Estimation Lower Bounds and Synchronization Issue in Single Carrier System

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Outline

Estimation Lower Bounds

Estimator Perfomance Breakdown

Relative Insufficiency of Deterministic Bounds

Bayesian Bounds of the Weiss-Weinstein Family

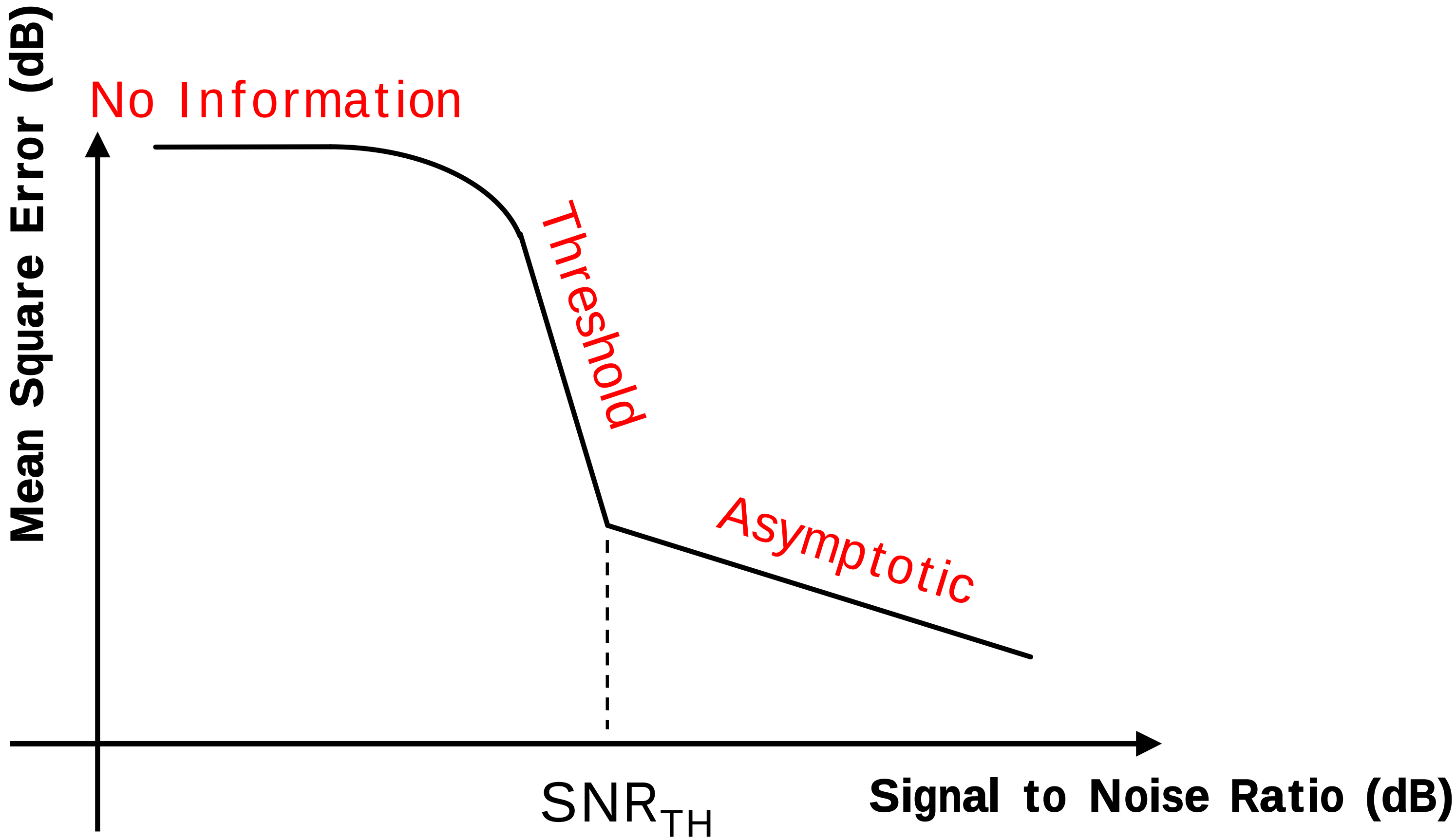
Synchronization Issue in Single Carrier System

