Analysis of Semiblind Channel Estimation for MIMO Systems

Tianbin Wo, Peter Adam Hoeher[1]
Ansgar Scherb, Karl-Dirk Kammeyer[2]

[1] Information and Coding Theory Lab (ICT),
University of Kiel, Germany
{wtb,ph}@tf.uni-kiel.de

[2] Department of Communications Engineering (ANT),
University of Bremen, Germany
{scherb,kammeyer}@ant.uni-bremen.de

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Overview

1. Motivation
2. Channel Model
3. Semiblind Channel Estimation
4. Cramer-Rao Lower Bound
5. Conclusion

Motivation

- Training-based channel estimation (TBCE)

$$\text{channel estimation quality} \xleftrightarrow{\text{trade-off}} \text{system throughput}$$

(bit error rate) — (bandwidth efficiency)

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$$\text{system throughput maximized} \xrightarrow{\text{but}} \text{ambiguity (permutation, phase) and local minima}$$

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- Semiblind channel estimation (SBCE)

training-based channel estimation + blind techniques

$\rightsquigarrow$ achieve high system throughput without sacrificing robustness

Channel Model

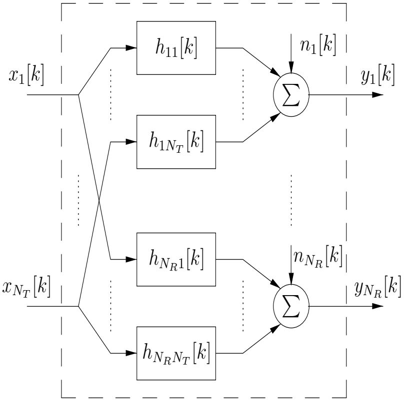
- Equivalent discrete-time channel model:

$$y_i[k] = \sum_{j=1}^{N_T} h_{ij}[k] \cdot x_j[k] + n_i[k],$$

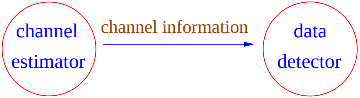
with $1 \leq i \leq N_R$

- Within a channel coherence interval, channel model can be rewritten in matrix form:

$$\mathbf{Y} = \mathbf{H} \cdot \mathbf{X} + \mathbf{N}$$



- Task: Given the observation $\mathbf{Y} = \mathbf{HX} + \mathbf{N}$ and the knowledge of training, $\mathbf{X}$ shall be detected
- Solution 1: estimate $\mathbf{H}$ first by training, then detect $\mathbf{X}$



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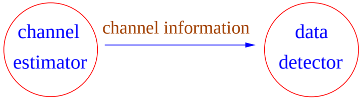
- Solution 1: estimate $\mathbf{H}$ first by training, then detect $\mathbf{X}$
- Solution 2: estimate $\mathbf{H}$ and $\mathbf{X}$ iteratively



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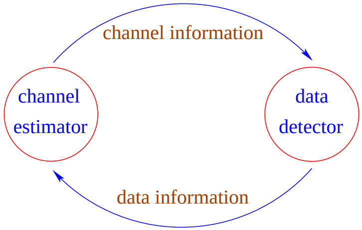
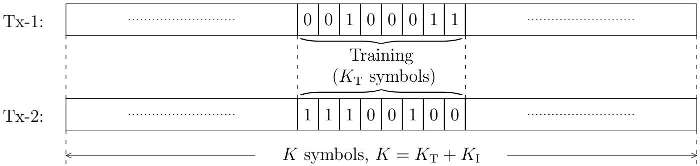
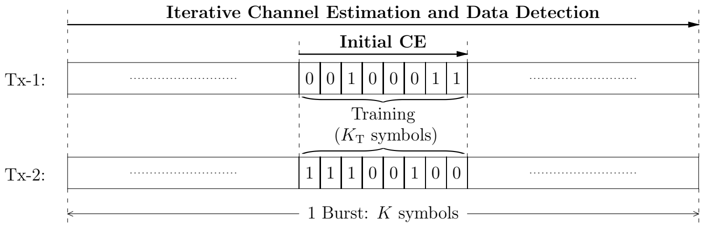


