Resource Allocation for Type-I HARQ-based Wireless Networks with Finite-Length Codes

Sébastien Marcille*†, Philippe Ciblat*, and Christophe J. Le Martret†

*Telecom ParisTech, Paris, France
†Thales Communications & Security, Gennevilliers, France

***Abstract*—Optimal bandwidth and power allocation is performed in a wireless clustered ad hoc network (CAHN) when Type-I Hybrid Automatic Repeat reQuest (HARQ) is used at the link layer along with an Orthogonal Frequency Division Multiple Access (OFDMA) scheme under individual data rate constraint. The considered network topology imposes the resource allocation manager to know only the channel state information (CSI) statistics. Gaussian codes with a finite block length are assumed in order to give the best achievable performance, with finite latency, that one could expect from this HARQ-based CAHN. The framework developed in this paper is well adapted to predict the performance of strong channel coding over the Rayleigh channel. Typically it gives useful insights on the behavior of the proposed HARQ-based CAHN when LDPC codes are implemented.**

I. Introduction

We focus on optimal resource allocation in the context of a wireless clustered ad hoc network (CAHN). The nodes of this network are managed by a cluster head (CH) collecting the requests of the transmitting nodes, and performing a centralized resource allocation accordingly. OFDMA is considered in order to allow simultaneous and interference-free peer to peer links in the cluster avoiding to concentrate all the traffic at the CH. To manage the CAHN, in addition to OFDMA, the transmissions also follow a Time Division Multiple Access (TDMA) scheme with specific slots reserved for signalling and data. An important consequence is that the CH is able to get *statistical* channel state information (CSI) only for the different links in the cluster. Indeed, since the resource allocation is centralized at the CH, the time to initiate a specific link transmission and to transfer back the CSI may last several frame periods resulting in a CSI which is completely outdated when available at the CH. The only possibility offered to the CH is to draw statistics from the received CSI along time and use it for resource allocation as in [1].

In order to handle efficiently the unknown variation of the channel at the transmitter side (since only channel statistics are available), we propose to consider

- Frequency Hopping (FH) for providing channel frequency diversity enough, leading to a equivalent fast fading channel,
- and (Type-I) Hybrid Automatic Repeat reQuest (HARQ) since it is a powerful mechanism enabling to accommodate the unknown channel variations by achieving a good trade-off between channel coding and retransmission [2].

When HARQ is used, it is now well-known [3], [4] that the relevant metric for data rate is the so-called goodput. In fact, the goodput performs a natural trade-off between capacity (as an information reward) and QoS (through packet error rate). In order to increase the lifetime of the cluster and to mitigate also the inter-cluster interference, we would like to find *resource allocation (occupied bandwidth and transmit power per user) minimizing the total power in the considered cluster when a minimal goodput per user is required assuming only statistical CSI at the resource allocation manager.*

Such an optimization problem has been already addressed in [1] assuming that the coding used in (Type-I) HARQ is the convolutive one. Here, we would like to obtain a benchmark when using the best code, namely, the finite-length Gaussian codes. We restrict to finite-length for delay purpose. Notice that if infinite-length Gaussian codes were used, the HARQ becomes useless and then the relevant data rate metric boils down to the ergodic capacity as in [5]. Moreover using optimal finite-length codes will provide a good insight on the resource allocation to be done when using the best known practical codes, such as LDPC [6].

In order to perform the resource allocation algorithm, the goodput requires the error probability of the considered codes (*i.e.*, the finite-length Gaussian codes) in Rayleigh channel. This can be obtained through the outage probability concept. Indeed, even if the concept of outage probability is usually dedicated to the analysis of the block fading channel [7], and more generally of non-ergodic channels, it can be also used in AWGN channels (and so, in ergodic channels) when finite size inputs and outputs are considered [8] since their mutual information is still a random variable. Then, an outage occurs whenever the coding rate R exceeds the mutual information $i(\boldsymbol{X};\boldsymbol{Y})$ between the transmitted codeword $\boldsymbol{X}$ and received codeword $\boldsymbol{Y}$ of length n [9], with probability $P_o := \Pr\{i(\boldsymbol{X},\boldsymbol{Y}) \leq R\}$. Like [9] in the context of AWGN channels, the outage probability can still represent the ultimate error probability of finite-length codes in the context of Rayleigh channel. The distribution of the mutual information in a (static) fading channel has been succinctly described in [8], but a closed-form expression of the outage probability has still to be obtained in Rayleigh fading channel. As a consequence, an additional contribution of the paper is to provide a new approximate closed-form expression for the outage probability in Rayleigh channel.

