Multi-bit Quantizer Design for Distributed Parameter Estimation

Yue Bi*†, Philippe Ciblat†, Yue Wu*, Cunqing Hua*
*School of Electronic Information and Electrical Engineering, Shanghai Jiao Tong University, China
{wuyue,cqhua}@sjtu.edu.cn
† LTCI, Telecom Paris, IP Paris, 91120 Palaiseau, France, {bi,ciblat}@telecom-paris.fr

***Abstract*—We consider sensors deployed in diverse locations measuring a common parameter through noisy observations. These observations are quantized to be sent to a fusion center doing the estimation of the common parameter. We design these quantizers to minimize the worst-case mean square error for common parameter estimation. Relying on an asymptotic regime in terms of sensors' number and on random multi-bit quantizer per sensor, we provide a relevant continuous distribution for the thresholds of these quantizers via signomial programming. Through numerical simulations, we show that the proposed quantizers outperform the uniformly-distributed one and some deterministic ones even when the number of sensors is limited.**

***Index Terms*—Distributed estimation, Minimax, Cramer-Rao bound, Quantization, Signomial programming.**

I. Introduction

In numerous surveillance applications, sensors are positioned in various locations with the aim of measuring a common phenomenon and, consequently, the same parameter [1]–[3]. The sensors, individually, do not carry out the final parameter estimation. Instead, they transmit their data after quantization through a propagation channel to a fusion center (FC), which conducts the estimation. Discovering multi-bit quantizers constitutes a key issue in estimation. The objective of this paper is to find relevant multi-bit quantizers.

More precisely, most works consider that *i)* the quantized version of a new arrival sample is sent to the FC according to a sequential Round-Robin (RR) technique, *ii)* then the FC collects all the quantized samples during one RR round to perform the estimation, and *iii)* so the performance are evaluated for one RR round. In that context, Cramer-Rao bound (CRB) or Mean Square Error (MSE) has been calculated under different assumptions [4]–[12]. In [11], [12], lower bounds for the estimation quality are provided and they are independent of any quantizer. Nevertheless, when compared to a practical quantizer, these bounds are too pessimistic. In other above-mentioned papers, only performance with specified quantizers are considered but without quantizer optimization.

In [13]–[20], the authors propose quantizer optimization for different configurations and assumptions. For instance, [13] proposes an optimal deterministic multi-bit quantizer for one sensor at low Signal-to-Noise Ratio (SNR) case. In [14], Bayesian CRB and a dynamic programming approach are considered to exhibit the optimal multi-bit deterministic quantizer in a single sensor context. In [15]–[17], the authors propose a deterministic quantizer obtained by minimizing the CRB when the parameter vanishes (i.e., at low SNR) with a particle swarm optimization algorithm. In [18], multi-sensors are considered but each equipped with a one-bit quantizer. The quantizer is assumed to be random at each sensor but the related threshold cumulative density function is linear piece-wise and data-dependent. The criterion is minimax, i.e., they minimize the worst CRB with respect to the parameter range. In [19], the same setup as [18] is considered but they find the best threshold distribution without the linear-piecewise structure assumption by optimizing the asymptotic relative efficiency which is equivalent to work on CRB. In [20], a scenario with multiple sensors is considered. Mathematically, the authors optimize the worst case CRB in an iterative way with respect to the threshold of each sensor.

In this paper, we consider multi-bit quantizers in multiple sensors scenarios for the minimax approach. We assume that each sensor has a random quantizer coming from a common distribution between sensors. This common distribution is optimized for the worst case (and not the average one as in Bayesian approach) when the number of sensors is large enough. In [19], the same approach was considered but for one-bit quantizer. Here, the challenge is to extend [19] to multi-bit quantizer. This extension is not straightforward since *i)* expressing our CRB in closed-form requires order statistics, and *ii)* the obtained optimization problem is not convex anymore but requires the use of signomial programming.

The remainder of this paper is organized as follows: Section II is devoted to the system model. Section III provides the CRB for finite number of sensors. In Section IV, the CRB is simplified when number of sensors goes to infinity. The optimization problem is given and solved in Section V. Numerical results showing the gain provided by our approach are drawn in Section VI. Concluding remarks are in Section VII.

II. System model

We consider a distributed estimation system with K sensors and one FC. The goal is to estimate a scalar parameter θ. Sensor $k \in \{1, \cdots, K\}$ collects a noisy sample y_k. This sample is transmitted after a quantization process with B bits. The FC so receives KB bits. Transmission channels between the sensors and the FC are assumed to be perfect.