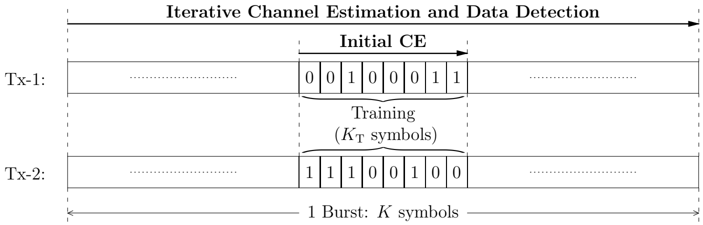
Solution 2



- One burst: K_T training symbols, K_I data symbols, $K = K_\mathsf{T} + K_\mathsf{I}$

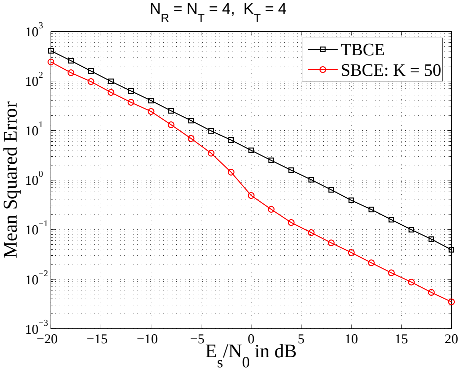


- Algorithm:
  1. Initial channel estimation by using the training only
  2. ∗ Given channel knowledge, perform data detection
     ∗ Given data decisions, perform channel estimation by taking the whole burst as a virtual training
  3. Repeat step 2 until a certain stopping criterion is reached



- ML-SBCE:
  1. $i = 0$: $\widehat{\mathbf{H}}_0 = \mathbf{Y}_\mathsf{T} \cdot \mathbf{X}_\mathsf{T}^\dagger$ ("i" denotes the iteration index)
  2. $i = i + 1$
     - $\widehat{\mathbf{X}}_i = \arg\min_{\widetilde{\mathbf{X}} \in \mathcal{X}} \left\{ \|\mathbf{Y} - \widehat{\mathbf{H}}_{i-1} \widetilde{\mathbf{X}}\|_\mathsf{F}^2 \right\}$ "ML data detection"
     - $\widehat{\mathbf{H}}_i = \mathbf{Y} \cdot \widehat{\mathbf{X}}_i^\dagger$ "LS channel estimation"
  3. Repeat 2 until $\left(\widehat{\mathbf{H}}_i, \widehat{\mathbf{X}}_i\right) = \left(\widehat{\mathbf{H}}_{i-1}, \widehat{\mathbf{X}}_{i-1}\right)$

- TBCE: training-based CE,
  SBCE: semiblind CE
- MSE curves are linear
  at high and low SNR
- Crossover SNR range
- longer burst,
  larger performance improvement



Semiblind Channel Estimation
Performance

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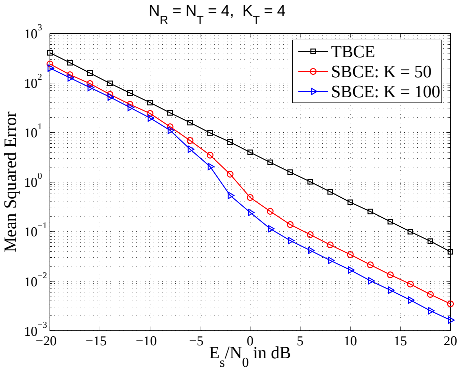

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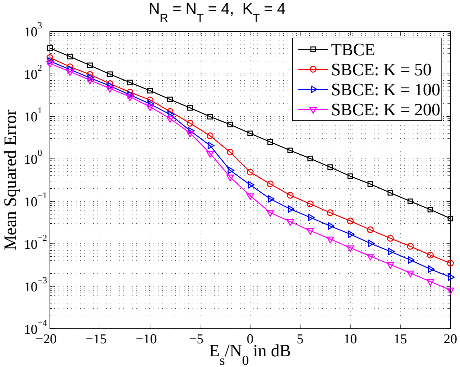

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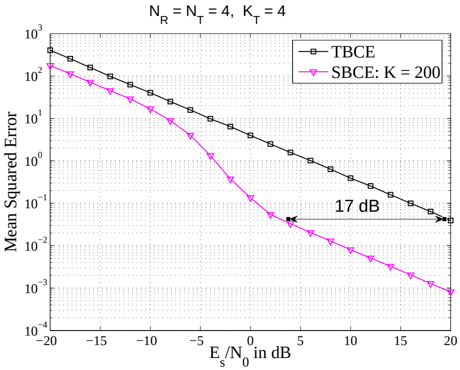

