OPTIMAL REUSE FACTOR AND RESOURCE ALLOCATION FOR DOWNLINK OFDMA WITH MULTICELL INTERFERENCE

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ABSTRACT

In this paper we investigate the issue of power control and subcarrier assignment in an OFDMA downlink system impaired by multicell interference. As done in Wimax, we assume that a certain part of the available bandwidth is likely to be reused by different base stations (and is thus subject to multicell interference) and that an other part of the bandwidth is used by one base station only (and is thus "protected" from multicell interference). Questions are: *i)* how many subcarriers should be assigned to a given user in the interference bandwidth, and how many in the protected bandwidth ? *ii)* What power should be allocated to a given user in each of these bands ? *iii)* What is the optimal frequency reuse factor ?

1. INTRODUCTION

Resource allocation in the downlink of OFDMA systems has been studied in a number of works ([2]-[5] for example). In [3], it was proven that the optimal resource allocation that maximizes the sum capacity of the users in the downlink of an OFDMA system with perfect *Channel State Information* (CSI) is *multiuser waterfilling*. This optimal allocation consists in assigning each subcarrier to the user who has the best channel gain on that subcarrier. The power allocated to each subcarrier is then calculated by waterfilling. In [4] an algorithm to minimize the transmit power needed to satisfy the users' rate requirements was proposed for the single-cell case and the base station (BS) was supposed to have full knowledge of the users channels. [5] proposes a solution to the multicell problem that involves coordination between the base stations and an exchange of all the channel state information of the users in the system. In [2], knowledge of only the statistics of the channels was assumed and an iterative algorithm for resource allocation between cells was proposed for the multicell case. In this algorithm a frequency (or subcarrier) reuse factor equal to one was chosen, which means that each cell is allowed to use all available subcarriers.

In this paper, we are interested in the imperfect CSI case where the BS only knows the statistics of users' channels. For that purpose, we use the same system model of [2] but with the main difference that, as done in practical system such as Wimax [7], a certain part of the available bandwidth is shared orthogonally between the adjacent base stations (and is thus "protected" from multicell interference) and the remaining part is reused by different base stations (and is thus subject to multicell interference). We also assume that each user has the possibility to modulate in each of these two parts of the bandwidth. According to this so-called reuse partitioning [8], we need to determine the optimal frequency reuse factor in a certain sense. Notice that in our model we do not assume a priori a cell partitioning scheme, but we manage to prove the optimality of such a scheme. Finally, we characterize the optimal value of the frequency reuse factor α. Strictly speaking, this value depends on all the parameters of the problem. (such as the number K of users in each cell, data rate requirements R_k and positions x_k of all users k). As it is not reasonable to assume such a dependency, we would like to determine an optimal, say, "universal", reuse factor. To that end, we consider the case when the number of users K grows to infinity. In the limit of an infinite K, it is possible to determine a value of the reuse factor α which is (asymptotically) optimal and which does not depend on the particular cell configuration. The obtained reuse factor only depends on an "average" cell configuration.

2. SYSTEM MODEL

2.1. OFDMA Signal Model

For sake of simplicity and due to lack of space, we only focus on two adjacent cells, say cell A and cell B. Extension of this work to the context of several interfering cells is straightforward and will be done in forthcoming journal version. We denote by D the radius of each cell. Each cell contains an arbitrary number of users, with arbitrary positions. From now on, we focus on cell A in order to provide a simple signal model. We denote by K the number of users in cell A. Base station A provides information to all K users of the cell following an OFDMA scheme (downlink). The total number of available subcarriers is denoted by N. For a given user $k \in 1, 2, \ldots, K$ in cell A, we denote by $\mathcal{N}_k$ the set of indices corresponding to the subcarriers modulated by k. $\mathcal{N}_k$ is a subset of $\{0, 1, \ldots, N-1\}$. By definition of OFDMA, two distinct users k, k' belonging to cell A are such that $\mathcal{N}_k \cap \mathcal{N}_{k'} = \emptyset$. For each user $k \in \{1, \ldots, K\}$ of cell A, the signal received by k at the nth subcarrier ($n \in \mathcal{N}_k$) and at the mth OFDM block is given by