This work has been supported by National Key R&D Program of China under Grant No 2023YFB2704903 and No 2020YFB1807504, and ERC Grant "CTOCom".

For the sake of simplicity, we consider the following model between θ and y_k:

$$y_k = \theta + w_k, \tag{1}$$

where w_k is a zero-mean white Gaussian noise with variance σ_w^2 and θ is assumed to be within the finite support $\mathcal{I} \subset \mathbb{R}$.

The output q_k of the quantizer Q_k at sensor k is

$$q_k = Q_k(y_k). \tag{2}$$

This quantizer Q_k transforms the continuous scalar y_k into B bits in such a way

$$Q_k(u) \triangleq \begin{cases} 0 & \text{if } u < \tau_{k,1}, \\ L & \text{if } u \geq \tau_{k,L}, \\ i & \text{if } \tau_{k,i} \leq u < \tau_{k,i+1}, \text{ for } i \in \{1, \cdots, L-1\}, \end{cases}$$

where $L = 2^B - 1$ and $\{\tau_{k,i}\}_{i \in \{1,\cdots,L\}}$ is a strictly increasing sequence of thresholds.

The goal of this paper is to find a relevant way to build the threshold sequence per sensor. For doing that, we rely on the derivations of the Cramer-Rao bound for θ at the FC.

III. Non-asymptotic Cramer-Rao Bound

The Cramer-Rao Bound for any unbiased estimation at the FC can be written as

$$\mathbb{E}[(\hat{\theta} - \theta)^2] \geq \text{CRB}(\theta) = \frac{1}{F(\theta)}, \tag{3}$$

where $F(\theta)$ is the Fisher Information associated with quantized bits $\mathbf{q} \triangleq \{q_k\}_{k \in \{1,\cdots,K\}}$ and is written as

$$F(\theta) \triangleq \mathbb{E}_{\mathbf{Q}|\theta}\left[\left(\frac{\partial \log p_{\mathbf{Q}|\theta}(\mathbf{q}|\theta)}{\partial \theta}\right)^2\right] \tag{4}$$

with $p_{\mathbf{Q}|\theta}$ the distribution of $\mathbf{q}$ for the parameter value θ. As data are independent between sensors, we have

$$F(\theta) = \sum_{k=1}^{K} F_k(\theta), \tag{5}$$

where $F_k(\theta)$ is the Fisher information provided by one quantized sample for the k-th sensor and

$$F_k(\theta) \triangleq \mathbb{E}_{Q|\theta}\left[\left(\frac{\partial \log p_{Q|\theta}(q_k, \theta)}{\partial \theta}\right)^2\right]. \tag{6}$$

According to [7], we get

$$F_k(\theta) = \sum_{i=0}^{L} \frac{1}{p_{Q|\theta}(q_k = i|\theta)} \cdot \left(\frac{\partial p_{Q|\theta}(q_k = i|\theta)}{\partial \theta}\right)^2 \tag{7}$$

where $p_{Q|\theta}(q_k = i|\theta)$ can be decomposed as follows

$$p_{Q|\theta}(q_k = i|\theta) = \int p_{Q|Y}(q_k = i|y_k) p_{Y|\theta}(y_k|\theta) dy_k \tag{8}$$

since the value of q_k depends only on y_k according to Eq. (2). As $y_k|\theta$ follows a Gaussian distribution with mean θ and variance σ_w^2, we deduce

$$p_{Q|\theta}(q_k = i|\theta) = \begin{cases} \Psi(\tau_{k,1}, \theta) & \text{if } i = 0 \\ 1 - \Psi(\tau_{k,L}) & \text{if } i = L \\ \Psi(\tau_{k,i+1}, \theta) - \Psi(\tau_{k,i}) & \text{otherwise}, \end{cases}$$

where

$$\Psi(\tau, \theta) \triangleq \Phi\left(\frac{\tau - \theta}{\sigma_w}\right) \tag{9}$$

with Φ the cumulative distribution function for standard normal distribution.