Cramer-Rao Lower Bound (CRLB)

- Assuming ML-SBCE is unbiased

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- Cramer-Rao lower bound

$$\mathbf{C}_{\widehat{\mathbf{H}}} - \mathbf{F}^{-1} \geqslant \mathbf{0}, \quad \text{where} \quad \mathbf{C}_{\widehat{\mathbf{H}}} = \mathsf{E}\left\{\text{vec}\{\mathbf{H} - \widehat{\mathbf{H}}\}\text{vec}\{\mathbf{H} - \widehat{\mathbf{H}}\}^H\right\}$$

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- Fisher information matrix is given by

$$\mathbf{F} = \frac{1}{|\mathcal{A}|\sigma_n^2} \int_{\mathbb{C}^{N_R \times K}} \frac{\sum\limits_{\mathbf{X}_i, \mathbf{X}_j \in \mathcal{A}} p(\mathbf{Y}|\mathbf{H}, \mathbf{X}_i) p(\mathbf{Y}|\mathbf{H}, \mathbf{X}_j)}{\sum\limits_{\mathbf{X}_i \in \mathcal{A}} p(\mathbf{Y}|\mathbf{H}, \mathbf{X}_i)} (\mathbf{X}_i \otimes \mathbf{I}_{N_R}) \underline{\mathbf{d}}_i^* \underline{\mathbf{d}}_j^T (\mathbf{X}_j^H \otimes \mathbf{I}_{N_R}) \, d\mathbf{Y}$$

with

$$\underline{\mathbf{d}}_i = \text{vec}\left\{\frac{\mathbf{Y} - \mathbf{H}\mathbf{X}_i}{\sigma_n}\right\}$$

- Minimum distance of noiseless channel outputs

$$\delta_{\mathsf{min}} = \min_{\mathbf{X}_i, \mathbf{X}_j \in \mathcal{X}} \left\| \mathbf{H}(\mathbf{X}_i - \mathbf{X}_j) \right\|_F^2, \quad i \neq j$$

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- For high SNR values where $\delta_{\text{min}} \gg \sigma_n^2$, we will obtain

$$\mathbf{F}_{\text{high}} = \frac{K}{\sigma_n^2} \mathbf{I}_{N_{\text{R}} N_{\text{T}}}$$

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$$\text{CRLB:} \quad \text{MSE} \geq \text{tr}\left\{ \mathbf{F}_{\text{high}}^{-1} \right\} = \frac{N_{\text{R}} N_{\text{T}}}{K} \sigma_n^2$$

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- CRLB at low SNR

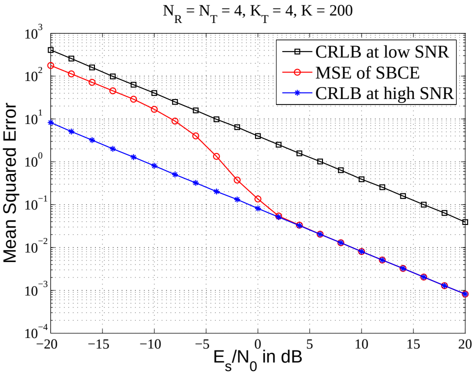
$$MSE \geq \frac{N_\mathsf{R} N_\mathsf{T}}{K_\mathsf{T}} \sigma_n^2$$

$\longrightarrow$ only training symbols are used for channel estimation

- CRLB at high SNR

$$MSE \geq \frac{N_\mathsf{R} N_\mathsf{T}}{K} \sigma_n^2$$

$\longrightarrow$ all data symbols are known and used for channel estimation



- Least squares channel estimator:

$$\widehat{\mathbf{H}} = \mathbf{Y}\mathbf{X}^H\left(\mathbf{X}\mathbf{X}^H\right)^{-1} = \left(\mathbf{H}\mathbf{X} + \mathbf{N}\right)\mathbf{X}^H\left(\mathbf{X}\mathbf{X}^H\right)^{-1}$$

It is easy to find that $\mathsf{E}\{\widehat{\mathbf{H}}\} = \mathbf{H}$, i.e. the estimator is unbiased

- Semiblind channel estimator:

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- Given large burst length, following approximations are valid:

$$\begin{aligned} \widehat{\mathbf{X}}\widehat{\mathbf{X}}^H &\approx K\mathbf{I}_{N_\mathsf{T}} \\ \mathsf{E}\{\mathbf{X}\widehat{\mathbf{X}}\} &\approx (K_\mathsf{T} + (1 - 2P_s)K_\mathsf{I})\mathbf{I}_{N_\mathsf{T}} \end{aligned}$$

