ZIV-ZAKAI BOUND FOR HARMONIC RETRIEVAL
IN MULTIPLICATIVE AND ADDITIVE GAUSSIAN NOISE

Philippe Ciblat[1] *and Mounir Ghogho*[2]

[1] Ecole Nationale Supérieure des Télécommunications (ENST), Paris, France
[2] Leeds University, Leeds, United Kingdom

Email: *philippe.ciblat@enst.fr, m.ghogho@leeds.ac.uk*

ABSTRACT

The paper addresses the problem of the lower performance bound evaluation for harmonic retrieval in multiplicative and additive noise, especially when the SNR is low and/or when the number of available samples is small. As a novelty, we express an accurate approximation of the Ziv-Zakai bound in closed-form. This enables us to analyse the influence of some signal parameters on performance and to characterise the so-called threshold region.

1. INTRODUCTION

We consider the following signal model

$$y(n) = a(n)e^{2i\pi(\varphi_0 + \varphi_1 n)} + b(n), \quad n = 0, \cdots, N-1$$

where

- $y(n)$ is the observed signal of which N samples are available.

- $b(n)$ is an additive noise, which is assumed to be a *circular* complex-valued white Gaussian stationary process with zero-mean and variance $\sigma_b^2 = \mathbb{E}[|b(n)|^2]$.

- $a(n)$ is a multiplicative noise, which is assumed to be a zero-mean Gaussian stationary process. This noise could be either complex or real-valued, *circular* or *non-circular*.

The parameters of interest φ_0 and φ_1 are the phase and the frequency respectively.

Such a signal model can be encountered in several application fields such as DOA estimation, radar detection and digital communications [1, 2, 3].

To characterise the lower performance bound on parameter estimation, the so-called Cramér-Rao bound (CRB) is often used, partly because of the simplicity of its derivation.

Unfortunately, it is now well-known that the CRB is not accurate at low SNR and/or when the number of samples N is small. Therefore a few other bounds, which are more complicated but tighter than the CRB, have been proposed in the literature. The widely spread ones are the Barankin bound (BB) and the Ziv-Zakai bound (ZZB) [4, 5].

The Barankin bound for harmonic retrieval in multiplicative and additive noise has been already derived in [6, 7, 8]. Therefore, this paper focuses on the derivation and the analysis of the Ziv-Zakai bound. Our work is motivated by the fact that the Ziv-Zakai Bound appears to be tighter that the Barankin bound (of order 1), as was reported in [9, 10] in the context of additive noise models.

In the literature, different Ziv-Zakai-type bounds have been introduced and derived for the problem of bearing [11] and time-delay [12] estimation. A special case of bearing estimation involves the problem of harmonic retrieval in additive noise (without multiplicative noise). To the best of our knowledge, the derivation of the Ziv-Zakai bound when a harmonic signal is disturbed by additive and multiplicative noise sources has not been addressed in the literature. The main purpose of this paper is to fill this gap.

2. REVIEW OF ZIV-ZAKAI BOUND

In this section, we recall the definition of the ZZB in the context of a multi-variate parameter of interest.

Let $\hat{\varphi}_0$ and $\hat{\varphi}_1$ be estimates of φ_0 and φ_1 respectively. The error vector is defined as follows $\boldsymbol{\varepsilon} = [\hat{\varphi}_0 - \varphi_0, \hat{\varphi}_1 - \varphi_1]^{\mathrm{T}}$ where the superscript $^{\mathrm{T}}$ stands for the transposition operator. Let $\boldsymbol{\varphi} = (\varphi_0, \varphi_1)$. We denote the error correlation matrix by

$$\mathbf{E}_{\boldsymbol{\varphi}} = \mathbb{E}[\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^{\mathrm{T}}].$$