Then, we have

$$\frac{\partial p_{Q|\theta}(q_k = i|\theta)}{\partial \theta} = \begin{cases} -\psi(\tau_{k,1}, \theta) & \text{if } i = 0, \\ \psi(\tau_{k,L}, \theta) & \text{if } i = L, \\ -\psi(\tau_{k,i+1}, \theta) + \psi(\tau_{k,i}, \theta) & \text{otherwise}, \end{cases}$$

where

$$\psi(\tau, \theta) \triangleq \frac{1}{\sigma_w} \phi\left(\frac{\tau - \theta}{\sigma_w}\right) \tag{10}$$

with ϕ the probability density function for standard normal distribution.

Finally, we obtain that

$$F_k(\theta) = \eta_1(\tau_{k,1}, \theta) + \eta_L(\tau_{k,L}, \theta) + \sum_{i=1}^{L-1} \eta(\tau_{k,i}, \tau_{k,i+1}, \theta)$$

with

$$\eta_1(\tau, \theta) \triangleq \frac{(\psi(\tau - \theta))^2}{\Psi(\tau - \theta)}, \tag{11}$$

$$\eta_L(\tau, \theta) \triangleq \frac{(\psi(\tau - \theta))^2}{1 - \Psi(\tau - \theta)}, \tag{12}$$

$$\eta(\tau, \tau', \theta) \triangleq \frac{(\psi(\tau' - \theta) - \psi(\tau - \theta))^2}{\Psi(\tau' - \theta) - \Psi(\tau - \theta)}. \tag{13}$$

IV. Asymptotic Cramer-Rao Bound

By considering large number of sensors, the CRB can be approximated by

$$\text{CRB}(\theta) \approx \frac{1}{K \cdot \underline{F}(\theta)} \tag{14}$$

where

$$\underline{F}(\theta) \triangleq \lim_{K \to \infty} \frac{1}{K} \sum_{k=1}^{K} F_k(\theta) \tag{15}$$

$$= \lim_{K \to \infty} \frac{1}{K} \sum_{k=1}^{K} \Big(\eta_1(\tau_{k,1}, \theta) + \eta_L(\tau_{k,L}, \theta) + \sum_{i=1}^{L-1} \eta(\tau_{k,i}, \tau_{k,i+1}, \theta) \Big). \tag{16}$$

Like [19], the evaluation of the term $\underline{F}(\theta)$ will be done by assuming that the thresholds $\{\tau_{k,i}\}_{k,i}$ correspond to a realization of a random variable. The realizations on k are iid but the realizations on i have to be sorted since $\tau_{k,i} \leq \tau_{k,i+1}$ by construction. Therefore, we rely on the order statistics [21]. Let λ be the probability distribution for the L ordered threshold for any sensor (the distribution is assumed to be the same whatever the sensor, so we skip the index k). We also define the marginal probability distribution for the first threshold (actually, $\tau_{k,1}$ for any sensor) as λ_1, the marginal probability distribution for the last threshold (actually, $\tau_{k,L}$ for any sensor) as λ_L, and the joint probability distribution between two consecutive thresholds (actually, $(\tau_{k,i}, \tau_{k,i+1})$ for any sensor with $i \in \{1, \cdots, L-1\}$) as λ_i. We thus obtain

$$\begin{aligned}\underline{F}(\theta) &= \int_{-\infty}^{\infty} \eta_1(\tau,\theta)\lambda_1(\tau)d\tau + \int_{-\infty}^{\infty} \eta_L(\tau,\theta)\lambda_L(\tau)d\tau \\ &+ \sum_{i=1}^{L-1} \int_{-\infty}^{\infty}\int_{-\infty}^{\tau'} \eta(\tau,\tau',\theta)\lambda_i(\tau,\tau')d\tau d\tau'. \qquad (17)\end{aligned}$$

The ordered thresholds are obtained thanks to an unique random variable whose probability density function is g and the cumulative distribution function is G. More precisely, for each sensor, we collect L realizations related to the random variable driven by g. Then these variables are ranked in order to provide the ordered thresholds. Consequently, the distributions for the ranked variables are given by

$$\lambda_1(\tau) = c[1-G(\tau)]^{L-1}g(\tau), \qquad (18)$$
$$\lambda_L(\tau) = c[G(\tau)]^{L-1}g(\tau), \qquad (19)$$
$$\lambda_i(\tau,\tau') = c_i[G(\tau)]^{i-1}[1-G(\tau')]^{L-i}g(\tau)g(\tau'), \qquad (20)$$

with $c = L$ and $c_i = L(L-1)\binom{L-2}{i-1}$.