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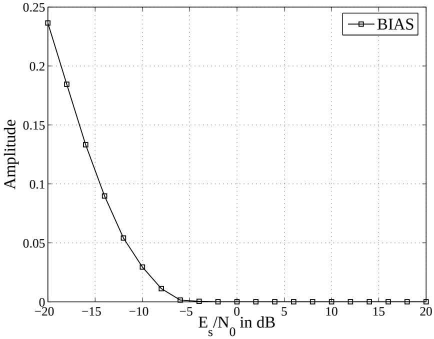
- Let $\mathbf{E} = \widehat{\mathbf{X}} - \mathbf{X}$ denote the detection error, we will obtain

$$\mathsf{E}\{\widehat{\mathbf{H}}\} \approx \frac{K_\mathsf{T} + (1-2P_s)K_\mathsf{I}}{K}\mathbf{H} + \frac{1}{K}\mathsf{E}\{\mathbf{N}\mathbf{E}^H\}$$

- Degree of biasing:

$$\mathbf{BIAS} \doteq \left\| \mathbf{H} - \mathsf{E}\{\widehat{\mathbf{H}}\} \right\|_F^2$$



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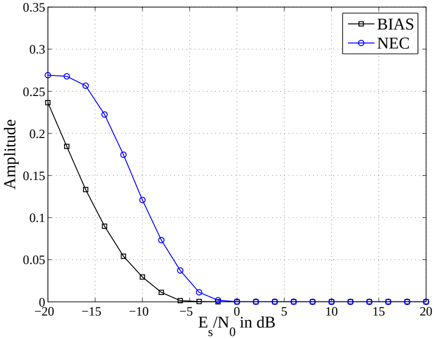

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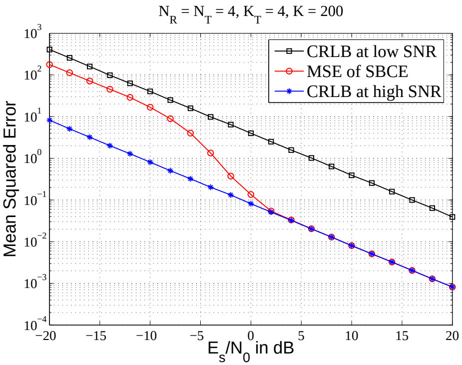

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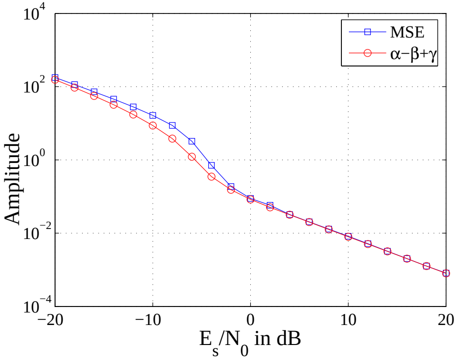
- Analytical approximation:

$$\mathsf{MSE} \approx \underbrace{\left(\frac{2P_sK_\mathsf{I}}{K}\right)^2 \|\mathbf{H}\|_\mathsf{F}^2}_{\alpha} - \underbrace{\frac{4P_sK_\mathsf{I}}{K^2}\mathrm{tr}\left\{\Re\left\{\mathsf{E}\left\{\mathbf{N}\mathbf{E}^H\right\}\mathbf{H}^H\right\}\right\}}_{\beta} + \underbrace{\frac{1}{K^2}\mathsf{E}\left\{\|\mathbf{N}\widehat{\mathbf{X}}^H\|_\mathsf{F}^2\right\}}_{\gamma}$$

- MSE-Anal. $= \alpha - \beta + \gamma$
- The analytical expression coincides with the true MSE at reasonable SNR values.