In [5, 10], it was proved that the following inequality holds for any vector $\mathbf{z}$ of size (2×1),

$$\mathbf{z}^{\mathrm{T}}\mathbf{E}_{\boldsymbol{\varphi}}\mathbf{z} \geq \int_0^\infty \Delta \left(\max_{\mathbf{h}|\mathbf{z}^{\mathrm{T}}\mathbf{h}=\Delta} f(\mathbf{h}) \right) d\Delta \qquad (1)$$

where $\mathbf{h} = [h_0, h_1]^{\mathrm{T}}$, and

$$f(\mathbf{h}) = \int \min(p(\boldsymbol{\varphi}), p(\boldsymbol{\varphi} + \mathbf{h})) P_e(\boldsymbol{\varphi}, \boldsymbol{\varphi} + \mathbf{h}) d\boldsymbol{\varphi}. \quad (2)$$

The function $p(.)$ is the *a priori* density function of the bi-variate parameter $\boldsymbol{\varphi}$, and $P_e(\boldsymbol{\varphi}, \boldsymbol{\varphi} + \mathbf{h})$ is the error probability when the optimal detector (namely, the ML detector) is used to decide between the following two hypotheses

$$\begin{cases} H_0 : & y(n) = a(n)e^{2i\pi(\varphi_0 + \varphi_1 n)} + b(n) \\ H_1 : & y(n) = a(n)e^{2i\pi((\varphi_0 + h_0) + (\varphi_1 + h_1)n)} + b(n) \end{cases}$$

where hypotheses H_0 and H_1 are equally likely.

The RHS of Equation (1) is called the Ziv-Zakai Bound (ZZB). Actually, it is only a (simplified) lower bound of the Ziv Zakai bound since the Valley-filling function defined in [10] is set here to be equal to the identity, and the above binary detection task is carried out under the assumption that H_0 and H_1 are equally-likely.

By inspecting Equation (1), one can remark that the likelihood of $\boldsymbol{\varphi}$ is scanned over the entire search interval of $\boldsymbol{\varphi}$, as is also the case for the Barankin bound [7]. This contrasts with the CRB where the likelihood function is only evaluated around the true point. Therefore the CRB is less accurate than the BB and the ZZB, especially at low SNR. Indeed at low SNR, several estimate's realizations, called "outliers", may be far away from the true point. The CRB is incapable of predicting or quantifying such phenomenon.

Since the outliers effect is particularly associated with frequency estimation, this paper focuses mainly on the estimation of φ_1. The ZZB for φ_1 is obtained by setting $\mathbf{z} = [0, 1]^{\mathrm{T}}$. The mean square error for φ_1, denoted MSE$_1$, is then bounded by

$$\mathrm{MSE}_1 \geq \int_0^\infty h_1 \left(\max_{h_0} f(h_0, h_1) \right) dh_1.$$

Notice that the terms $f(h_0, h_1)$ and $f(\mathbf{h})$ refer to the same function.

3. DERIVATION OF ZIV-ZAKAI BOUND

The key task now is to express the function $\mathbf{h} \mapsto f(\mathbf{h})$ in closed-form. According to (2), we need to explicitly express $\min(p(\boldsymbol{\varphi}), p(\boldsymbol{\varphi} + \mathbf{h}))$ and $P_e(\boldsymbol{\varphi}, \boldsymbol{\varphi} + \mathbf{h})$. We first focus on the derivation of the error probability $P_e(\boldsymbol{\varphi}, \boldsymbol{\varphi} + \mathbf{h})$ which can be split into two terms as follows

$$P_e(\boldsymbol{\varphi}, \boldsymbol{\varphi} + \mathbf{h}) = \frac{1}{2} P_e(\boldsymbol{\varphi} \to \boldsymbol{\varphi} + \mathbf{h}) + \frac{1}{2} P_e(\boldsymbol{\varphi} + \mathbf{h} \to \boldsymbol{\varphi})$$

where $P_e(\boldsymbol{\varphi} \to \boldsymbol{\varphi} + \mathbf{h})$ is the probability that, given the true parameter $\boldsymbol{\varphi}$ (i.e., H_0), the test decision is $\boldsymbol{\varphi} + \mathbf{h}$ (i.e., H_1), and $P_e(\boldsymbol{\varphi} + \mathbf{h} \to \boldsymbol{\varphi})$ is defined similarly.

