Markovian Decision Process (MDP): theory and applications to wireless networks

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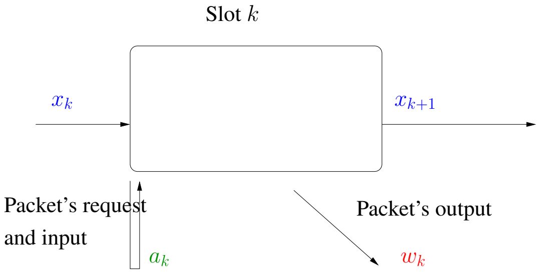
Outline

- A few examples
- Markov Decision Process
  - Markov chain
  - Discounted approach
  - Bellman's equation and fixed-point theorem
  - Practical algorithms
- Applications to wireles networks
  - Channel exploration
  - Hybrid ARQ optimization
  - Scheduling with Energy Harvesting

Part 1 : A few examples

Example 1 : Inventory control



- x_k stock at the beginning of slot k
- a_k stock ordered and immediatly delivered at the beginning of slot k
- w_k request of stock during slot k (iid process)

Goal : optimizing $\{a_k\}$? but
a_k too large (cost) ; a_k too small (stock outage with $x_{k+1} = 0$)

Example 1 : Inventory control (cont'd)

Mathematical model 1 :

- Reward : $r(x_k, a_k)$

$$\alpha\text{-Discounted (long-term) reward : } r = \lim_{N\to\infty} \sum_{n=0}^{N} \alpha^n r(x_n, a_n)$$

<u>Additive</u> cost over time

- Sequential dynamic :

$$x_{k+1} = f(x_k, a_k, w_k) = (x_k + a_k - w_k)^+$$

$\{x_k\}$ **<u>Markov</u> chain with transition probability** $Q(x_{k+1}|x_k, a_k)$

- Policy : $a_{k+1} = \mu_{(k)}(x_k)$

$$\{\mu_{(k)}\}? \min \mathbb{E}_{|x_0,\mu} \left[\lim_{N\to\infty} \sum_{n=0}^{N} \alpha^n r(x_n, \mu_{(n)}(x_n)) \right]$$

Markov Decision Process (MDP)

Example 1 : Inventory control (cont'd)

Mathematical model 2

- Reward/Cost : $c(x_k)$
- Outage : $\tilde{x}_k = 0$ if $x_k + a_k - w_k \geq 0$, 1 otherwise

$$o(\tilde{x}_k) = \tilde{x}_k \Rightarrow \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N} o(\tilde{x}_n) < O_t$$

- Let $s_k = [x_k, \tilde{x}_k]$: system state. Optimal policy :

$$\{\mu_{(k)}\}? \min \mathbb{E}_{|s_0,\mu} \left[\lim_{N\to\infty} \sum_{n=0}^{N} \alpha^n r(s_n, \mu_{(n)}(s_n)) \right]$$

s.t.

$$\mathbb{E}_{|s_0,\mu} \left[\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N} o(s_n) \right] < O_t$$

Constrained Markov Decision Process (CMDP)

Example 2 : Water reservoir control

Reservoir with C finite capacity : maximize simultaneously the stock and water release

- x_k water volume at time k
- a_k water released volume at time k (for electricity or irrigation)
- w_k random inflow during slot k (rain and tributary rivers) : iid process

We have

$$x_{k+1} = \min(x_k - a_k + w_k, C)$$

- Cost : $r(x_k, a_k) = x_k + a_k$
- x_k not perfectly known, but estimated. So we know

$$y_k = g(x_k, \zeta_k)$$

Partially Observable Markov Decision Process (POMDP)

Part 2 : Markov Decision Process

Markov chain

Let $\{x_k\}$ a random sequence/process.

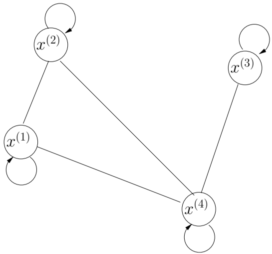
Definition

A sequence is said a Markov chain iff

$$P(x_{k+1}|x_k, \cdots, x_0) = P(x_{k+1}|x_k), \ \forall k$$

If x_k takes a finite number of values $\mathcal{X} = \{x^{(1)}, \cdots, x^{(V)}\}$,

- transition probability matrix :
  $T = (t_{m,n})_{1 \leq m,n \leq V}$ with
  $t_{m,n} = \text{Prob}(x_{k+1} = x^{(m)} | x_k = x^{(n)}) \geq 0$
  and $\sum_{m=1}^{V} t_{m,n} = 1$.
- Graph theory
- (Stochastic) non-negative matrix theory