Problem Setup

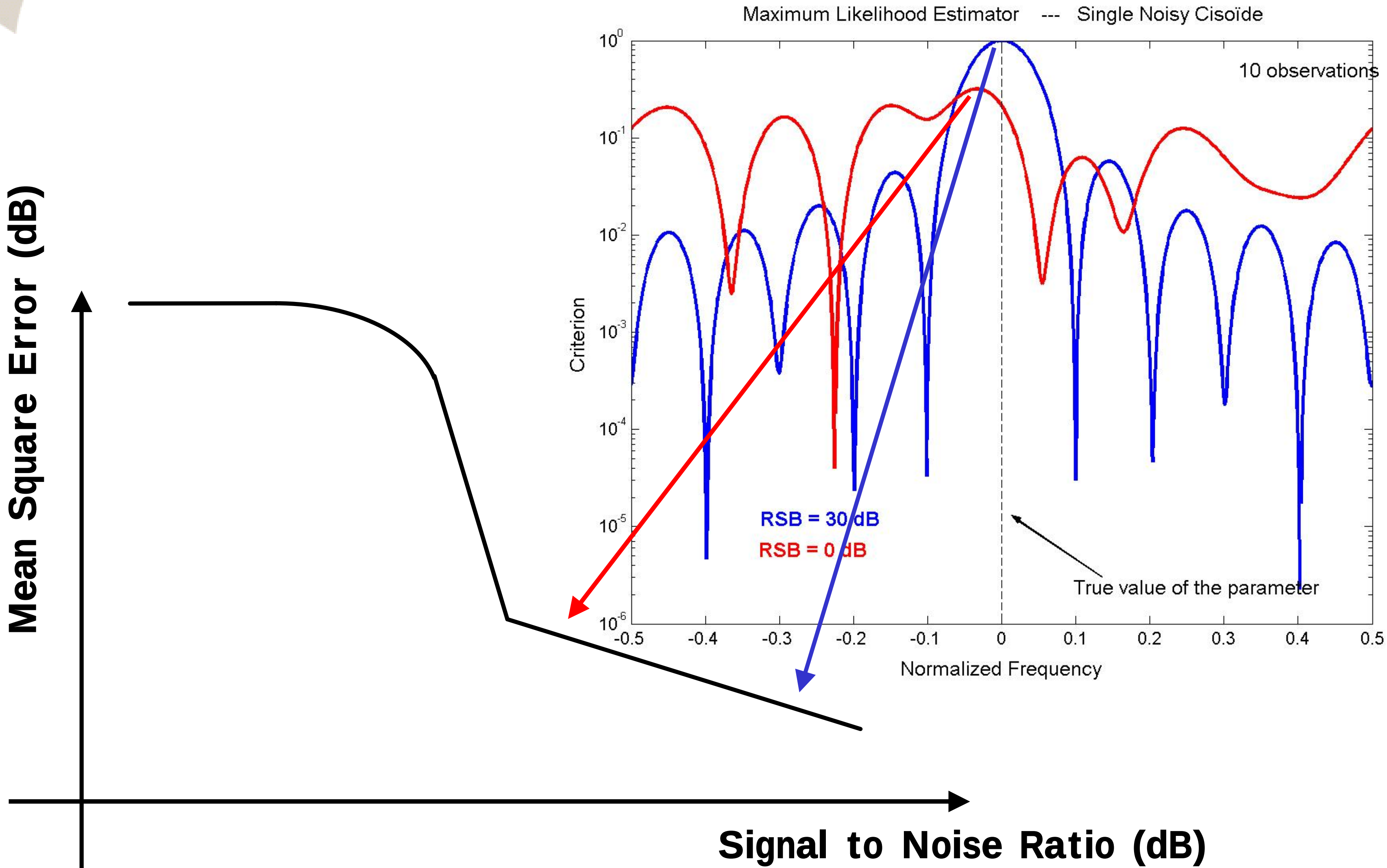
Deterministic Bounds

Weiss & Weinstein Family Bounds

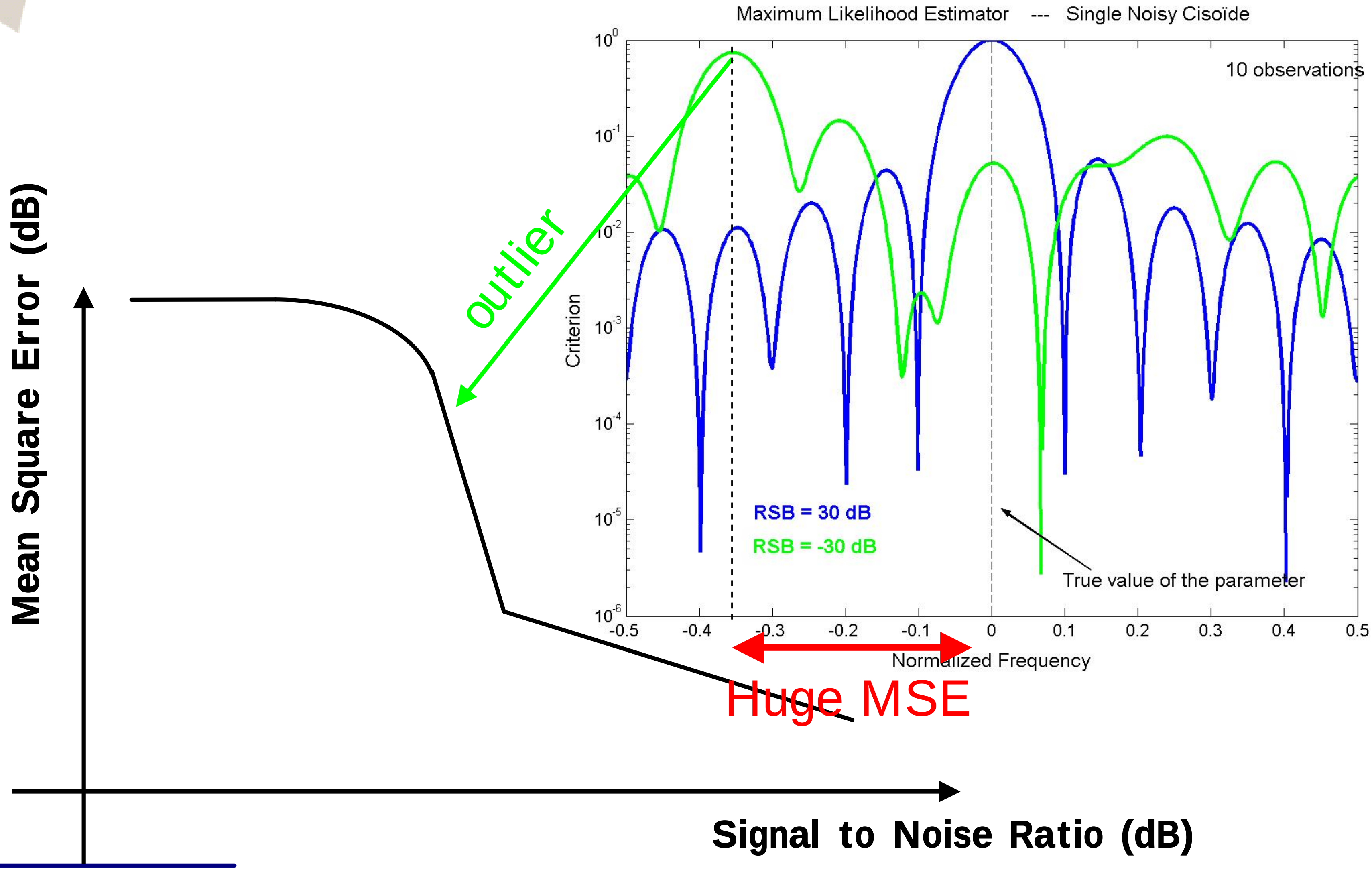
Estimator Perfomance Breakdown



Estimator Perfomance Breakdown