Before developing any further, we infer the following lemma. The proof is provided hereafter.

Lemma 1 *The error probabilities $P_e(\boldsymbol{\varphi} \to \boldsymbol{\varphi} + \mathbf{h})$ and $P_e(\boldsymbol{\varphi} + \mathbf{h} \to \boldsymbol{\varphi})$ are identical. Furthermore, $P_e(\boldsymbol{\varphi}, \boldsymbol{\varphi} + \mathbf{h})$ is independent of $\boldsymbol{\varphi}$.*

Proof – Let $y(n) = a(n)e^{2i\pi(\phi^{(0)} + \phi^{(1)}n)} + b(n)$ and $\boldsymbol{\phi} = [\phi^{(0)}, \phi^{(1)}]^{\mathrm{T}}$. The vector $\mathbf{y} = [y(0), \cdots, y(N-1)]^{\mathrm{T}}$ is then a zero-mean complex-valued Gaussian vector with correlation matrix $\mathbf{R}_\phi = \mathbb{E}[\mathbf{y}\mathbf{y}^{\mathrm{H}}]$ where the superscript $^{\mathrm{H}}$ stands for the conjugate transposition, and with pseudo-correlation matrix $\mathbf{U}_\phi = \mathbb{E}[\mathbf{y}\mathbf{y}^{\mathrm{T}}]$. The matrix $\mathbf{U}_\phi$ is also referred to as the conjugate correlation matrix. If $\mathbf{U}_\phi \neq 0$, the Gaussian process is said to be non-circular or unproper.

Let $\tilde{\mathbf{y}} = [\mathbf{y}^{\mathrm{T}} \mathbf{y}^{\mathrm{H}}]^{\mathrm{T}}$. We have that

$$\tilde{\mathbf{y}} = \tilde{\boldsymbol{\Gamma}}_\phi \tilde{\mathbf{a}} + \tilde{\mathbf{b}}$$

where

$$\tilde{\boldsymbol{\Gamma}}_\phi = \left[\begin{array}{cc} \boldsymbol{\Gamma}_\phi & \mathbf{0}_{N,N} \\ \mathbf{0}_{N,N} & \overline{\boldsymbol{\Gamma}_\phi} \end{array} \right]$$

with $\boldsymbol{\Gamma}_\phi = \mathrm{diag}(e^{2i\pi(\phi^{(0)} + \phi^{(1)}n)}, n = 0, \cdots, N-1)$ and where $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$ are defined in the same way as $\tilde{\mathbf{y}}$. As in [13], one can remark that

$$\tilde{\mathbf{y}} = \tilde{\boldsymbol{\Gamma}}_\phi \tilde{\mathbf{x}} \quad (3)$$

with $\tilde{\mathbf{x}} = \tilde{\mathbf{a}} + \tilde{\boldsymbol{\Gamma}}_\phi^{-1} \tilde{\mathbf{b}}$.

The vector $\tilde{\mathbf{x}}$ is Gaussian with zero-mean and correlation matrix $\tilde{\mathbf{R}}_x$ independent of ϕ since we have that

$$\tilde{\mathbf{R}}_x = \tilde{\mathbf{R}}_a + \sigma_b^2 \mathbf{I}_{2N}$$

where $\tilde{\mathbf{R}}_a$ the correlation matrix of $\tilde{\mathbf{a}}$ and $\mathbf{I}_{2N}$ is the $(2N \times 2N)$ identity matrix.