State, Action, and Policy

- State : x_k (system information at time k)
- Action : a_k
- Disturbance : w_k iid process

$$x_{k+1} = f(x_k, a_k, w_k)$$

Transition kernel :

$$\begin{aligned} Q(x_{k+1}|x_k, a_k) &= \text{Prob}(x_{k+1} = x^{(m)}|x_k = x^{(n)}, a_k = a) \\ &= \int t_{m,n}(a, w)p_w(w)dw \end{aligned}$$

- Reward : $r(x_k, a_k)$

$$\mathbb{E}_{|x_0}\left[\sum_{n=0}^{\infty} \alpha^n r(x_n, a_n)\right] \text{ (}\textit{discount}\text{) or } \mathbb{E}_{|x_0}\left[\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N} r(x_n, a_n)\right]$$

- Policy : $\mu_k(x_k) \in \mathcal{A}$ with $\mathcal{A}$ set of actions
  - deterministic (μ_k is a function) or random (μ_k is a pdf)
  - stationary (μ_k independent of k) or nonstationary

Bellman's equation

- Let μ be a deterministic stationary policy
- Let $R_\mu(x_0)$ be the discounted reward assuming used policy μ and initialized at x_0

$$\begin{aligned}
R_\mu(x_0) &= \mathbb{E}_{|x_0}\left[\sum_{n=0}^{\infty} \alpha^n r(x_n, \mu(x_n))\right] \\
&= r(x_0, \mu(x_0)) + \alpha \mathbb{E}_{|x_0}\left[\sum_{n=0}^{\infty} \alpha^n r(x_{n+1}, \mu(x_{n+1}))\right] \\
&= r(x_0, \mu(x_0)) + \alpha \mathbb{E}_{|x_0}\left[R_\mu(x_1)\right] \\
&= r(x_0, \mu(x_0)) + \alpha \int R_\mu(x) Q(x|x_0, \mu(x_0)) dx, \ \forall x_0
\end{aligned}$$

Bellman's equation

$$R_\mu = T_\mu(R_\mu) \quad \text{(fixed-point)}$$

$$x_0 \mapsto T_\mu(f)(x_0) = r(x_0, \mu(x_0)) + \alpha \int f(x) Q(x|x_0, \mu(x_0)) dx$$

Fixed-point theorem

We also define

$$T(f)(x_0) = \max_{a \in \mathcal{A}} \left\{ r(x_0, a) + \alpha \int f(x) Q(x|x_0, a) dx \right\}$$

Lemma

It exits an unique function R (resp. R_μ) such that

$$R = T(R) \quad \text{and} \quad R_\mu = T_\mu(R_\mu)$$

Sketch of proof :

- Let $\|f\|_\infty = \sup_x |f(x)|$ (within set of bounded function)
- Assuming $r(\bullet, \bullet)$ bounded function for any state and action

$$\begin{aligned} \|T(f) - T(g)\|_\infty &\leq \alpha \max_{a \in \mathcal{A}} \left\{ \int f(x) Q(x|x_0, a) dx - \int g(x) Q(x|x_0, a) dx \right\} \\ &\leq \alpha \|f - g\|_\infty \end{aligned}$$

T is a α-contraction, so Banach's fixed-point theorem applies

Optimal deterministic policy

Theorem

A (deterministic stationary) policy $\mu^\star$ is said optimal

- $R_{\mu^\star}(x_0) = R^\star(x_0), \ \forall x_0$ with $R^\star(x_0) = \sup_\mu R_\mu(x_0)$
- Equivalently, $R_{\mu^\star}$ is the fixed point of T

Sketch of proof : Let $\mu^\star$ a policy s.t. $R_{\mu^\star}$ is the fixed point of T

- $R_{\mu^\star}(x_0) \geq r(x_0, a) + \alpha \int R_{\mu^\star}(x) Q(x|x_0, a) dx, \ \forall (x_0, a) \in \mathcal{X} \times \mathcal{A}$
- $R_{\mu^\star}(x_n) \geq r(x_n, a_n) + \alpha \int R_{\mu^\star}(x) Q(x|x_n, a_n) dx$
- $\mathbb{E}_{|x_0,\mu}[\alpha^n R_{\mu^\star}(x_n)] - \alpha^{n+1} \mathbb{E}_{|x_0,\mu}[\int R_{\mu^\star}(x) Q(x|x_n, a_n) dx] \geq \mathbb{E}_{|x_0,\mu}[\alpha^n r(x_n, a_n)]$ $(\times \alpha^n$+expectation)
- $\mathbb{E}_{|x_0,\mu}[\alpha^n \int R_{\mu^\star}(x) Q(x|h_{n-1}) dx] - \mathbb{E}_{|x_0,\mu}[\alpha^{n+1} \int R_{\mu^\star}(x) Q(x|h_n) dx] \geq \mathbb{E}_{|x_0,\mu}[\alpha^n r(x_n, a_n)]$ (Markov)
- $R_{\mu^\star}(x_0) - \alpha^{N+1} \mathbb{E}_{|x_0,\mu}[\int R_{\mu^\star}(x) Q(x|h_N) dx] \geq \mathbb{E}_{|x_0,\mu}[\sum_{n=0}^{N} \alpha^n r(x_n, a_n)]$ (sum over n)
- $R_{\mu^\star}(x_0) \geq \mathbb{E}_{|x_0,\mu}[\sum_{n=0}^{\infty} \alpha^n r(x_n, a_n)] = R_\mu(x_0), \ \forall \mu$ (Q.E.D)