The rest of the paper is organized as follows: In Section II, the error probability of finite-length Gaussian codes is computed in closed-form over the Rayleigh channel. Based on this new result, optimal power and bandwidth allocation is performed in Section III. Finally, some numerical results are given in Section IV and Section V concludes the paper.

II. The error probability of finite-length Gaussian codes over the Rayleigh channel

It is more convenient to deal with single link for deriving the error performance of the finite-length Gaussian codes. The extension to multiuser links will be treated in Section III. Denoting by $\boldsymbol{Y} \in \mathbb{C}^n$ the channel output:

$$\boldsymbol{Y} = \boldsymbol{H}\boldsymbol{X} + \boldsymbol{N}, \tag{1}$$

where $\boldsymbol{X}$ and $\boldsymbol{N}$ are random vectors of length n with i.i.d. elements X_k and N_k, respectively. $X_k \sim \mathcal{CN}(0, E_s)$, whereas $N_k \sim \mathcal{CN}(0, N_0)$, and $\boldsymbol{H}$ is a $n \times n$ diagonal matrix with i.i.d. elements $H_k \sim \mathcal{CN}(0, \sigma_h^2)$. Thus, the channel gains $|H_k|$ are Rayleigh distributed and a random SNR can be defined as:

$$\mathrm{SNR}_k = \frac{|H_k|^2 E_s}{N_0}. \tag{2}$$

SNR_k is exponentially distributed with parameter $1/\overline{\mathrm{SNR}}$, where $\overline{\mathrm{SNR}} = \sigma_h^2 E_s/N_0$ is the average SNR.

For finite n, the mutual information rate $i(\boldsymbol{X}, \boldsymbol{Y})$ is a random variable denoted by Z_n. In [8], it was shown that $Z_n = (1/n)\sum_{k=1}^{n} i_k$ with $(i_k)_{k\in\{1\ldots,n\}}$ an i.i.d. random process given by:

$$i_k = \log\left(1 + |H_k|^2 \frac{E_s}{N_0}\right) + \sqrt{\frac{|H_k|^2 E_s/N_0}{1 + |H_k|^2 E_s/N_0}}\, W_k,$$

where W_k are i.i.d. Laplace random variables with mean zero and parameter 1, so that $\mathbb{E}[W_k] = 0$ and $\mathrm{Var}(W_k) = 2$, $|H_k|^2$ is exponentially distributed with parameter $1/\sigma_h^2$, and H_k is independent of W_k.

For the sake of simplicity, we resort to a Gaussian approximation of Z_n for the Rayleigh channel, which was shown to be accurate in [9] for the AWGN channel. Thus, as the sum of n i.i.d. random variables, Z_n is approximated with a Gaussian random variable $\mathcal{N}(m, \sigma_n^2)$, where $m := \mathbb{E}[Z_n]$ and $\sigma_n^2 := \mathrm{Var}(Z_n)$. The mean $m = \mathbb{E}[\log(1 + \mathrm{SNR}_k)]$ is easily obtained since Z_n is the sum of i.i.d. random variables, and since $\mathbb{E}[W_k] = 0$. It is well known that this expectation leads to:

$$m = e^{1/\overline{\mathrm{SNR}}} E_1(1/\overline{\mathrm{SNR}}), \tag{3}$$

where $E_1(x) := \int_1^{\infty} e^{-xu}/u\mathrm{d}u = \int_x^{\infty} e^{-t}/t\mathrm{d}t$ is known as the exponential integral [10]. The variance can be computed from the conditional variance formula [11], and we have

$$\sigma_n^2 = \frac{1}{n}\left(\varsigma^2 + \kappa^2\right). \tag{4}$$

with $\varsigma^2 := 2\mathbb{E}[\mathrm{SNR}_k/(1 + \mathrm{SNR}_k)]$ and $\kappa^2 := \mathrm{Var}(\log(1 + \mathrm{SNR}_k))$.