By definition, $P_e(\boldsymbol{\varphi} \to \boldsymbol{\varphi} + \mathbf{h})$, which corresponds to the error probability of the ML detector, can be written as follows

$$P_e(\boldsymbol{\varphi} \to \boldsymbol{\varphi} + \mathbf{h}) = \mathrm{Prob}\left(\frac{p(\mathbf{y}|\boldsymbol{\varphi} + \mathbf{h})}{p(\mathbf{y}|\boldsymbol{\varphi})} > 1 \big| \tilde{\mathbf{y}} = \tilde{\boldsymbol{\Gamma}}_\varphi \tilde{\mathbf{x}} \right).$$

Since $\mathbf{y}$ is Gaussian, the likelihood function is given by

$$p(\mathbf{y}|\boldsymbol{\phi}) \propto \frac{1}{\sqrt{|\det(\tilde{\mathbf{R}}_\phi)|}} e^{-\frac{1}{2}\tilde{\mathbf{y}}^{\mathrm{H}}\tilde{\mathbf{R}}_\phi^{-1}\tilde{\mathbf{y}}}$$

where

$$\begin{aligned} \tilde{\mathbf{R}}_\phi &:= \mathbb{E}[\tilde{\mathbf{y}}\tilde{\mathbf{y}}^{\mathrm{H}}] \\ &= \left[\begin{array}{cc} \mathbf{R}_\phi & \mathbf{U}_\phi \\ \overline{\mathbf{U}_\phi} & \overline{\mathbf{R}_\phi} \end{array} \right] \end{aligned}$$

and where the overline stands for the complex conjugate operator. According to Eq. (3), we have that

$$\tilde{\mathbf{R}}_\phi = \tilde{\boldsymbol{\Gamma}}_\phi \tilde{\mathbf{R}}_x \tilde{\boldsymbol{\Gamma}}_\phi^{\mathrm{H}} \quad (4)$$

which implies that $\det(\tilde{\mathbf{R}}_\phi)$ is independent of ϕ.

Consequently

$$P_e(\boldsymbol{\varphi} \to \boldsymbol{\varphi} + \mathbf{h}) = \mathrm{Prob}\left(e^{-\frac{1}{2}\tilde{\mathbf{y}}^H \tilde{\mathbf{W}}_{\varphi,h}\tilde{\mathbf{y}}} > 1 \big| \tilde{\mathbf{y}} = \tilde{\boldsymbol{\Gamma}}_\varphi \tilde{\mathbf{x}}\right)$$

with

$$\tilde{\mathbf{W}}_{\varphi,\mathbf{h}} = \tilde{\mathbf{R}}_{\varphi+\mathbf{h}}^{-1} - \tilde{\mathbf{R}}_\varphi^{-1}.$$

Using $\tilde{\mathbf{y}} = \tilde{\boldsymbol{\Gamma}}_\varphi \tilde{\mathbf{x}}$ and Eq. (4), we obtain

$$P_e(\boldsymbol{\varphi} \to \boldsymbol{\varphi} + \mathbf{h}) = \mathrm{Prob}\left(e^{-\frac{1}{2}\tilde{\mathbf{x}}^H \tilde{\mathbf{W}}_h \tilde{\mathbf{x}}} > 1\right) \quad (5)$$

with

$$\begin{aligned}
\tilde{\mathbf{W}}_\mathbf{h} &:= \tilde{\mathbf{W}}_{0,h} \\
&= \tilde{\mathbf{R}}_\mathbf{h}^{-1} - \tilde{\mathbf{R}}_0^{-1} \\
&= \tilde{\boldsymbol{\Gamma}}_\mathbf{h}\tilde{\mathbf{R}}_x^{-1}\tilde{\boldsymbol{\Gamma}}_\mathbf{h}^H - \tilde{\mathbf{R}}_x^{-1}.
\end{aligned}$$

Thus, $P_e(\boldsymbol{\varphi} \to \boldsymbol{\varphi} + \mathbf{h})$ is independent of $\boldsymbol{\varphi}$ since the statistics of $\tilde{\mathbf{x}}$ are independent of $\boldsymbol{\varphi}$. Using the same reasoning as above, $P_e(\boldsymbol{\varphi} + \mathbf{h} \to \boldsymbol{\varphi})$ can be shown to be given by the same expression as in the RHS of eq. (5). This concludes the proof. ∎