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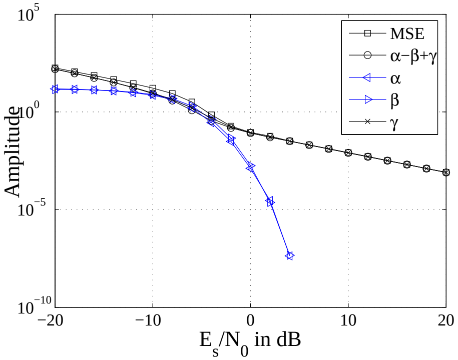
$$\alpha = \left(\frac{2P_s K_\mathsf{I}}{K}\right)^2 \mathrm{tr}\left\{\Re\{\mathbf{H}\mathbf{H}^H\}\right\}$$

$$\beta = \frac{4P_s K_\mathsf{I}}{K^2}\mathrm{tr}\left\{\Re\{\mathsf{E}\{\mathbf{N}\mathbf{E}^H\}\mathbf{H}^H\}\right\}$$

$\Rightarrow$ pattern of noise-error correlation is determined by the channel matrix



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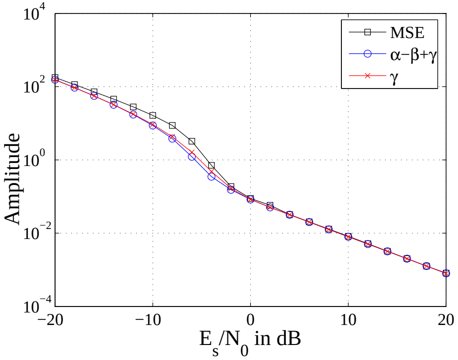

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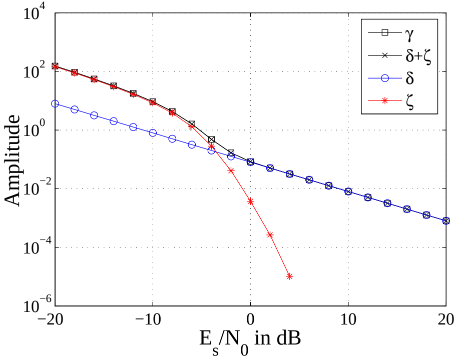

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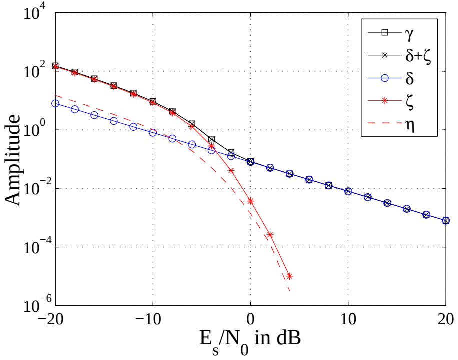

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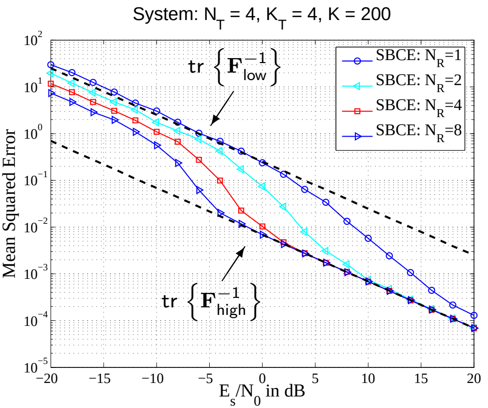
"CRLB at high SNR"

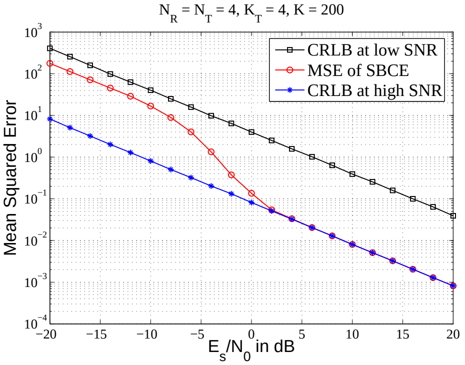
- Approximation of CRLB at high SNR fits for all cases
- As N_R increases, the cross-over SNR range shifts to the left
- Reason:
  $\min_{i \neq j} \left\{ \left\| \mathbf{H}\widetilde{\mathbf{X}}_i - \mathbf{H}\widetilde{\mathbf{X}}_j \right\|_F^2 \right\}$ increases,
  $\rightarrow$ higher capability of error correction
- For $N_\mathrm{R} < N_\mathrm{T}$,
  the reduction of noise-error correlation may be of practical interests