$$y_k(n, m) = H_k(n, m)x_k(n, m) + w_k(n, m), \tag{1}$$

where $x_k(n, m)$ represents the data symbol transmitted by the base station A. Process $w_k(n, m)$ is an additive noise which encompasses the thermal noise and the possible multicell interference. Coefficient $H_k(n, m)$ is the frequency response of the channel at the subcarrier n and the OFDM block m. Random variables $H_k(n, m)$ are assumed to be Rayleigh distributed with variance $\rho_k = E[|H_k(n, m)|^2]$. For a given user k, $H_k(n, m)$ are identically distributed w.r.t. n, m, but are not supposed to be independent. Channel coefficients are supposed to be perfectly known at the receiver side, and unknown at the base station side. However, variances ρ_k are supposed to be known at the base station. As usual, we assume that ρ_k vanishes with the distance between base station A and user k, based on a given path loss model. In the sequel, it is convenient to assume (without restriction) that users $k = 1, 2, \dots, K$ are numbered from the nearest to the base station to the farthest. Therefore,

$$\rho_1 > \rho_2 > \dots > \rho_K. \quad (2)$$

2.2. Frequency Reuse

In practical cellular OFDMA systems, it is usually assumed that certain subcarriers $n \in \{0, \dots, N-1\}$ used by the base station A are reused by the adjacent cell B. Denote by $\mathcal{I}$ this set of "$\mathcal{I}$nterfering" subcarriers, $\mathcal{I} \subset \{0, \dots, N-1\}$. If user k modulates such a subcarrier $n \in \mathcal{I}$, the additive noise $w_k(n,m)$ contains both thermal noise of variance σ^2 and interference. Therefore, the variance of $w_k(n,m)$ depends on k and is crucially related to the position of user k. We thus define[1]

$$\forall n \in \mathcal{I}, \mathbb{E}[|w_k(m,n)|^2] = \sigma_k^2 \,.$$

In the case of two interfering cells A and B considered in this paper, it can be assumed that (if users $k = 1, 2 \dots, K$ are numbered from the nearest to the base station to the farthest):

$$\sigma_1^2 < \sigma_2^2 < \dots < \sigma_K^2 \,. \quad (3)$$

The reuse factor α is defined as the ratio between the number of reused subcarriers and the total number of available subcarriers: so that $\mathcal{I}$ contains αN subcarriers. The remaining $(1-\alpha)N$ subcarriers are shared by the two cells A and B in an orthogonal way. We assume that $\frac{1-\alpha}{2}N$ of these subcarriers are used by base station A only and are forbidden for B. Denote by $\mathcal{P}_A$ this set of "$\mathcal{P}$rotected" subcarriers. If user k modulates such a subcarrier $n \in \mathcal{P}_A$, the additive noise $w_k(n,m)$ contains only thermal noise. In other words, subcarrier n does not suffer from multicell interference. Then we simply write $\mathbb{E}[|w_k(n,m)|2] = \sigma^2$. Similarly, we denote by $\mathcal{P}_B$ the remaining $\frac{1-\alpha}{2}N$ subcarriers, such that each subcarrier $n \in \mathcal{P}_B$ is only used by station B, and is not used by A. Finally, $\mathcal{I} \cup \mathcal{P}_A \cup \mathcal{P}_B = \{0, \dots, N-1\}$.