Thanks to Lemma 1, we are now able to simplify $f(\mathbf{h})$ as follows [10]

$$f(\mathbf{h}) = A(\mathbf{h})P_e(\mathbf{h})$$

where

$$A(\mathbf{h}) = \int \min(p(\boldsymbol{\varphi}), p(\boldsymbol{\varphi} + \mathbf{h}))d\boldsymbol{\varphi}$$

and $P_e(\mathbf{h})$ stands for $P_e(\boldsymbol{\varphi}, \boldsymbol{\varphi} + \mathbf{h})$. Next, we derive $A(\mathbf{h})$ and $P_e(\mathbf{h})$.

Since we have no *a priori* information on $\boldsymbol{\varphi}$, we consider that φ_0 and φ_1 are uniformly distributed over $[0, 1/2]$, i.e. the *a priori* distribution of the parameters of interest $p(\boldsymbol{\varphi})$ is flat. The upper bound $1/2$ is chosen since the phase and the frequency can only be estimated modulo $1/2$ when multiplicative noise occurs [10]. Consequently

$$A(\mathbf{h}) = (1/2 - h_0)(1/2 - h_1)$$

This leads to

$$\mathrm{MSE}_1 \geq \int_0^{1/2}(1/2 - h_1)h_1(\max_{h_0}(1/2 - h_0)P_e(h_0, h_1))dh_1$$

where $P_e(h_0, h_1)$ represents the same function as $P_e(\mathbf{h})$.

The rest of the paper deals with the evaluation of $P_e(\mathbf{h})$. According to Eq. (5), we have that

$$P_e(\mathbf{h}) = \mathrm{Prob}(\tilde{\mathbf{x}}^H \tilde{\mathbf{W}}_\mathbf{h} \tilde{\mathbf{x}} < 0) \quad (6)$$

In words, we wish to calculate the probability that a quadratic form, which is neither positive nor negative, of a Gaussian vector is negative.

Firstly, we change the complex-valued vector $\tilde{\mathbf{x}}$ into its associated real-valued vector $\check{\mathbf{x}} = [\Re(\mathbf{x}^T), \Im(\mathbf{x}^T)]^T$ where $\Re(.)$ and $\Im(.)$ are the real and imaginary part of a complex-valued number respectively. Consequently, Eq. (6) becomes

$$P_e(\mathbf{h}) = \mathrm{Prob}(\check{\mathbf{x}}^T \check{\mathbf{W}}_\mathbf{h} \check{\mathbf{x}} < 0) \quad (7)$$

with

$$\check{\mathbf{W}}_\mathbf{h} = \check{\mathbf{R}}_\mathbf{h}^{-1} - \check{\mathbf{R}}_0^{-1}$$

and

$$\check{\mathbf{R}}_\phi = \mathbb{E}[\check{\mathbf{y}}\check{\mathbf{y}}^T].$$

Secondly, in order to obtain a simple expression for $P_e(\mathbf{h})$, we proceed by whitening the process $\check{\mathbf{x}}$. This is achieved by diagonalising $\check{\mathbf{R}}_x := \mathbb{E}[\check{\mathbf{x}}\check{\mathbf{x}}^T]$. According to Eq. (4), we prove that $\check{\mathbf{R}}_0 = \check{\mathbf{R}}_x$ where $\check{\mathbf{R}}_0 = \check{\mathbf{R}}_{\phi_{|\phi=0}}$. This matrix can be decomposed as follows

$$\check{\mathbf{R}}_0 = \mathbf{D}_0^T \boldsymbol{\Lambda}_0 \mathbf{D}_0$$

where $\mathbf{D}_0$ is an orthogonal real-valued matrix of size $(2N \times 2N)$ and $\boldsymbol{\Lambda}_0$ is a positive real-valued diagonal matrix. Therefore, the multi-variate random variable $\check{\mathbf{z}} = \boldsymbol{\Lambda}_0^{-1/2}\mathbf{D}_0\check{\mathbf{x}}$ is a white Gaussian vector.