V. Proposed Quantizer

Our goal is to minimize the asymptotic CRB with respect to g. According to Eq. (14), minimizing the CRB is equivalent to maximizing $\underline{F}$. Therefore by incorporating Eqs. (18)-(20) into Eq. (17) and over the infimum of θ, we obtain the following minimax optimization problem.

Problem 1 (Functional optimization problem)**.** *Assuming $g(\tau) \geq 0$, we solve*

$$\max_g \inf_\theta \quad f(g,\theta)$$

s.t. $\int g(\tau)d\tau = 1$, *with*

$$\begin{aligned}f(g,\theta) &= \int \eta_1(\tau,\theta)\cdot c[1-G(\tau)]^{L-1}g(\tau)d\tau \\ &+ \int \eta_L(\tau,\theta)\cdot c[G(\tau)]^{L-1}g(\tau)d\tau \\ &+ \sum_{i=1}^{L-1}\iint^{\tau'} \eta(\tau,\tau',\theta)\cdot c_i[G(\tau)]^{i-1} \\ &\quad \cdot[1-G(\tau')]^{L-i}g(\tau)g(\tau')d\tau d\tau'. \qquad (21)\end{aligned}$$

We propose to replace this functional optimization problem with a vector optimization problem by discretizing g. Before going further, we assume that the support of θ is symmetric about the origin, i.e. $\mathcal{I} = [-W_0, W_0]$, with $W_0 > 0$. Then, we discretize the interval $\mathcal{I}$ into $N(>0)$ regular subintervals $\{\mathcal{J}_j \triangleq [u_{j-1}, u_j]\}_{j\in\{1,\cdots,N\}}$ with $u_0 = -W_0$ and $u_N = W_0$ whose the middle of each interval is m_j for $j \in \{1,\ldots,N\}$. The discrete version of g at m_j is given by its normalized value i.e.,

$$a_j = \frac{g(m_j)}{\sum_{j'=1}^{N} g(m_{j'})} \quad \forall j \in \{1,\cdots,N\}. \qquad (22)$$

We now define the cumulative sequence of $\mathbf{a} \triangleq \{a_j\}_{j\in\{1,\cdots,N\}}$ as

$$A_\ell = \sum_{j=1}^{\ell} a_j,$$

and the complement cumulative sequence as $R_\ell = 1 - A_\ell$. By convention, we also put $A_0 = 0$. Define also the sequence $\{\theta_j\}_{j\in\{1,\cdots,M\}}$ with $\theta_1 = -W_0$ and $\theta_M = W_0$ representing a quantization of the parameter range.

We discretize the function $f(g,\theta)$ into a set of functions $\tilde{f}_j(\mathbf{a})$ as follows

$$\begin{aligned}\tilde{f}_j(\mathbf{a}) &= \sum_{\ell=1}^{N} d_{\ell,j}^{(1)} R_\ell^{L-1} a_\ell + \sum_{\ell=1}^{N} d_{\ell,j}^{(L)} A_\ell^{L-1} a_\ell \\ &+ \sum_{i=1}^{L-1}\sum_{\ell_2=1}^{N}\sum_{\ell_1=1}^{\ell_2} d_{\ell_1,\ell_2,j}^{(i)} A_{\ell_1}^{i-1} a_{\ell_1} R_{\ell_2}^{L-i} a_{\ell_2} \qquad (23)\end{aligned}$$

with $d_{\ell,j}^{(1)} = c\cdot\eta_1(u_\ell,\theta_j)$, $d_{\ell,j}^{(L)} = c\cdot\eta_L(u_\ell,\theta_j)$, and $d_{\ell_1,\ell_2,j}^{(i)} = c_i\cdot\eta(u_{\ell_1},u_{\ell_2},\theta_j)$. The discretized version of Problem 1 is straightforward. But in terms of optimization, we have the following objective function to maximize $\min_j \tilde{f}_j(\mathbf{a})$. In order to handle the minimum operator easily, we introduce a new variable x bounding all functions $\tilde{f}_j(\mathbf{a})$. We then obtain the following optimization problem.