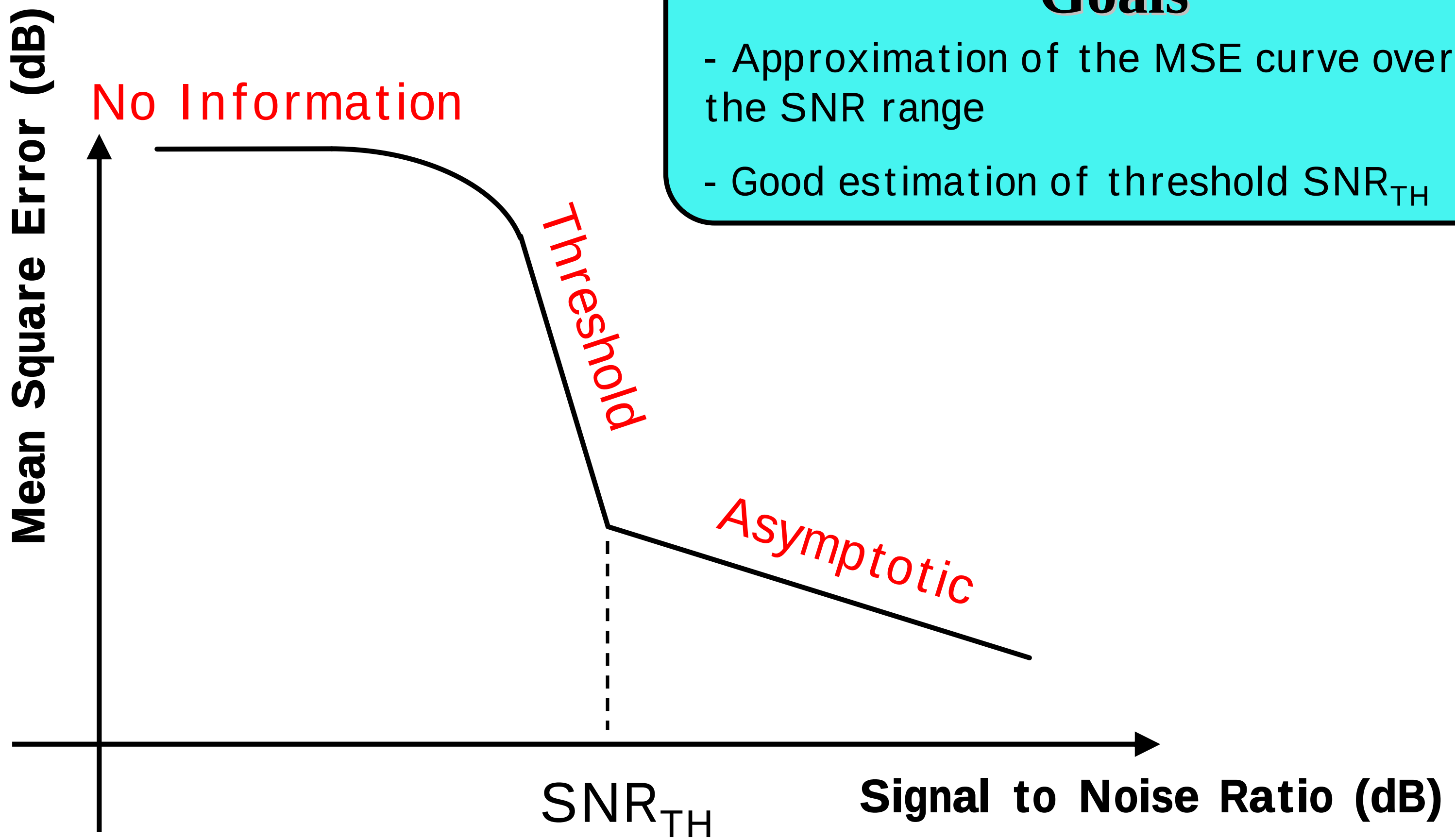
Estimator Perfomance Breakdown



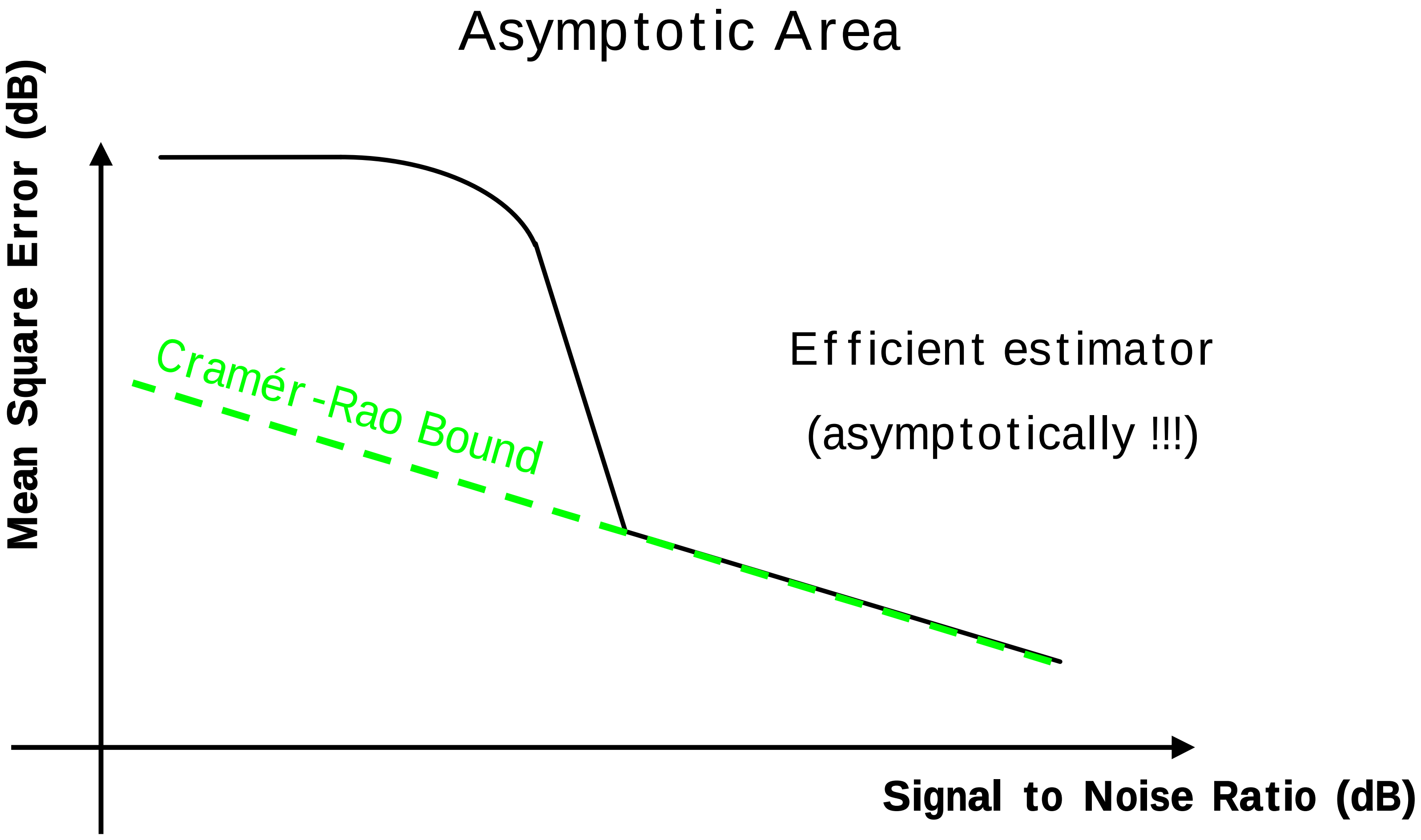
Estimator Perfomance Breakdown

Goals

- Approximation of the MSE curve over all the SNR range
- Good estimation of threshold SNR_{TH}