Thirdly, we concentrate on the term $\check{\mathbf{x}}^T \check{\mathbf{W}}_\mathbf{h} \check{\mathbf{x}}$. Thanks to the decomposition of $\check{\mathbf{R}}_0$, we get

$$\check{\mathbf{x}}^T \check{\mathbf{W}}_\mathbf{h} \check{\mathbf{x}} = \check{\mathbf{z}}^T \mathbf{V} \check{\mathbf{z}}$$

with

$$\mathbf{V}_\mathbf{h} = \boldsymbol{\Lambda}_0^{1/2}\mathbf{D}_0\check{\mathbf{W}}_\mathbf{h}\mathbf{D}_0^T\boldsymbol{\Lambda}_0^{1/2}.$$

One can observe that $\mathbf{V}_\mathbf{h}$ is a symmetric matrix which can be decomposed as

$$\mathbf{V}_\mathbf{h} = \mathbf{D}_\mathbf{h}^T \boldsymbol{\Lambda}_\mathbf{h} \mathbf{D}_\mathbf{h} \quad (8)$$

where $\mathbf{D}_\mathbf{h}$ is an orthogonal real-valued matrix of size $2N \times 2N$ and $\boldsymbol{\Lambda}_\mathbf{h}$ is a diagonal real-valued matrix. Notice that the matrix $\boldsymbol{\Lambda}_\mathbf{h}$ is neither positive nor negative. Therefore the set of its eigenvalues can be split into two parts: the positive ones and the negatives ones. We consider that the first m eigenvalues are negative and are denoted by $-\lambda_n^{(2)}$ with $\lambda_n^{(2)} > 0$ for $n = 0, \cdots, m-1$. The positive or null eigenvalues are denoted $\lambda_n^{(1)}$ with $\lambda_n^{(1)} \geq 0$ for $n = m, \cdots, 2N-1$. Thus, we get

$$\boldsymbol{\Lambda}_\mathbf{h} = \mathrm{diag}(-\lambda_0^{(2)}, \cdots, -\lambda_{m-1}^{(2)}, \lambda_m^{(1)}, \cdots, \lambda_{2N-1}^{(1)}).$$

Recall that these eigenvalues depend on $\mathbf{h}$.

Let $\mathbf{u} = [u_0, \cdots, u_{2N-1}]^T = \mathbf{D}_\mathbf{h}\check{\mathbf{z}}$ be a zero-mean unit-variance Gaussian vector. By construction, $\mathbf{u}$ is white, so its statistics do not depend on $\mathbf{h}$. By replacing $\check{\mathbf{z}}$ and $\mathbf{V}_\mathbf{h}$ with $\mathbf{u}$ and Eq. (8) respectively, we obtain

$$\check{\mathbf{x}}^T \check{\mathbf{W}}_\mathbf{h} \check{\mathbf{x}} = \mathbf{u}^T \boldsymbol{\Lambda}_\mathbf{h} \mathbf{u}.$$

Finally, we obtain

$$P_e(\mathbf{h}) = \text{Prob}\left(\sum_{n=m}^{2N-1} \lambda_n^{(1)} u_n^2 < \sum_{n=0}^{m-1} \lambda_n^{(2)} u_n^2\right). \quad (9)$$

where $\{u_n\}$ is a real-valued i.i.d. Gaussian random process with zero-mean and unit-variance.

We now wish to derive a closed-form expression for the following term

$$P_e(\mathbf{h}) = \text{Prob}\left(p_1 < p_2\right) \quad (10)$$

where $p_m = \sum_n \lambda_n^{(m)} u_n^2$ is a weighted sum of squared independent Gaussian variable. Notice that, by construction, p_1 and p_2 are independent.