Algorithm 1 : Value iteration (VI)

According to fixed-point thorem, we have

$$R_{\mu^\star} = \lim_{N\to\infty} \underbrace{T \circ ... \circ T}_{N \text{ times}}(R), \ \forall R$$

Let $\{R_n\}$ wityh any function R_0 s.t.

$$\begin{aligned} R_{n+1} &= T(R_n) \\ R_{n+1}(x) &= \max_a \left\{ r(x,a) + \alpha \int R_n(y) Q(y|x,a) dy \right\} \\ \mu_n(x) &= \arg\max_a \left\{ r(x,a) + \alpha \int R_n(y) Q(y|x,a) dy \right\} \end{aligned}$$

Theorem

$$\lim_{N\to\infty} \mu_N(x) = \mu^\star(x), \ \forall x$$

Algorithm 2 : Linear Programming (LP)

- Finite set of states $\mathcal{X} = \{x^{(0)}, \cdots, x^{(V)}\}$
- Finite set of actions $\mathcal{A} = \{a^{(0)}, \cdots, a^{(V')}\}$

Lemma

Let T be the Bellman's operator on vector s.t. $\tilde{W} = T(W)$

$$\tilde{W}(m) = \max_a \left\{ r(x^{(m)}, a) + \alpha \sum_{n=0}^{V} W(n) Q(x^{(n)} | x^{(m)}, a) \right\}$$

Let $\geq$ be the elementwise "greater than".

- If $U \geq V$, then $T(U) \geq T(V)$
- Let $W^\star$ be fixed point of T and W s.t. $W \geq T(W)$, then $W \geq W^\star$
- As $W^\star \geq T(W^\star)$, we get

$$W^\star = \arg\min_W \sum_{m=0}^{V} W(m) \text{ s.t. } W \geq T(W)$$

Algorithm 2 : Linear Programming (LP) (cont'd)

$$\text{Function } R_{\mu^\star} \Leftrightarrow \text{Vector } \mathbf{R}^\star = [R_{\mu^\star}(x^{(0)}), \cdots, R_{\mu^\star}(x^{(V)})]$$
$$\Leftrightarrow \mathbf{R}^\star \text{ fixed point of } T$$

Linear programming algorithm

$$\mathbf{R}^\star = \arg\min_R \sum_{m=0}^{V} R(n)$$

s.t.

$$R \geq T(R) \Leftrightarrow R(m) \geq \max_a \left\{ r(x^{(m)}, a) + \alpha \sum_{n=0}^{V} R(n) Q(x^{(n)} | x^{(m)}, a) \right\}$$

i.e.

$$R(m) \geq r(x^{(m)}, a^{(t)}) + \alpha \sum_{n=0}^{V} R(n) Q(x^{(n)} | x^{(m)}, a^{(t)}), \forall m, t$$

Extension : CMDP

- Deterministic stationary policy not optimal anymore
- It exists a random stationary policy
- Computation of optimal policy still through linear programming

Part 3 : Applications to wireless networks

Multiband Exploration [Lun15]

- One secondary user may use N_c non-contiguous channels (in 4G with carrier aggregation)
- Can explore only N_e channels simultaneously

Issue : which ones selecting at any time ?

Partially Observable Markov Decision Process (POMDP)

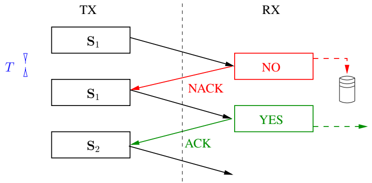
- States (at time n) -not all known- ......empty/occupied channels
- Action.................................N_e channels to explore
- Transition kernel $Q(s_{n+1}|s_n, a_n)$ ....Markovian process
- Reward $r(s_n, a_n)$ ....#empty tested channel
- Policy $a_n = \pi(s_n)$ ....

$$\text{Policy maximizing } \mathbb{E}\left[\sum_n \alpha^n r(s_n, a_n)\right]$$

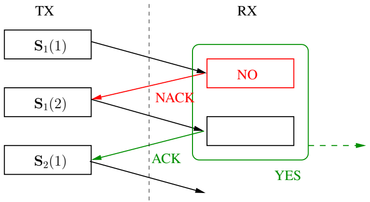
Extension : Competition between secondary users $\Rightarrow$ stochastic game

