0\}}\int_h^0 \mathbf{1}_{\{w \leqslant x < w-h+u\}} du + \mathbf{1}_{\{h \geq 0\}}\int_0^h \mathbf{1}_{\{w-h+u \leqslant x < w\}} du. \end{aligned}$$

As a consequence,

$$\begin{aligned} |J_2| &\leqslant \mathbf{1}_{\{h<0\}}(-h)\mathbf{1}_{[w,w-h)}(x) + \mathbf{1}_{\{h\geq 0\}} h \mathbf{1}_{[w-h,w)}(x) \\ &= h\left(\mathbf{1}_{[w-h,w)}(x) - \mathbf{1}_{[w,w-h)}(x)\right) = |h| \left(\mathbf{1}_{[w-h,w)}(x) + \mathbf{1}_{[w,w-h)}(x)\right). \end{aligned} \tag{6.12}$$

Using (6.11) and (6.12) in (6.10) yields the conclusion. □

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