Like in [9], the error probability $P_e^{(n)}$ is computed as the cumulative distribution P_o of the mutual information rate Z_n, *i.e.* $P_e^{(n,R)} = \Pr\{Z_n \leq R\}$. Since the Gaussian random variable $\mathcal{N}(m, \sigma_n^2)$ has been introduced as an approximation to Z_n, the distribution of Z_n can be replaced with the Gaussian distribution, leading to:

$$P_e^{(n,R)} \approx Q\left(\frac{m - R}{\sigma_n}\right), \tag{5}$$

where m and σ_n^2 are known explicitly from Eqs. (3)-(4).

Actually, one can easily prove that

$$\varsigma^2 = 2 - \frac{2}{\overline{\mathrm{SNR}}} e^{1/\overline{\mathrm{SNR}}} E_1(1/\overline{\mathrm{SNR}}). \tag{6}$$

In addition, a simple approximation for the term κ^2 can be obtained at low SNR regime, and is given by:

$$\kappa^2 \approx \log^2(1 + \overline{\mathrm{SNR}}) - m^2 \quad (\text{for } \overline{\mathrm{SNR}} \text{ small enough}). \tag{7}$$

In Fig. 1, the closed-form expression given in Eq. (5) with m, ς and κ given by Eqs. (3)-(6)-(7) respectively is compared to the empirical outage probabilities for several values of rates R and blocklength n.

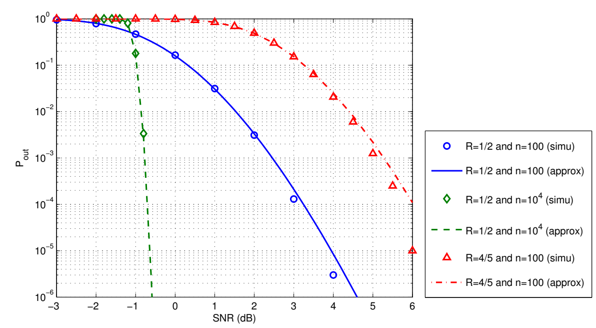

Figure 1. Approximate and empirical outage probability vs SNR.

III. Resource allocation algorithm

In all the paper, the superscript ^{T} stands for the transposition operator, and the (multi-variate) complex-valued circular Gaussian distribution with mean a and covariance matrix Σ is denoted $\mathcal{CN}(a, \Sigma)$.

A. System model and optimization problem

Each link is a frequency-selective channel and OFDM with N subcarriers is used to compensate for the frequency selectivity. It is assumed that the channel remains constant over one OFDM symbol but may change between two consecutive OFDM symbols. The channel corresponds to the link between the transmitting user k and any receiving node in the network, including the cluster head. Let $\boldsymbol{h}_k(i) = [h_k(i,0), \ldots, h_k(i, M-1)]^T$ be the channel impulse response of link k associated with OFDM symbol i, where M is the number of taps. Let us denote by $\boldsymbol{H}_k(i) = [H_k(i,0), \ldots, H_k(i, N-1)]^T$ the Fourier Transform of $\boldsymbol{h}_k(i)$. Assuming well-designed OFDM cyclic prefix and FH pattern, the received signal at OFDM symbol i and subcarrier n for link k is:

$$Y_k(i,n) = H_k(i,n)X_k(i,n) + Z_k(i,n), \tag{8}$$

where $X_k(i,n) \sim \mathcal{CN}(0, E_k W/N)$ is the transmitted symbol by link k at subcarrier n of OFDM symbol i, and the additive noise $Z_k(i,n) \sim \mathcal{CN}(0, N_0 W/N)$ where N_0 is the noise power spectral density and W is the total bandwidth. It is assumed that each channel is an independent random process with possibly different variances $\varsigma_{k,m}^2$ for each tap, *i.e.* $\boldsymbol{h}_k(i) \sim \mathcal{CN}(0, \Sigma_k)$ with $\Sigma_k := \mathrm{diag}_{M\times M}(\varsigma_{k,m}^2)$. Direct calculations show that the diagonal elements of $\boldsymbol{H}_k(i)$ are identically distributed [5], *i.e.* $H_k(i,n) \sim \mathcal{CN}(0, \varsigma_k^2)$ with $\varsigma_k^2 = \mathrm{Tr}(\Sigma_k)$.