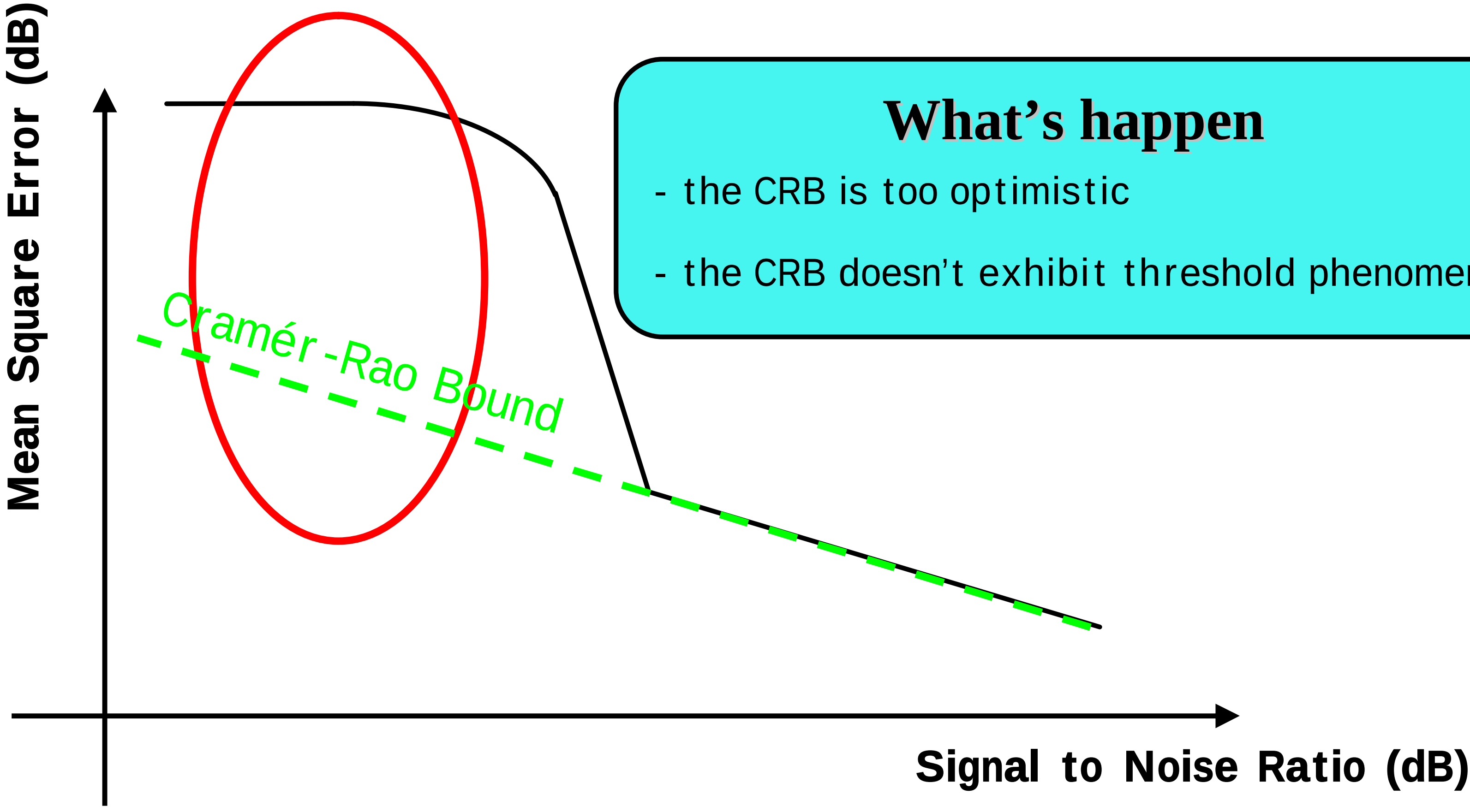


Insufficiency of the Cramér-Rao Bound



Insufficiency of the Cramér-Rao Bound

Non-asymptotic Area



What's happen

- the CRB is too optimistic
- the CRB doesn't exhibit threshold phenomenon

Relative Insufficiency of Deterministic Bounds

The parameters are assumed to be deterministic

Unification of deterministic lower bounds

Bounds = Minimum MSE of an estimator in θ_0 with contraints on the bias (in possible other points)

$$\begin{cases} \min & \int \left(\hat{\theta} - \theta_0\right)^2 p\left(\mathbf{x}|\theta_0\right) d\mathbf{x} \\ s.t. & \int \left(\hat{\theta} - \theta_0\right) p\left(\mathbf{x}|\theta_0\right) g_k\left(\mathbf{x}\right) d\mathbf{x} = c_k \end{cases}$$