2.3. Resource allocation parameters

Of course, for a given user k of cell A, the noise variance σ_k^2 depends on the particular resource allocation used in the adjacent cell B. For the sake of simplicity, we assume that σ_k^2 is fixed and known at base station A. Then we concentrate on the resource allocation in cell A, given a *known multicell interference* $(\sigma_k^2)_{k=1\dots K}$. We assume that a given user may use subcarriers in both the "interference" bandwidth $\mathcal{I}$ and the "protected" bandwidth $\mathcal{P}_A$. We denote by $\gamma_{k,1}N$ (resp. $\gamma_{k,2}N$) the number of subcarriers modulated by user k in the set $\mathcal{I}$ (resp. $\mathcal{P}_A$). In other words, $\gamma_{k,1} = \text{card}(\mathcal{I} \cap \mathcal{N}_k)/N$ and $\gamma_{k,2} = \text{card}(\mathcal{P}_A \cap \mathcal{N}_k)/N$. Note that by definition of $\gamma_{k,1}$ and $\gamma_{k,2}$, $\sum_k \gamma_{k,1} \le \alpha$ and $\sum_k \gamma_{k,2} \le \frac{1-\alpha}{2}$. Furthermore, we assume that a given user k of cell A can modulate in both bands $\mathcal{I}$ and $\mathcal{P}_A$ using distinct powers in each band. For any modulated subcarrier $n \in \mathcal{N}_k$, we define $P_{k,1} = E[|x_k(n,m)|^2]$ if $n \in \mathcal{I}$, $P_{k,2} = E[|x_k(n,m)|^2]$ if $n \in \mathcal{P}_A$. Moreover, let $g_{k,1}$ (resp. $g_{k,2}$) be the channel Gain to Noise Ratio (GNR) in band $\mathcal{I}$ (resp. $\mathcal{P}_A$), namely $g_{k,1} = \rho_k/\sigma_k^2$ (resp. $g_{k,2} = \rho_k/\sigma^2$) and let $W_{k,i} = \gamma_{k,i}P_{k,i}$ be the average power transmitted to user k in $\mathcal{I}$ if $i = 1$ and in $\mathcal{P}_A$ if $i = 2$. "Setting a resource allocation for cell A" means setting a value for parameters $\{\gamma_{k,1}, \gamma_{k,2}, P_{k,1}, P_{k,2}\}_{k=1\dots K}$, or equivalently for parameters $\{\gamma_{k,1}, \gamma_{k,2}, W_{k,1}, W_{k,2}\}_{k=1\dots K}$.

[1] We assume that σ_k^2 is a constant w.r.t. the subcarrier index n. This assumption is valid in a large number of OFDMA multicell systems using frequency hopping or random subcarrier assignment as in Wimax.

3. RESOURCE ALLOCATION FOR SINGLE CELL A

3.1. Optimization Problem

Assume that the resource allocation in cell B is fixed. Assume that each user k has a rate requirement of R_k bits/s/Hz. Our aim is to optimize the resource allocation for cell A which *i)* allows to satisfy all target rates R_k of all users, and *ii)* minimizes the power used by base station A in order to achieve these rates. In frequency-hopping OFDMA scheme used in Wimax, it makes sense to assume that a code word encounters channel coefficients $H_k(n,m)$ for several m and several n. Moreover assuming that the channel varies OFDM symbol by OFDM symbol or that the channel is sufficiently frequency-selective, successful transmission at rate R_k is then provided by $R_k < C_k$, where C_k denotes the ergodic capacity associated with user k. Furthermore, assuming $(w_k(n,m))_{n,m}$ a zero mean independent Gaussian process, we show that

$$C_k = \sum_{i=1}^{2} \gamma_{k,i} \mathbb{E}\left[\log\left(1 + g_{k,i}\frac{W_{k,i}}{\gamma_{k,i}}X\right)\right] \quad (4)$$

where X represents a standard Chi-Square distributed random variable with two degrees of freedom. The power Q spent by base station A during one OFDM block is

$$Q = \sum_k (W_{k,1} + W_{k,2})$$

The optimal resource allocation problem for cell A can be formulated as follows. If $(w_k(n,m))_{n,m}$ is not Gaussian, Eq. (4) becomes a lower bound of the ergodic capacity.

Problem 1. *Minimize Q w.r.t. $\{\gamma_{k,1}, \gamma_{k,2}, W_{k,1}, W_{k,2}\}_k$ under the following constraints.*

C1 : $\forall k, R_k \le C_k$	**C4** : $\gamma_{k,1} \ge 0, \gamma_{k,2} \ge 0$
C2 : $\sum_{k=1}^K \gamma_{k,1} \le \alpha$	**C5** : $W_{k,1} \ge 0, W_{k,2} \ge 0.$
C3 : $\sum_{k=1}^K \gamma_{k,2} \le \frac{1-\alpha}{2}$	

This problem is convex in $\{\gamma_{k,1}, \gamma_{k,2}, W_{k,1}, W_{k,2}\}_k$ and can be solved with the help of the classical Lagrange Karush-Kuhn-Tucker (KKT) conditions. Obviously, we then are able to to finding the optimum $\{\gamma_{k,1}, \gamma_{k,2}, P_{k,1}, P_{k,2}\}_k$.