If $\lambda_n = \lambda$ for all n, then p_1 (resp. p_2) obeys a χ^2 distribution with $(2N-m)$ (resp. m) degrees of freedom. However, if the weighting coefficients are different, the p_m's are not χ^2 distributed anymore. Further, expressing the distribution of p_m in closed-form is not tractable. Nevertheless, it can be well approximated by means of the Gamma distribution [14]. We recall that the Gamma distribution, denoted $\mathcal{G}(\alpha, \theta)$, is defined as follows

$$P_{\alpha,\theta}(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)\theta^\alpha} e^{-x/\theta}$$

where $\Gamma(.)$ is the so-called Gamma function.

Hence, the distribution of p_m is next approximated by the Gamma distribution whose first and second moments are equal to those of p_m. We thus obtain

$$p_1 \sim \mathcal{G}(\alpha_1, \theta_1) \quad \text{and} \quad p_2 \sim \mathcal{G}(\alpha_2, \theta_2)$$

with

$$\alpha_1 = \frac{1}{2} \frac{(\sum_{n=m}^{2N-1} \lambda_n^{(1)})^2}{\sum_{n=m}^{2N-1} \lambda_n^{(1)2}} \text{ and } \theta_1 = 2 \frac{\sum_{n=m}^{2N-1} \lambda_n^{(1)2}}{\sum_{n=m}^{2N-1} \lambda_n^{(1)}},$$

and

$$\alpha_2 = \frac{1}{2} \frac{(\sum_{n=0}^{m-1} \lambda_n^{(2)})^2}{\sum_{n=0}^{m-1} \lambda_n^{(2)2}} \text{ and } \theta_2 = 2 \frac{\sum_{n=0}^{m-1} \lambda_n^{(2)2}}{\sum_{n=0}^{m-1} \lambda_n^{(2)}}.$$

As p_m is now assumed Gamma distributed, Eq. (10) can be simplified. Indeed, by using the fact that the square root of a Gamma distributed random variable is Nakagami distributed, and then by using Eq. (46) in [15], we have that

$$P_e(\mathbf{h}) = \frac{\theta^{\alpha_1}}{\alpha_1} B(\alpha_1, \alpha_2)\,_2F_1(\alpha_1 + \alpha_2, \alpha_1, \alpha_1 + 1; -\theta)$$

where

- $\theta = \theta_1/\theta_2$,

- $B(\alpha_1, \alpha_2) = \Gamma(\alpha_1 + \alpha_2)/\Gamma(\alpha_1)$ is called either the Euler's first integral or the Beta function,

- $_2F_1(\alpha, \beta, \gamma; x)$ is the so-called hyper-geometric function.

The above expression for $P_e(\mathbf{h})$ represents the main result of this paper. Although this expression is not interpretable, its numerical computation will provide interesting results.

4. SIMULATIONS

The multiplicative noise $a(n)$ is hereafter assumed to be white non-circular Gaussian process with zero-mean, unit-variance, and pseudo-variance $\rho = \mathbb{E}[a(n)^2]$. For sake of simplicity, we also assume that the real part of $a(n)$ is independent of its imaginary part. This implies that ρ is real-valued. If $\rho = 0$, then $a(n)$ is circular; if $\rho = 1$, then $a(n)$ is real-valued. Thus, ρ quantifies the non-circularity rate of $a(n)$. In each figure, we display four curves: dashed line corresponds to the empirical mean square error (MSE) for the well-known NLS estimate [1, 2, 16]. Solid lines with star point represent the ZZB. Solid lines with triangular point and with circle point plot represent the BB and the CRB respectively [16, 7].