Hybrid ARQ optimization

- **Type-I HARQ :** Let **S** be a packet composed by N coded symbols



- **Type-II HARQ :** Memory at RX and non-identical packets at TX



$$\begin{aligned} Y_1 &= S_1(1) + N_1 \\ Y_2 &= S_1(2) + N_2 \end{aligned} \Rightarrow \text{detection on } Y = [Y_1, Y_2] \text{ (Coding gain)}$$

Case 1 : Power optimization [Taj13]

We would like to adapt the power packet per packet

After the n-th transmission, if NACK is received, we have

- New action A_{n+1} to do : here choosing P_{n+1}
- Available information : accumulated mutual information equal to

$$I_n = \sum_{\ell=1}^{n} \log_2(1 + G_\ell P_\ell)$$

- $K_n \in \{$ACK and new round, 1 attempt and NACK received, $\cdots$, L attempts and NACK received$\}$

$$P(K_{n+1}, I_{n+1} | K_n, I_n, \cdots) = P(K_{n+1}, I_{n+1} | K_n, I_n)$$

$$\Rightarrow \textbf{Markov Chain : } S_n = (K_n, I_n)$$

A policy μ = how selecting P_{n+1} given S_n

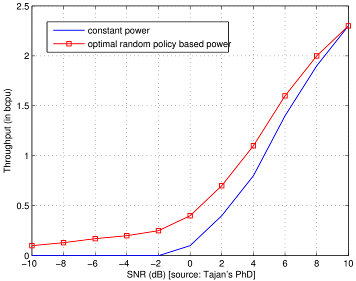
Optimization problem and numerical results

$$\max_{\mu} \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \underbrace{reward_{\mu}(S_n, A_n)}_{\text{\#information bits after ACK}}$$

s.t.

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} power(S_n, A_n) \leq P_{\max}$$

- Optimal random policy exists
- Optimal pdf obtained through linear programming

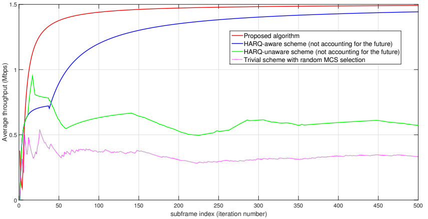


Case 2 : Best Modulation and Coding scheme [Ksa15]

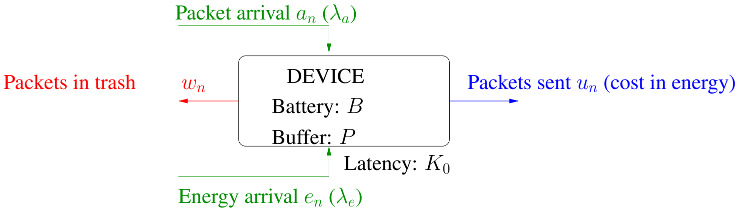
- Action : Best modulation and coding for the first packet transmission
- Action : Best modulation for the retransmission
  - if BPSK : a few redondancy bits sent but well protected
  - if QAM : a lot of redondancy bits sent but not well protected
- State : current channel impulse response, number of HARQ attempts, effective SNR

Numerical result

- LTE set-up
- Correlated channel (25km/h)



Scheduling with energy harvesting [Faw17]



Action : u_n sent packets (of length L) using the following energy

$$C_n = \frac{\sigma^2}{g_n}\left(2^{u_n L} - 1\right)$$

States : $\mathbf{s}_n$

- $\mathbf{k}_n$: age of the packet within the buffer
- b_n : battery level $\rightarrow b_{n+1} = \min(b_n - C_n + e_{n+1}, B)$
- g_n : channel gain

Scheduling with energy harvesting (cont'd)

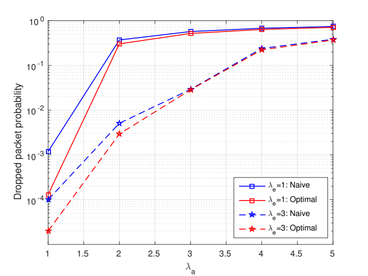
Reward : packet loss (buffer overflow, delay non-fulfillment)

$$\min \lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}\sum_{n=0}^{N}(\varepsilon_o(\mathbf{s}_n, u_n) + \varepsilon_d(\mathbf{s}_n, u_n))\right]$$

with

- ε_o number of dropped packets due to buffer overflow
- ε_d number of dropped packets due to delay non-fulfillment

Numerical results



Not treated issues

- Curse of dimensionality
- Competition between users (game theory [Lun15])
- Unknown Q : reinforcment learning, Q-learning
- Closely related to Dynamic Programming (Viterbi algorithm for channel decoding or ISI management)

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