Problem 2 (Discretized optimization problem)**.** *Assuming $a_\ell \geq 0$ for $\ell \in \{1,\cdots,N\}$ and $x \geq 0$, we solve*

$$\max_{\mathbf{a},x} x \qquad (24a)$$
$$\text{s.t.} \quad \tilde{f}_j(\mathbf{a}) \geq x \;\forall j \in \{1,\cdots,M\}, \qquad (24b)$$
$$\sum_{\ell=1}^{N} a_\ell = 1. \qquad (24c)$$

Focusing on Problem 2 rather than on Problem 1 leads to a loss in performance which may be controlled by the choice of N. We can find a stationary point of Problem 2 by Signomial Geometric Programming (SGP). However, SGP faces certain difficulties when addressing optimization problems with equality constraints. We thus change the optimization variables from $\mathbf{a}$ to $\mathbf{A} \triangleq \{A_1, A_2, \cdots, A_{N-1}\}$ (A_0 and A_N are excluded from the variable vector as they are always equal to 0 and 1 respectively). Functions in Eq. (23) are rewritten as

$$\begin{aligned}&f_j(\mathbf{A}) \\ &= \sum_{\ell=1}^{N} d_{\ell,j}^{(1)} R_\ell^{L-1}(A_\ell - A_{\ell-1}) + \sum_{\ell=1}^{N} d_{\ell,j}^{(L)} A_\ell^{L-1}(A_\ell - A_{\ell-1}) \\ &+ \sum_{i=1}^{L-1}\sum_{\ell_2=1}^{N}\sum_{\ell_1=1}^{\ell_2} d_{\ell_1,\ell_2,j}^{(i)} A_{\ell_1}^{i-1}(A_{\ell_1} - A_{\ell_1-1}) \\ &\times R_{\ell_2}^{L-i}(A_{\ell_2} - A_{\ell_2-1}). \qquad (25)\end{aligned}$$

The function f_j is a signomial and can be written as a difference of two posynomials [22]. Let's write this decomposition below

$$f_j(\mathbf{A}) \triangleq f_{j,d}(\mathbf{A}) - f_{j,n}(\mathbf{A}), \qquad (26)$$

where $f_{j,d}$ and $f_{j,n}$ are two posynomials.

Thanks to Eqs. (25)-(26), Problem 2 can be rewritten as Problem 3 which is a SGP and so can be solved easily (see [23] for more details on the practical algorithms).

Problem 3 (Final optimization problem)**.** *Assuming* $A_\ell \geq 0$ *for* $\ell \in \{1, \cdots, N\}$ *and* $x \geq 0$*, we solve*

$$\min_{\mathbf{A},x} x^{-1} \tag{27a}$$

$$\text{s.t.} \quad \frac{x + f_{j,n}(\mathbf{A})}{f_{j,d}(\mathbf{A})} \leq 1, \forall j \in \{1, \cdots, M\}, \tag{27b}$$

$$\frac{A_{\ell-1}}{A_\ell} \leq 1, \qquad \forall \ell \in \{2, \cdots, N-1\}, \tag{27c}$$

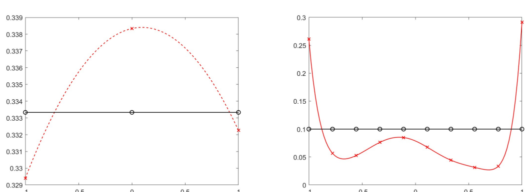
$$A_\ell \leq 1, \qquad \forall \ell \in \{1, \cdots, N-1\}. \tag{27d}$$

Once Problem 3 is solved, we obtain the points $\{a_\ell^\star\}_{\ell=1,\cdots,N}$. Then we compute $g(\tau)$ by doing a Lagrangian-polynomial interpolation on the set $\{(m_\ell, a_\ell^\star)\}_{\ell=1,\cdots,N}$ followed by a normalization step. Finally, each sensor builds its quantizer as follows: it obtains $(L-1)$ realizations from the random variable whose distribution is $g(\tau)$. Each sensor sorts its realizations to be employed as its thresholds.

VI. Numerical Results

We assume that each sensor sends 2 bits ($L = 4$) and $M = 8$. Problem 3 has been numerically solved by algorithm described in [24]. Two cases for discretization problem have been computed: $N = 3$ or $N = 10$. We also examine the performance of the uniformly-distributed (i.e. $a_\ell = 1/N, \forall \ell \in \{1, \cdots, N\}$) and the regular deterministic quantizers.

In Fig. 1, we plot the optimized continuous distribution $g(\tau)$. Two cases have been considered: $N = 3$ and $N = 10$. We observe that the solutions are different from the uniform distribution, which implies the necessity of the optimization. Moreover, $N = 3$ does not match with $N = 10$ exhibiting better performance, as shown later in Fig. 2 and Fig. 3. This implies that $N = 3$ is insufficient.