Let us assume that the resource allocator only knows the CSI statistics given by:

$$G_k := \frac{\varsigma_k^2}{N_0}. \tag{9}$$

Since G_k is independent of i, the resource allocation algorithm will not distinguish between the subcarriers for a given link, and thus cannot allocate which subcarriers link k will use, but only how many. So the bandwidth proportion occupied by link k is equal to $\gamma_k = n_k/N$ where n_k is the number of subcarriers assigned to link k. Thus γ_k corresponds to the bandwidth parameter to be optimized. Due to the independence of G_k with respect to the subcarrier index, it is natural for link k to use the same average energy E_k on each subcarrier, which is the energy parameter to be optimized.

We remind that we would like to minimize the total energy consumption under individual data rate constraints. The energy consumed for sending one OFDM symbol is actually equal to $\sum_{k=1}^{K} \gamma_k E_k$. In addition, as HARQ is used, the goodput will be the meaningful metric for the data rate. We will denote by $\eta_k(\gamma_k, E_k)$ the goodput related to link k with respect to the bandwidth γ_k and the energy E_k. As a consequence, the considered optimization problem can be written as follows.

Problem 1.

$$\min_{(\boldsymbol{\gamma},\boldsymbol{E})} \sum_{k=1}^{K} \gamma_k E_k, \tag{10a}$$

$$\text{s.t.} \quad \log \eta_k(\gamma_k, E_k) \geq \log \eta_k^{(0)}, \ \forall k, \tag{10b}$$

$$\sum_{k=1}^{K} \gamma_k \leq 1, \tag{10c}$$

$$\gamma_k \geq 0, \ E_k \geq 0, \ \forall k. \tag{10d}$$

Notice that the $\log$ in Eq. (10b) has been introduced for algorithmic concerns and will be explained in detail in next the Subsection.

B. Resource allocation algorithm

First of all, on each subcarrier, link k undergoes an average SNR given by:

$$\overline{\mathrm{SNR}}_k = G_k E_k. \tag{11}$$

Then, in Type-I HARQ, the goodput η_k takes the following form [1]

$$\eta_k(\gamma_k, E_k) = \gamma_k r_k (1 - P_e^{(n,r_k)}(G_k E_k)), \tag{12}$$

where $P_e^{(n,R)}(\overline{\mathrm{SNR}})$ is the error probability with respect to $\overline{\mathrm{SNR}}$, of a (n,R) Gaussian code, and r_k is the information rate of link k. Thanks to Eq. (5), we know

$$P_e^{(n,R)}(\overline{\mathrm{SNR}}) = Q\left(\frac{\sqrt{n}(m(\overline{\mathrm{SNR}}) - R)}{\sigma(\overline{\mathrm{SNR}})}\right) \tag{13}$$

where $m(\overline{\mathrm{SNR}})$ is given in Eq. (3) and $\sigma(\overline{\mathrm{SNR}}) = \sqrt{n\sigma_n^2}$ with σ_n^2 given in Eq. (4). It is assumed that the information rate r_k is fixed during the optimization procedure, and the choice of r_k will be discussed in Section IV.

Following the same reasoning as in [1], one can see that Problem 1 is feasible if, and only if:

$$\sum_{k=1}^{K} \frac{\eta_k^{(0)}}{r_k} < 1. \tag{14}$$

In the rest of the paper, it is assumed that Eq. (14) holds.

Before going further, let us introduce the Conjecture 1.

Conjecture 1. *$\forall k$, the function $\log(1/\eta_k)$ is biconvex in (γ_k, E_k).*

In order to be convinced, one can remark, from Eq. (12), that

$$\log \eta_k(\gamma_k, E_k) = \log \gamma_k + \log r_k + \log(1 - P_e^{(n,r_k)}(G_k E_k)). \tag{15}$$

The log-goodput is clearly concave in $\gamma_k > 0$. To analyze the concavity of the log-goodput with respect to E_k, we need to study the function $x \mapsto f(x) := \log(1 - Q(u(x)))$, where Q is the Gaussian tail, and u is the function given by $u(x) := \sqrt{n}(m(x) - R)/\sigma(x)$. One can easily check that if u is concave, then f is concave, and the log-goodput too. Since $u''(x) = \sqrt{n}\,\Delta(x)/\sigma^4(x)$, where

$$\Delta(x) = \sigma^2(x)\Big(m''(x)\sigma(x) - \sigma''(x)(m(x)-R) - m'(x)\sigma'(x) - m(x)\sigma'(x)\Big) + 2(\sigma'(x))^2\sigma(x)(m(x) - R).$$

The sign of u'' is thus driven by those of Δ. Proving that $\Delta(x) \leq 0$ for all $x > 0$ is difficult. Nevertheless it can be conjectured from Fig. 2.