3.2. Optimization Procedure

Define the following functions on $\mathbb{R}_+$ as

$$f(x) = \frac{\mathbb{E}[\log(1+xX)]}{\mathbb{E}\left[\frac{X}{1+xX}\right]} - x \,, \qquad F(x) = \mathbb{E}\left[\frac{X}{1+f^{-1}(x)X}\right].$$

where f^{-1} is the inverse of f with respect to composition.

Theorem 1. *Assume that the K users are such that the ordering conditions* (2) *and* (3) *hold. It exists a unique integer $L \in \{1, \dots, K\}$ and two unique positive numbers β_1, β_2 such that the optimal resource allocation for Problem 1 is given by:*

1. *For each $k < L$,*

$$P_{k,1} = g_{k,1}^{-1} f^{-1}(g_{k,1}\beta_1), \quad \gamma_{k,1} = \frac{R_k}{\mathbb{E}[\log(1 + g_{k,1}P_{k,1}X)]} \quad \Bigg| \quad P_{k,2} = 0, \quad \gamma_{k,2} = 0$$

2. *For each $k > L$,*

$$P_{k,1} = 0 \quad \Big| \quad P_{k,2} = g_{k,2}^{-1} f^{-1}(g_{k,2}\beta_2)$$

$$\gamma_{k,1} = 0 \quad \Big| \quad \gamma_{k,2} = \frac{R_k}{\mathbb{E}\left[\log\left(1 + g_{k,2}P_{k,2}X\right)\right]}$$

3. *For* $k = L$

$$P_{k,1} = g_{k,1}^{-1} f^{-1}(g_{k,1}\beta_1) \quad \Big| \quad P_{k,2} = g_{k,2}^{-1} f^{-1}(g_{k,2}\beta_2)$$

$$\gamma_{k,1} = \alpha - \sum_{l=1}^{k-1} \gamma_{l,1} \quad \Big| \quad \gamma_{k,2} = \frac{1-\alpha}{2} - \sum_{l=k+1}^{K} \gamma_{l,2}.$$

The principle of the proof of this theorem is to derive and to simplify the KKT conditions applied to the Lagrangian of our optimization problem. This proof will be omitted here for lack of space. The above result shows that the resource allocation depends on three unknown parameters β_1, β_2 and L. The optimal selection of these parameters is provided in the next subsection. Before discussing the choice of β_1, β_2 and L, some comments are useful.

Comments on Theorem 1:

a) Theorem 1 states that the optimal resource allocation scheme is "binary": except for at most one user ($k = L$), any user receives data either in the interference bandwidth $\mathcal{I}$ or in the protected bandwidth $\mathcal{P}_A$, but not in both. Intuitively, it seems clear that users who are the farthest from the base station should mainly receive data in the protected bandwidth $\mathcal{P}_A$, as they are subject to an important multi-cell interference and hence need to be protected. Theorem 1 proves the optimality of this intuitive solution.

b) The geographic separation point between both groups of users is provided by the position of user L. The characterization of this key position is provided in the following subsection. Due to the nature of the considered problem, this pivot position depends on the various target rates $R_1, \ldots, R_K$ of the users and depends on the particular positions of the users in the cell via parameters ρ_k, σ_k^2.

c) Of course, it would have been convenient from an implementation point of view that this optimal pivot position were independent of the particular target rates or the particular users positions. The definition of a separating position which would be fixed but nevertheless relevant in "most situations" will be investigated later on (see Section 5).

d) As expected, the optimal resource allocation depends on the resource allocation in cell B via parameters $\sigma_1^2, \ldots, \sigma_K^2$. Joint optimization of the resource allocation in both cells A and B is investigated in Section 4.