(a) $N = 3$ (b) $N = 10$

Fig. 1: Proposed distribution $g(\tau)$ obtained by polynomial fitting based on discrete solution of Problem 3.

In Fig. 2, we plot the average CRB per sensor (obtained as the inverse of the average Fisher information $\min_{j'} \frac{1}{K} \sum_{k=1}^{K} F_k(\theta'_{j'})$ where the set $\{\theta'_{j'}\}_{j' \in \{1,\cdots,M'\}}$ with $M' = 100$ covers the parameter range well, and actually much more than the set chosen for optimization) versus K. The SNR (defined as $1/\sigma_w^2$) is put at 8dB. This evaluation is executed for 20 different realizations of quantizers per sensor.

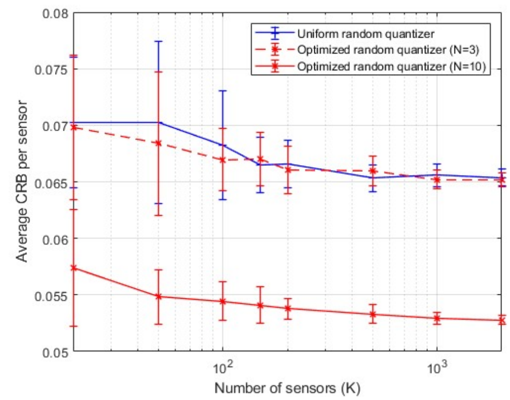


Fig. 2: Average CRB vs K (SNR=8dB).

Related standard deviation is also plotted. We observe that even for small values of K, our optimization obtained through an asymptotic approach is still valid. The information provided by each sensor at the system is higher with the optimized version of the random quantizer. Only the standard deviation decreases with K.

In Fig. 3, we plot the CRB (obtained as the inverse of the Fisher information $\min_j \sum_{k=1}^{K} F_k(\theta'_j)$) versus SNR for six quantizers including those of [15] (IDEA) and [14], [16] (PSOA) and the unquantized case obtained by setting $L = \infty$ and uniform quantizer in Eq. (16). The optimized random quantizer with $N = 10$ gives improvement compared to the other ones especially at high SNR. For instance, at mid/high SNR, the gain is around 1dB.

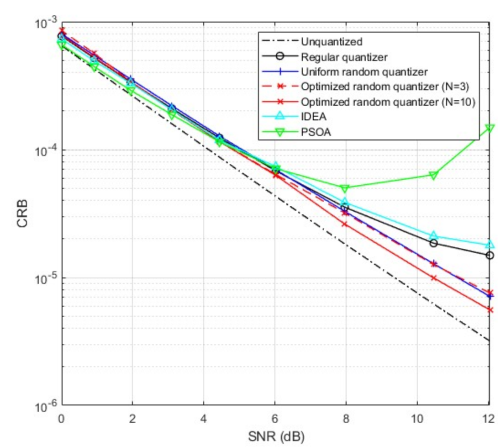


Fig. 3: CRB vs SNR ($K = 2000$).

VII. Conclusion

Our work proposed a worst-case random quantizer in the context of quantized communications of observations to a fusion center in order to estimate a common parameter. The proposed quantizer outperforms uniformly-distributed, regular deterministic and previously-proposed quantizers for mid/high SNR scenarios.

REFERENCES

[1] F. Pasqualetti, R. Carli, and F. Bullo, "Distributed estimation via iterative projections with application to power network monitoring," *Automatica*, vol. 48, no. 5, pp. 747–758, May 2012.

[2] X. Ge, Q.-L. Han, X.-M. Zhang, L. Ding, and F. Yang, "Distributed event-triggered estimation over sensor networks: A survey," *IEEE Transactions on Cybernetics*, vol. 50, no. 3, pp. 1306–1320, Mar. 2020.

[3] S. He, H.-S. Shin, S. Xu, and A. Tsourdos, "Distributed estimation over a low-cost sensor network: A review of state-of-the-art," *Information Fusion*, vol. 54, pp. 21–43, Feb. 2020.

[4] A. Ribeiro and G. Giannakis, "Bandwidth-constrained distributed estimation for wireless sensor networks-part I: Gaussian case," *IEEE Transactions on Signal Processing*, vol. 54, no. 3, pp. 1131–1143, Mar. 2006.