In order to facilitate the optimization procedure, we would like to find particular structure to Problem 1. One can firstly see that the objective function given in Eq. (10a) is biconvex in $(\boldsymbol{\gamma}, \boldsymbol{E})$ [12]. The constraints in Eqs.(10c)-(10d) are trivially convex and thus biconvex. Thanks to the Conjecture 1, one can assume that constraint given by Eq. (10b) is also biconvex. As a consequence, we can assume that Problem 1 corresponds to the minimization of a biconvex objective function over a biconvex set, and thus falls within the class of **biconvex optimization problems**. Notice that if the constraint given by Eq. (10b) would be written directly on the goodput (and not on its $\log$), Problem 1 were not biconvex anymore since

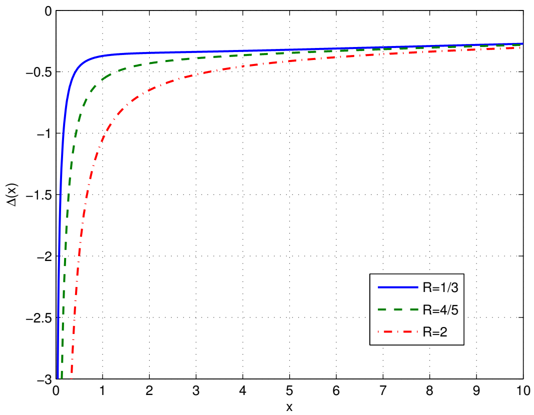

Figure 2. $x \mapsto \Delta(x)$ for different values of R.

we have checked that the goodput is quasi-convex (and not convex, unfortunately) with respect to the energy. Taking the Conjecture 1 as true, we can deduce that our optimization problem is biconvex. Finding optimal solutions to biconvex optimization problem has been solved in [13]. The optimal solution can be actually exhibited using a modified primal-dual approach called Global OPtimization (GOP) algorithm. We use this GOP algorithm in the next Section to solve Problem 1.

IV. Numerical results

Due to the complexity of GOP, only $K = 2$ communication links are considered with average SNRs configured to 10 dB and 30 dB, respectively. The transmitters use Gaussian codes of given length n and given rates r_k. The coding rates choice will be explained later. For the sake of simplicity the goodput request is uniform $\eta_k^{(0)} = \eta_T/K$, with η_T the total goodput demand of the cluster (in bit/s/Hz). The total goodput requirement η_T is related to the sum-rate of the cluster ρ (in bit/s) using the bandwidth W (in Hz) $\rho = \eta_T W$.

A. *GOP results versus increasing sum-rate demand*

First of all, the rate r_k is arbitrarily fixed to a value that satisfies Eq. (14). If $\eta_T < 1/2$ then $r_k = 1/2$, hence $\sum_{k=1}^K \eta_k^{(0)}/r_k = K(\eta_T/K)/(1/2) < 1$. Else if $1/2 \leq \eta_T < 1$ then $r_k = 1$, hence $\sum_{k=1}^K \eta_k^{(0)}/r_k = \eta_T < 1$, and so on.

In Fig. 3, we plot the power consumption ($\sum_{k=1}^K \gamma_k E_k$) resulting from GOP versus the goodput request η_T for different code lengths. The ergodic capacity based algorithm from [5] is displayed as a benchmark. We observe, as expected, that the power consumption decreases when the block length n increases but the performance gain with increasing n is nonetheless limited.

In Fig. 4, we plot the occupied bandwith ($\sum_{k=1}^K \gamma_k$) resulting from GOP versus the goodput request η_T for different code lengths. We see that the bandwidth is not totally occupied when the goodput request is low. This enables us to reduce the inter-cluster interference. In contrast, the ergodic capacity based algorithm allocates the entire bandwidth for any $\eta_T \in [0, 1]$.

Let us now select r_k in order to still improve the performance. In Fig. 5, we plot the total power consumption versus the goodpout request when Gaussian finite-length codes (with $n = 512$) is used and when convolutive codes (with rate-$1/2$ and length $n = 512$) is used. The optimization problem and algorithm associated with the convolutive code is described in [1] and is different from those proposed here since $P_e^{(n,R)}$ expresses very differently in closed-form. In addition, optimal choice for r_k (when Gaussian codes are used) and optimal choice of the modulation (when the convolutive code is used) are also simulated. We can observe that these practical convolutive codes perform at about 4 dB from the Gaussian codes. We can also remark that choosing optimally r_k also strongly improves the performance.