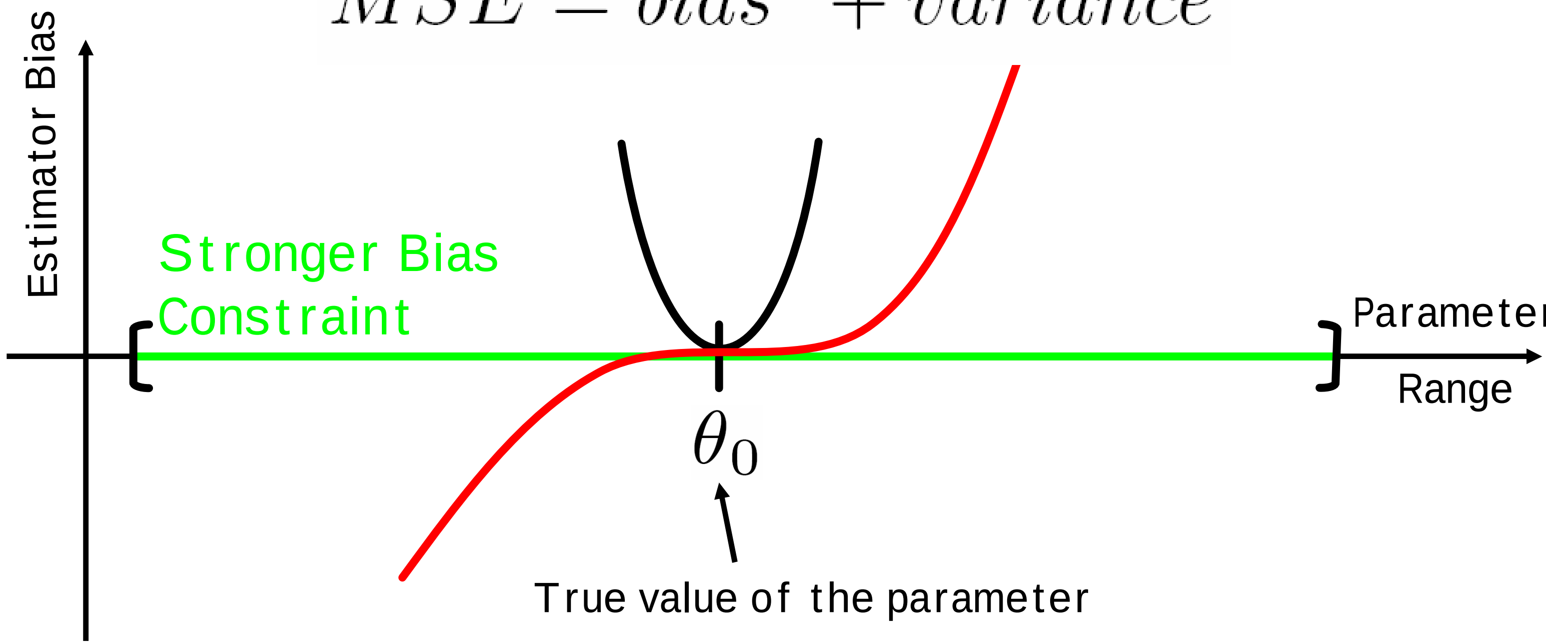
MSE

$$k = 0, ..., K-1$$

Bias constraints

Relative Insufficiency of Deterministic Bounds

$$MSE = bias^2 + variance$$

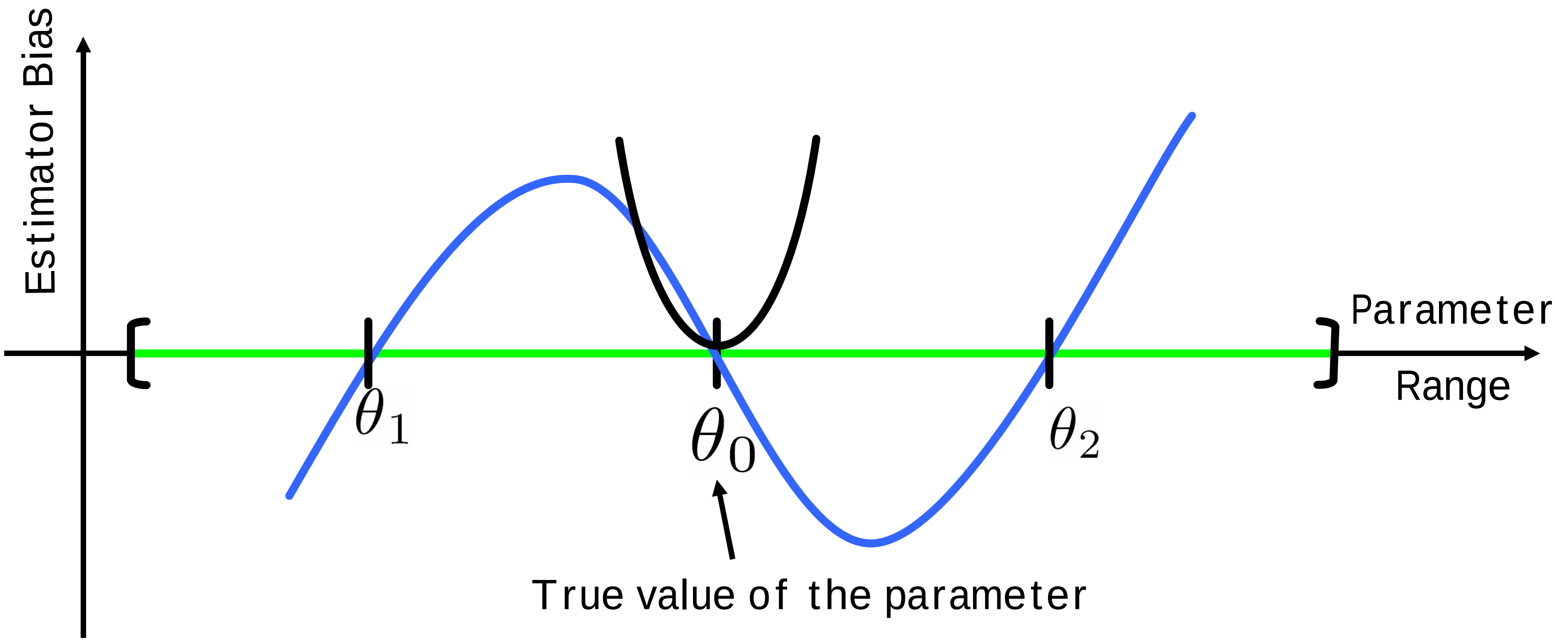


Solution of the constraints optimization problem ⇒

Cramér-Rao bound

Bhattacharyya bound

Relative Insufficiency of Deterministic Bounds



N test points: Barankin bound

1 test point: Chapman-Robins bound

Relative Insufficiency of Deterministic Bounds

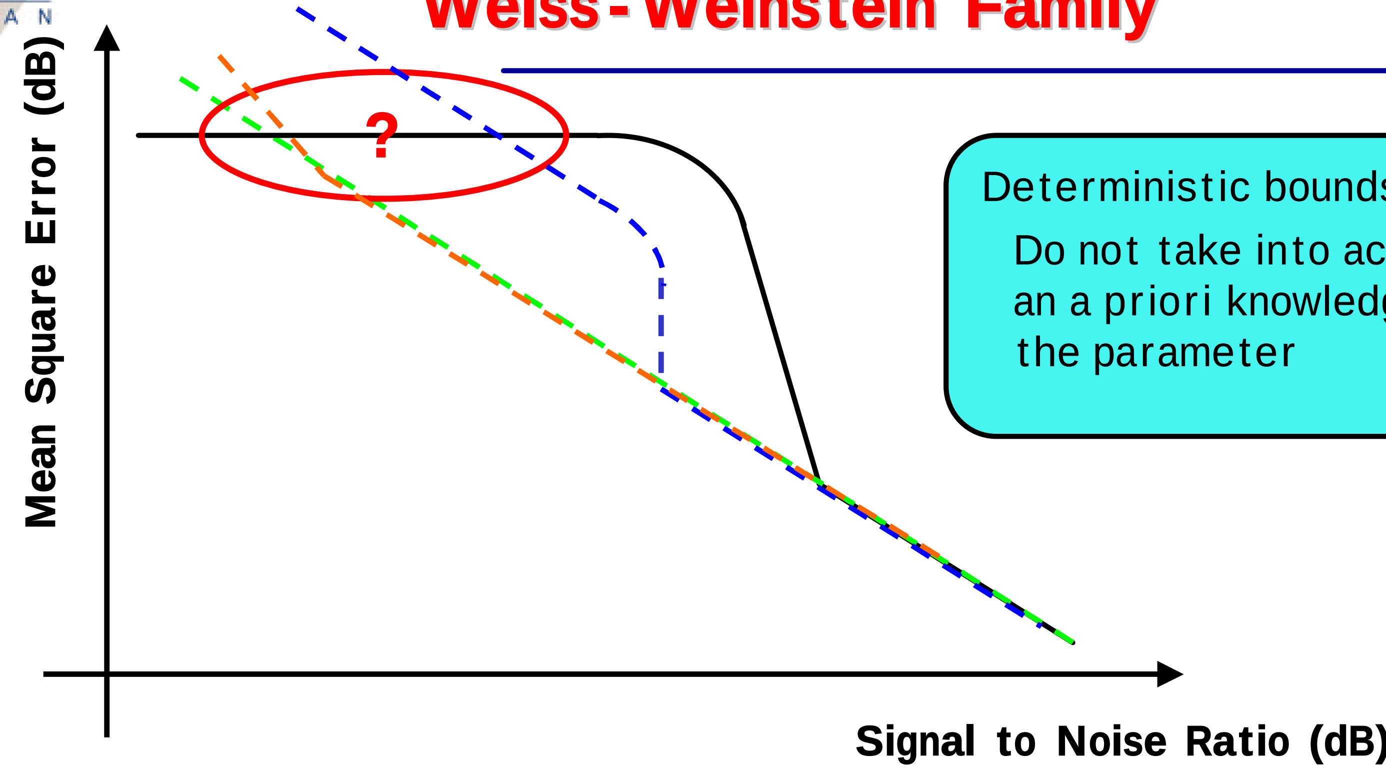
Test points ???