In Figure 1, we plot all the curves versus SNR with $N = 64$, $\rho = 1$. We observe that the ZZB is significantly tighter than the BB. The SNR threshold value predicted by the ZZB is quite close to that obtained empirically with the NLS estimate.
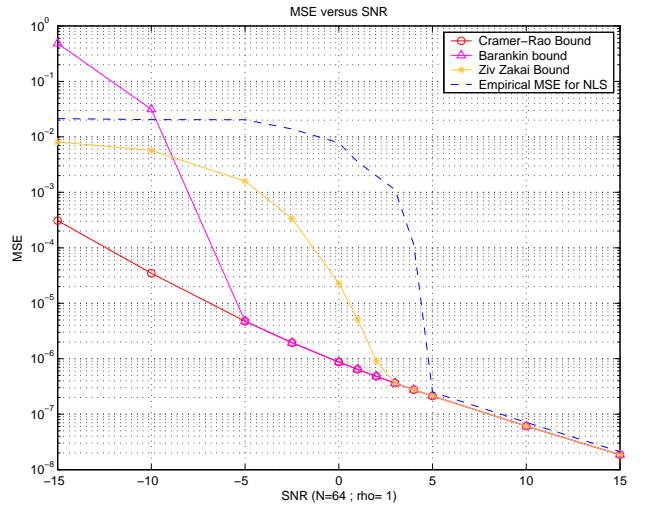

Fig. 1. MSE versus SNR

In Figure 2, we plot the curves versus N with SNR $=$ 10dB, $\rho = 0.9$. Even though the ZZB offers a more realistic value for the N threshold than the BB, the mismatch

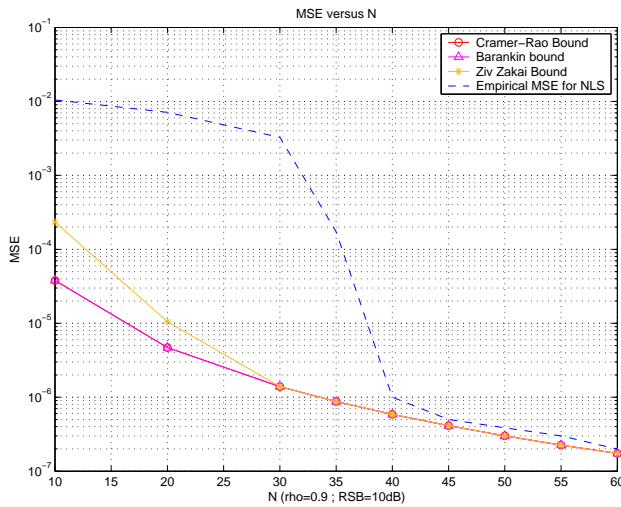between the ZZB and the NLS performance is still quite large.

Fig. 2. MSE versus N

In Figure 3, the curves are displayed versus ρ with $N = 64$, SNR $= 10$dB. The figure confirms that accurate frequency estimation is really difficult to achieve when the signal is not non-circular enough.
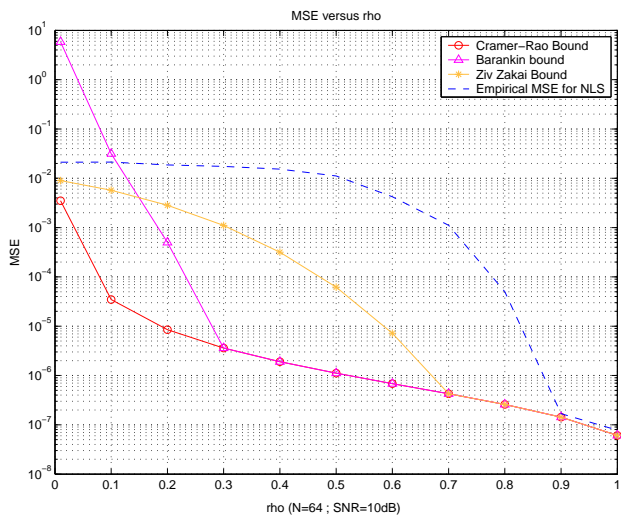

Fig. 3. MSE versus ρ

5. CONCLUSION

This paper developed an approximate closed-form expressions for the Ziv-Zakai bound on frequency estimation in the context of both additive and multiplicative noise. These expressions were used in an attempt to shed some light on the threshold effect problem associated with low SNR, small number of samples, or weak non-circularity of the multiplicative noise.

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