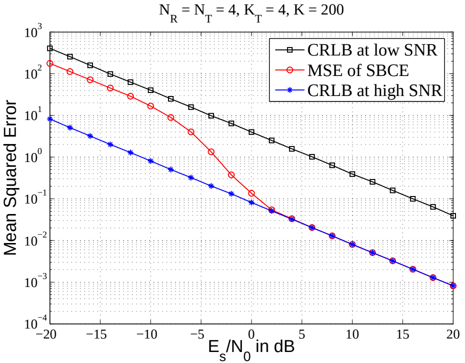


Conclusion

- SBCE is effective in improving channel estimation quality
- ML-SBCE is biased at low SNR and tends to be unbiased at high SNR
  Reasons of biasing: Erroneous data detection
  Noise-error correlation
- ML-SBCE is well bounded by CRLB at high SNR,
  while may exceed CRLB at low SNR due to biasing
- Semi-analytically: $\mathsf{MSE} \approx \frac{1}{K^2}\mathsf{E}\{\|\mathbf{N}\mathbf{E}^H\|_\mathsf{F}^2\}$
- As N_R increases, the cross-over range shifts to lower SNRs
  $\rightarrow$ N_R plays an important role in the performance of SBCE
- Further improvements: biasing compensation, MSE reduction

Cross-Over Range

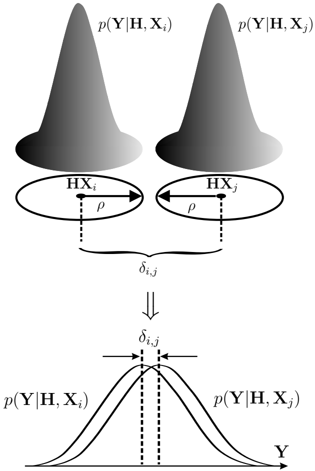




$$\mathbf{F} = \frac{1}{|\mathcal{A}|\sigma_n^2} \int_{\mathbb{C}^{N_R \times K}} \frac{\sum_{\mathbf{X}_i, \mathbf{X}_j \in \mathcal{A}} p(\mathbf{Y}|\mathbf{H}, \mathbf{X}_i) p(\mathbf{Y}|\mathbf{H}, \mathbf{X}_j)}{\sum_{\mathbf{X}_i \in \mathcal{A}} p(\mathbf{Y}|\mathbf{H}, \mathbf{X}_i)} (\mathbf{X}_i \otimes \mathbf{I}_{N_R}) \underline{\mathbf{d}}_i^* \underline{\mathbf{d}}_j^T (\mathbf{X}_j^H \otimes \mathbf{I}_{N_R}) \, d\mathbf{Y}$$

- The SNR range where $\mathbf{F}$ transits from $\mathbf{F}_{\text{high}}$ to $\mathbf{F}_{\text{low}}$

$$\alpha \cdot \delta_{\text{min}} \leq \sigma_n^2 \leq \beta \cdot \delta_{\text{min}}$$



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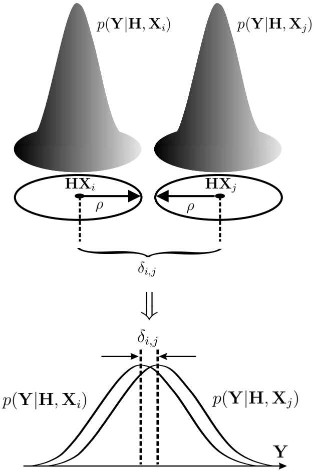
- Open issue:

$$\begin{aligned} \alpha &= ? \\ \beta &= ? \end{aligned}$$



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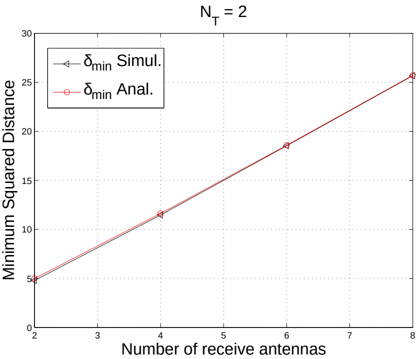
- Minimum channel output distance

$$\delta_{\text{min}} = 4N_{\text{T}} \int_0^{\infty} \frac{\delta^{N_{\text{R}}} e^{-\delta} \left[\Gamma(N_{\text{R}}, \delta)\right]^{N_{\text{T}}-1}}{\left[(N_{\text{R}}-1)!\right]^{N_{\text{T}}}} d\delta$$



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- Observation: as N_R increases,

δ_{min} increases

$\longrightarrow$ saturation point shifts to the leftside