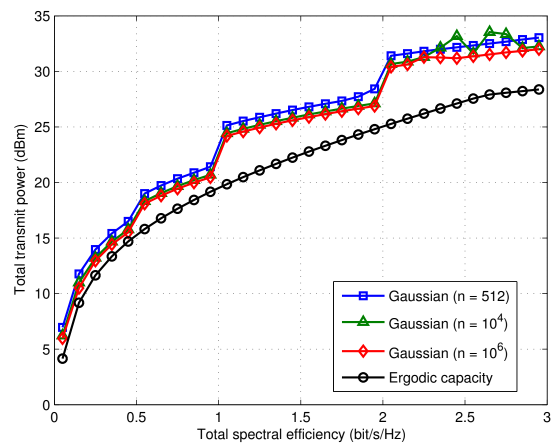

Figure 3. Total transmit power of the proposed algorithm vs. η_T.

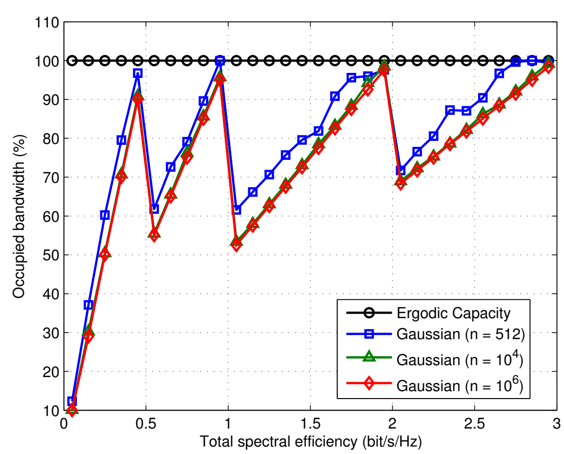

Figure 4. Occupied bandwidth of the proposed algorithm vs. η_T.

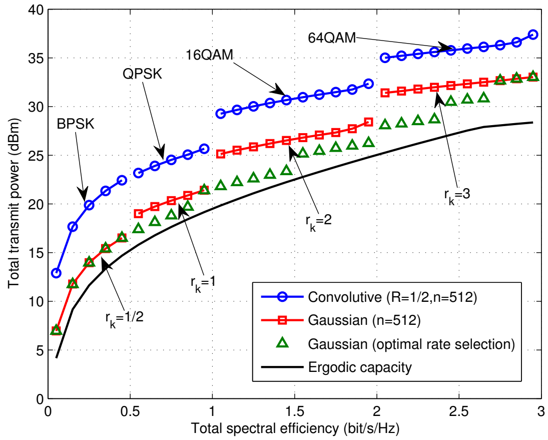

Figure 5. Total transmit power vs η_T for different rate selections ($n = 512$).

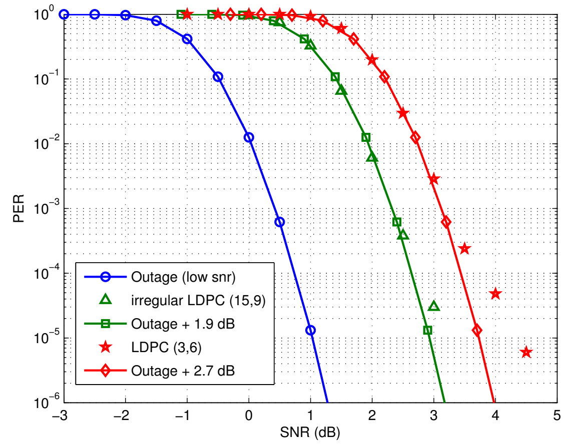

Figure 6. PER versus SNR for several FEC codes ($n = 504$ and $R = 1/2$).

B. How close are powerful FEC codes?

Finally, we show that the outage probability developed within this framework can be used to predict the performance of powerful FEC codes over the Rayleigh channel (other practical coding schemes, such as convolutive codes, were treated in [1]). The benefits of the LDPC codes come from the relaxation of the ML-receiver assumption, and from the fact that the LLR messages are well modeled by Gaussian random variables under iterative decoding [14]. This observation motivates us to describe the waterfall in the LDPC error performance, which is the capacity-achieving region of these codes, using finite-length Gaussian codes.