$$bound = f\left(h_1, h_2, ..., h_K\right)$$

$$bound_{best} = \max_{h_1, h_2, ..., h_K} f\left(h_1, h_2, ..., h_K\right)$$

Huge computational cost

Bayesian Bounds of the Weiss-Weinstein Family



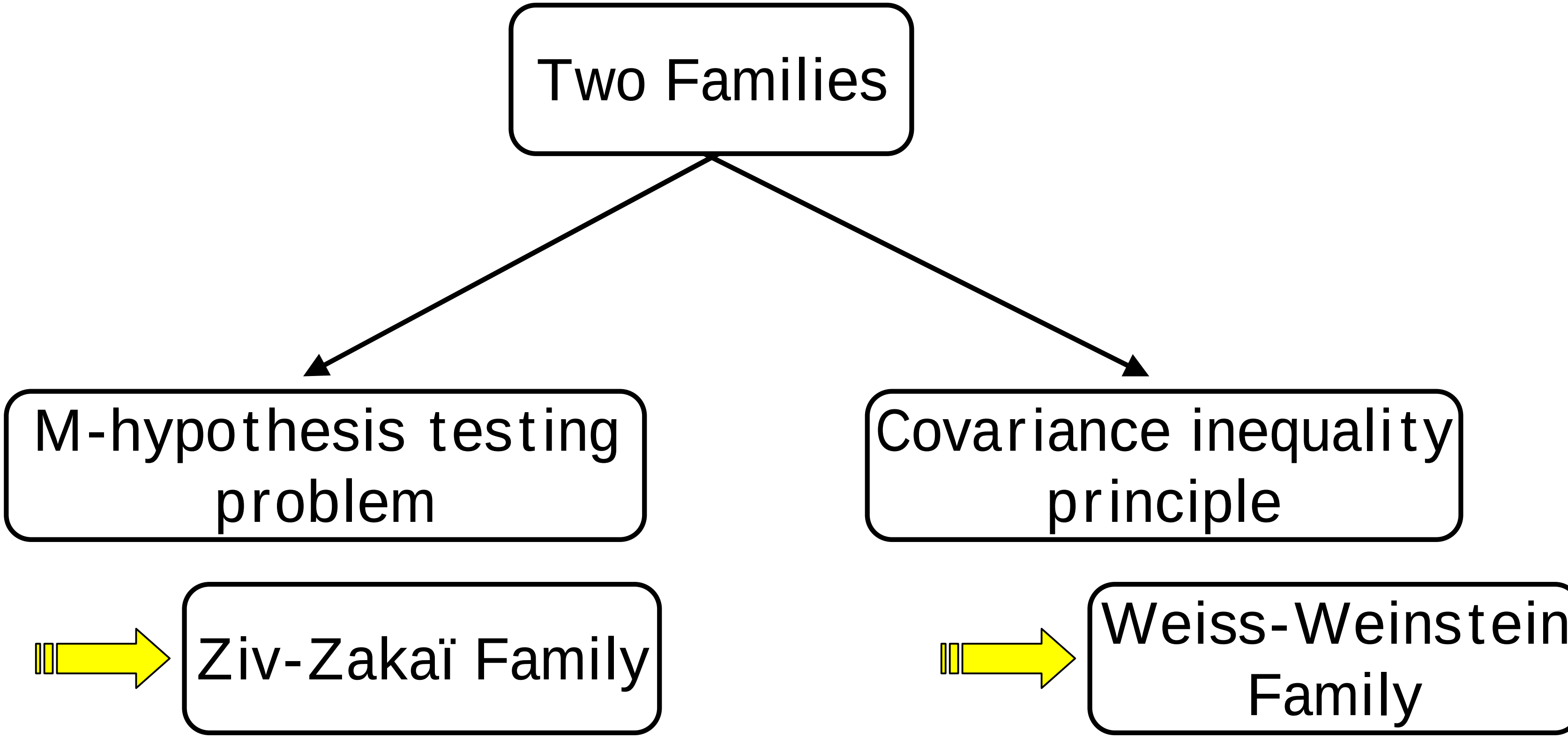
Deterministic bounds

Do not take into account an a priori knowledge of the parameter

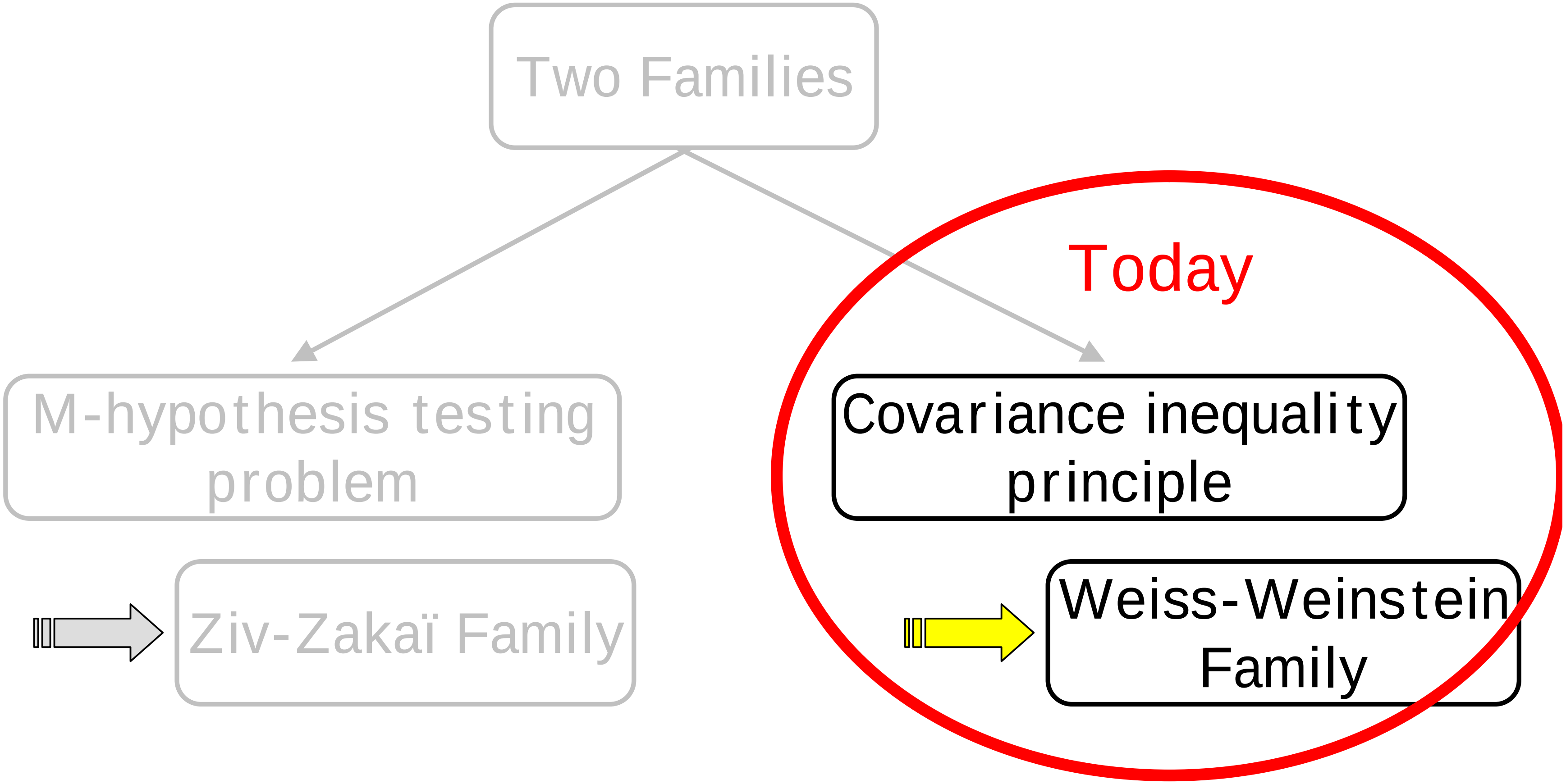
solution ⇒ **Bayesian Bounds**

The parameters are assumed to be random (take into account the a priori pdf)

Bayesian Bounds of the Weiss-Weinstein Family



Bayesian Bounds of the Weiss-Weinstein Family



Bayesian Bounds of the Weiss-Weinstein Family

Unification of Bayesian lower bounds

In the Bayesian context, the best Bayesian bound is given by the Conditional Mean Estimator

$$\hat{\theta}_{MMSEE} = \int \theta p\left(\theta|\mathbf{x}\right) d\theta$$

which is the solution of

$$\min \int_{\Omega}\int_{\Theta} \left(\hat{\theta}\left(\mathbf{x}\right) - \theta\right)^2 p\left(\mathbf{x},\theta\right) d\theta d\mathbf{x}$$ Global MSE

Bayesian Bounds of the Weiss-Weinstein Family

Unification of Bayesian lower bounds

$$\min \int_{\Omega}\int_{\Theta} \left(\hat{\theta}(\mathbf{x}) - \theta\right)^2 p(\mathbf{x},\theta)\, d\theta d\mathbf{x}$$

$$\Updownarrow$$

$$\begin{cases} \min\limits_{v} \int_{\Omega}\int_{\Theta} v^2(\mathbf{x},\theta)\, p(\mathbf{x},\theta)\, d\theta d\mathbf{x} \\ \text{s. t.} \int_{\Omega}\int_{\Theta} v(\mathbf{x},\theta) \left[\left(\frac{p(\mathbf{x},\theta+h)}{p(\mathbf{x},\theta)}\right)^{s} - \left(\frac{p(\mathbf{x},\theta-h)}{p(\mathbf{x},\theta)}\right)^{1-s}\right] p(\mathbf{x},\theta)\, d\theta d\mathbf{x} \end{cases}$$

$$= h \int_{\Omega}\int_{\Theta} \left(\frac{p(\mathbf{x},\theta-h)}{p(\mathbf{x},\theta)}\right)^{1-s} p(\mathbf{x},\theta)\, d\theta d\mathbf{x}$$

$\forall h$ and $\forall s$

$h \in parameter \quad range$

$s \in [0,1]$

Bayesian Bounds of the Weiss-Weinstein Family

Unification of Bayesian lower bounds

$\forall h$ and $\forall s$ ⟺ Infinite number of constraints

$\not\forall h$ and $\not\forall s$ ⇒ Something less than the best Bayesian bound (due to the constraints relaxation)