[5] A.-G. Zebadua, P.-O. Amblard, E. Moisan, and O. Michel, "Compressed and quantized correlation estimator," *IEEE Transactions on Signal Processing*, vol. 65, no. 1, pp. 56–68, Jan. 2017.

[6] P. Stoica, X. Shang, and Y. Cheng, "The Cramér–Rao bound for signal parameter estimation from quantized data [lecture notes]," *IEEE Signal Processing Magazine*, vol. 39, no. 1, pp. 118–125, Jan. 2022.

[7] A. Sani and A. Vosoughi, "On performance limit for bandwidth-constrained distributed parameter estimation systems," in *Conference on Information Sciences and Systems (CISS)*, Mar. 2022, pp. 78–83.

[8] Y. Han, A. Ozgur, and T. Weissman, "Geometric lower bounds for distributed parameter estimation under communication constraints," *IEEE Transactions on Information Theory*, vol. 67, no. 12, pp. 8248–8263, Dec. 2021.

[9] E. Meir and T. Routtenberg, "Cramér-Rao bound for estimation after model selection and its application to sparse vector estimation," *IEEE Transactions on Signal Processing*, vol. 69, pp. 2284–2301, 2021.

[10] L. Liu, C. Hua, J. Xu, G. Leus, and Y. Wang, "Greedy sensor selection: Leveraging submodularity based on volume ratio of information ellipsoid," *IEEE Transactions on Signal Processing*, vol. 71, pp. 2391–2406, 2023.

[11] L. P. Barnes, Y. Han, and A. Ozgur, "Lower bounds for learning distributions under communication constraints via Fisher information," *Journal of Machine Learning Research*, vol. 21, no. 236, pp. 1–30, 2020.

[12] J. C. Duchi, M. I. Jordan, M. J. Wainwright, and Y. Zhang, "Optimality guarantees for distributed statistical estimation," *arXiv preprint arXiv:1405.0782*, 2014.

[13] A. Mezghani, F. Antreich, and J. A. Nossek, "Multiple parameter estimation with quantized channel output," in *ITG Workshop on Smart Antennas (WSA)*, Feb. 2010.

[14] Y. Cheng, X. Shang, J. Li, and P. Stoica, "Interval design for signal parameter estimation from quantized data," *IEEE Transactions on Signal Processing*, vol. 70, pp. 6011–6020, 2022.

[15] F. Gao, L. Guo, H. Li, J. Liu, and J. Fang, "Quantizer design for distributed GLRT detection of weak signal in wireless sensor," *IEEE Transactions on Wireless Communications*, vol. 14, no. 4, pp. 2032–2042, Apr. 2015.

[16] X. Cheng, D. Ciuonzo, and P. Rossi, "Multibit decentralized detection through fusing smart and dumb sensors based on rao test," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 56, no. 2, pp. 1391–1405, Feb. 2019.

[17] S. Yang, W. Yi, A. Jakobsson, Y. Wang, and H. Xiao, "Weak signal detection with low-bit quantization in colocated mimo radar," *IEEE Transactions on Signal Processing*, vol. 71, no. 2, pp. 447–460, Feb. 2023.

[18] S. Kar, H. Chen, and P. K. Varshney, "Optimal identical binary quantizer design for distributed estimation," *IEEE Transactions on Signal Processing*, vol. 60, no. 7, pp. 3896–3901, Jul. 2012.

[19] A. Kipnis and J. Duchi, "Mean estimation from one-bit measurements," *IEEE Transactions on Information Theory*, vol. 68, no. 9, pp. 6276–6296, Sep. 2022.

[20] P. Venkitasubramaniam, L. Tong, and A. Swami, "Quantization for Maximin ARE in Distributed Estimation," *IEEE Transactions on Signal Processing*, vol. 55, no. 7, pp. 3596–3605, Jul. 2007.

[21] H. A. David and H. N. Nagaraja, *Order Statistics*. Wiley Series in Probability and Statistics, 2003.

[22] M. Avriel and A.-C. Williams, "Complementary geometric programming," *SIAM Journal of Applied Mathematics*, vol. 19, pp. 125–141, 1970.

[23] A. Louchart, "Resource allocation adapted to nonlinear impairments for cognitive satellite networks," Ph.D. dissertation, Institut Polytechnique de Paris, 2021.

[24] G. Xu, "Global optimization of signomial geometric programming problems," *European Journal of Operational Research*, vol. 233, no. 3, pp. 500–510, Mar. 2014.