In Fig. 6, we plot the PER of two BPSK modulated ($n = 504$, $R = 1/2$) LDPC codes versus SNR. We used a $(3, 6)$-regular code and an irregular PEG code [15]. The error probability of a ($n = 504$, $R = 1/2$) Gaussian code is plotted too, as well as its shifted versions using some SNR gaps. It is very interesting to observe that we obtain a tight approximation of the LDPC performance by shifting the finite-length Gaussian code error probability with a gap, whereas it was impossible to directly resort to gaps on the ergodic capacity function [1]. Approximation is very tight for PER between 10^{-3} and 1. The difference is noticeable beyond 10^{-3}, and may be explained by the floor behavior of LDPC, which is not present for Gaussian coding. This result corroborates one of [3] which tells that the operating point of FEC when optimizing the goodput is generally high (PER of about 10^{-1}). We infer from Fig. 6 that the allocated power in the CAHN when using this irregular LDPC code will be only 1.9 dB away from the best achievable performance.

V. Conclusion

We firstly computed in closed-form the error probability of Gaussian codes with finite-length over the Rayleigh channel. Based on this new result, we found the optimal power and bandwidth allocation of a Type-I HARQ-based OFDMA network relying on statistical CSI only. Next, we showed the performance are very close to that obtained by the ergodic capacity metric. In addition, this framework can serve as a basis for Type-I HARQ-based OFDMA resource allocation when powerful FEC is used, typically LDPC coding. Finally, future research towards approximate solutions could be of great interest to speed up the allocation procedure.

References

[1] S. Marcille, P. Ciblat, and C. J. Le Martret, “Optimal resource allocation in HARQ-based OFDMA wireless networks,” in *Military Communications Conference (Milcom)*. IEEE, Oct. 2012.

[2] S. Lin and D. J. Costello, *Error Control Coding: Fundamentals and Applications*. Englewood Cliffs, NJ: Prentice-Hall, 1983.

[3] P. Wu and N. Jindal, “Coding versus ARQ in fading channels: how reliable should the PHY be ?” *IEEE Trans. Commun.*, vol. 59, no. 12, pp. 3363–3374, Dec. 2011.

[4] B. Devillers, J. Louveaux, and L. Vandendorpe, “Bit and power allocation for goodput optimization in coded parallel subchannels with ARQ,” *IEEE Trans. Signal Process.*, vol. 56, no. 8, pp. 3652–3661, Aug. 2008.

[5] S. Gault, W. Hachem, and P. Ciblat, “Performance of OFDMA on Rayleigh fading channels in a multi-cell environment,” *IEEE Trans. Commun.*, vol. 55, no. 4, pp. 740–751, Apr. 2007.

[6] R. G. Gallager, “Low-Density Parity-Check codes,” Ph.D. dissertation, Massachusetts Institute of Technology, Cambridge, MA, 1963.

[7] D. N. Tse and P. Viswanath, *Fundamentals of Wireless Communications*. Cambridge University Press, 2005.

[8] J. N. Laneman, “On the distribution of mutual information,” in *Workshop on Information Theory and its Applications (ITA)*, Feb. 2006.

[9] D. S. Buckingham and M. C. Valenti, “The information-outage probability of finite-length codes over AWGN channels,” in *Conference on Information Sciences and Systems (CISS)*, Nov. 2008.

[10] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions, with formulas, graphs, and mathematical tables*, 20th ed., ser. Applied Mathematics Series. National Bureau of Standards, Jun. 1972, vol. 55.

[11] S. M. Ross, *Introduction to probability models*, 9th ed. Academic Press, 2007.

[12] J. Gorski, F. Pfeuferret, and K. Klamroth, “Biconvex sets and optimization with biconvex functions: a survey and extensions,” *Mathematical Methods of Operations Research*, vol. 66, no. 3, pp. 373–407, 2007.

[13] C. A. Floudas and V. Visweswaran, “A primal-relaxed dual global optimization approach,” *Journal of Optimization Theory and Applications*, vol. 78, no. 2, p. 187, 1993.

[14] W. E. Ryan and S. Lin, *Channel Codes: Classical and Modern*. Cambridge University Press, 2009.

[15] D. MacKay, “Some error correcting codes,” http://www.inference.phy.cam.ac.uk/mackay/codes.