Minimal bounds on the MSE

Bayesian Bounds of the Weiss-Weinstein Family

Unification of Bayesian lower bounds

$\forall h$ and $\forall s$ ⟺ Infinite number of constraints

$\cancel{\forall} h$ and $\cancel{\forall} s$ ⟹ Something less than the Conditional Mean Estimator MSE (due to the constraints relaxation)

⇓

Solution of the constrained optimization problem

<u>Degrees of freedom: choice of h and s</u>

Bayesian Bounds of the Weiss-Weinstein Family

Unification of Bayesian lower bounds

$$s = 1$$
$$h \to 0$$
$$\frac{\partial^K}{\partial \theta^K}$$

Bayesian Bhattacharyya bound

solution $= [1, 0, \cdots, 0]\, \mathbf{B}^{-1} [1, 0, \cdots, 0]^{\mathrm{T}}$

With $B_{i,j} = \int_\Omega \int_\Theta \frac{1}{p(\mathbf{x}, \theta)} \frac{\partial^i p(\mathbf{x}, \theta)}{\partial \theta^i} \frac{\partial^j p(\mathbf{x}, \theta)}{\partial \theta^j} d\theta d\mathbf{x}$

Bayesian Cramér-Rao bound

Particular case

$K = 1$ solution $= \left(\int_\Omega \int_\Theta \frac{1}{p(\mathbf{x}, \theta)} \frac{\partial p(\mathbf{x}, \theta)}{\partial \theta} d\theta d\mathbf{x} \right)^{-1}$

Bayesian Bounds of the Weiss-Weinstein Family

Unification of Bayesian lower bounds

$$s = 1$$

$$\mathbf{h} = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_K \end{bmatrix}$$

Reuven-Messer bound (Bayesian Barankin bound)

solution $= \mathbf{h}^{\mathrm{T}} \left(\mathbf{D} - \mathbf{1}\mathbf{1}^{\mathrm{T}} \right)^{-1} \mathbf{h}$

With $D_{i,j} = \int\limits_{\Omega} \int\limits_{\Theta} \frac{p(\mathbf{x}, \theta + h_i) \, p(\mathbf{x}, \theta + h_j)}{p(\mathbf{x}, \theta)} d\theta d\mathbf{x}$

Particular case

Bobrovsky-Zakaï bound

(Bayesian Chapman-Robins bound)

$K = 1$ → solution $= \dfrac{h^2}{\int\limits_{\Omega} \int\limits_{\Theta} \frac{p^2(\mathbf{x}, \theta + h)}{p(\mathbf{x}, \theta)} d\theta d\mathbf{x} - 1}$

Bayesian Bounds of the Weiss-Weinstein Family

Unification of Bayesian lower bounds

$$\mathbf{h} = [h_1, h_2, \cdots, h_K]^{\mathrm{T}}$$
$$\mathbf{s} = [s_1, s_2, \cdots, s_K]^{\mathrm{T}}$$

Weiss-Weinstein Bound

solution $= \boldsymbol{\xi}^{\mathrm{T}} \mathbf{W}^{-1} \boldsymbol{\xi}$

$$W_{i,j} = E\left[\left(L^{s_i}(\mathbf{x}|\theta + h_i, \theta) - L^{1-s_i}(\mathbf{x}|\theta - h_i, \theta)\right)\left(L^{s_j}(\mathbf{x}|\theta + h_j, \theta) - L^{1-s_j}(\mathbf{x}|\theta - h_j, \theta)\right)\right]$$

$$L(\mathbf{x}|\theta_1, \theta_2) \triangleq \frac{p(\mathbf{x}, \theta_1)}{p(\mathbf{x}, \theta_2)}$$

$$\boldsymbol{\xi} = \begin{bmatrix} h_1 E\left[L^{1-s_1}(\mathbf{x}|\theta - h_1, \theta)\right] \\ h_2 E\left[L^{1-s_2}(\mathbf{x}|\theta - h_2, \theta)\right] \\ \vdots \\ h_K E\left[L^{1-s_K}(\mathbf{x}|\theta - h_K, \theta)\right] \end{bmatrix}$$

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Problem Setup

Data Model

$$x_k = \boxed{\rho a_k e^{jk\theta} + n_k} \quad \text{with} \quad k = 0, \ldots, N-1$$

$\{a_k\}$: training sequence $\qquad \theta$: parameter of interest

$$\{n_k\} \sim \mathcal{N}_c\left(\mathbf{0}, \mathbf{I}_N\right) \qquad \rho^2 = SNR$$

Bayesian case

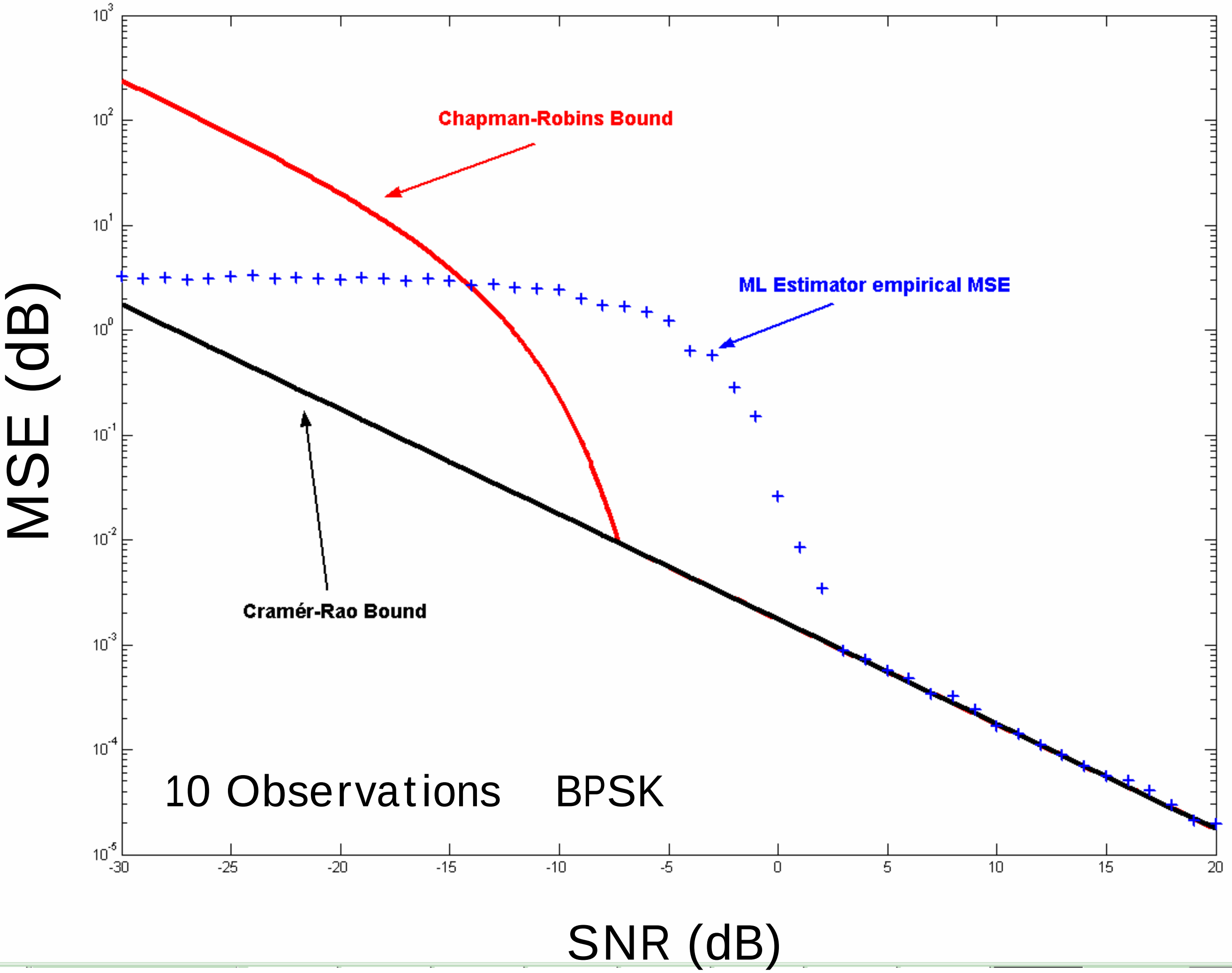
$$\theta \sim N\left(0, \sigma_\theta^2\right)$$

Deterministic Bounds

$$CRB(\theta_0) = \frac{1}{2\rho^2 \sum_{k=0}^{N-1} |a_k|^2 k^2}$$

$$ChRB(\theta_0) = \sup_{0 \le h \le \pi} \frac{h^2}{e^{4\rho^2 \sum_{k=0}^{N-1} |a_k|^2 (1-\cos(kh))} - 1}$$

Deterministic Bounds



Weiss & Weinstein Family Bounds

$$BCRB = \frac{\sigma_\theta^2}{2\sigma_\theta^2 \rho^2 \sum_{k=0}^{N-1} |a_k|^2 k^2 + 1}$$

$$BZB = \sup_h \frac{h^2}{\sqrt{2}\sigma_\theta e^{4\rho^2 \sum_{k=0}^{N-1} |a_k|^2 (1-\cos(hk)) - 2h^2 \left(\frac{1}{2\sigma_\theta^2} - 2 \right)} - 1}$$

$$\begin{cases} WWB = \sup_{h,s} \frac{h^2 \eta^2 (s,h)}{\eta(2s,h) + \eta(2-2s,-h) - 2\eta(s,2h)} \\ \eta(\alpha,\beta) = \sqrt{2}\sigma_\theta e^{-2\rho^2 \alpha(1-\alpha) \sum_{k=0}^{N-1} |a_k|^2 (1-\cos(k\beta)) - \alpha\beta^2 \left(\frac{1}{2\sigma_\theta^2} - \alpha \right)} \end{cases}$$

Weiss & Weinstein Family Bounds