3.3. Determination of β_1,β_2 and L and implementation issues

Define $C(x) = \mathbb{E}[1 + f^{-1}(x)X]$ for each $x \geq 0$. Our aim is to characterize the unique user L which is likely to modulate in both bands $\mathcal{I}$ and $\mathcal{P}_A$, and to find the values of Lagrange multipliers β_1, β_2. Assume for the sake of clarity that $\gamma_{L,1}, \gamma_{L,2}$ are strictly positive for this key user L. Using KKT conditions, we prove that the desired three parameters (β_1, β_2, L) coincide with the unique solution of the following equations in $(l, \tilde{\beta}_1, \tilde{\beta}_2) \in \{1 \ldots K\} \times \mathbb{R}_+ \times \mathbb{R}_+$:

$$g_{l,1}F(g_{l,1}\tilde{\beta}_1) = g_{l,2}F(g_{l,2}\tilde{\beta}_2) \tag{5}$$

$$R_l = \gamma_{l,1}C(g_{l,1}\tilde{\beta}_1) + \gamma_{l,2}C(g_{l,2}\tilde{\beta}_2) \tag{6}$$

where $\gamma_{l,1}, \gamma_{l,2}$ are simply given by

$$\gamma_{l,1} + \sum_{k<l} \frac{R_k}{C(g_{k,1}\tilde{\beta}_1)} = \alpha \tag{7}$$

$$\gamma_{l,2} + \sum_{k>l} \frac{R_k}{C(g_{k,2}\tilde{\beta}_2)} = \frac{1-\alpha}{2} \tag{8}$$

Here equation (5) reflects the fact that the user L to be characterized modulates in both bands $\mathcal{I}$ and $\mathcal{P}_A$. Equations (6), (7) and (8) are nothing else than the constraints **C1**, **C2** and **C3** (holding with equality), only rewritten with the help of Theorem 1. Plugging (7) and (8) into (6) leads to

$$\begin{aligned} R_l \quad = \quad & \left(\alpha - \sum_{k<l} \frac{R_k}{C(g_{k,1}\tilde{\beta}_1)}\right) C(g_{l,1}\tilde{\beta}_1) \\ & + \left(\frac{1-\alpha}{2} - \sum_{k>l} \frac{R_k}{C(g_{k,2}\tilde{\beta}_2)}\right) C(g_{l,2}\tilde{\beta}_2) \,. \end{aligned} \tag{9}$$

For any fixed l, (5) and (9) represent a system of two equations with two unknown $(\tilde{\beta}_1, \tilde{\beta}_2)$. L can be defined as the only integer l such that this system has a solution, and (β_1, β_2) coincide with this (unique) solution. The main focus is therefore on the practical computation of L, β_1, β_2. The most immediate way would be to use an exhaustive search on all $l = 1, \ldots, K$. L would be defined as the unique l for which the set of equations (5) and (9) has a solution, and (β_1, β_2) as the corresponding solution. Of course, this method is rather complex. Instead, we propose to simplify the search for L as follows. For each l, we respectively denote by a_l and b_l the unique positive numbers such that:

$$\sum_{k=1}^{l} \frac{R_k}{C(g_{k,1}a_l)} = \alpha \text{ and } \sum_{k=l+1}^{K} \frac{R_k}{C(g_{k,2}b_l)} = \frac{1-\alpha}{2},$$

with the convention $b_K = 0$. Existence and uniqueness of a_l and b_l are due to the fact that function C is increasing from 0 to ∞ on $\mathbb{R}_+$. One can show the following proposition:

Proposition 1. *User L can be defined as*

$$L = \min\left\{l = 1 \ldots K \big/ g_{l,1}F\left(g_{l,1}a_l\right) \leq g_{l,2}F\left(g_{l,2}b_l\right)\right\}. \tag{10}$$

Note that the above set contains at least integer K (indeed, we have $F\left(g_{K,1}a_K\right) \leq F\left(0\right)$ due to the decreasing character of F, and $g_{K,1} \leq g_{K,2}$) and is thus nonempty. In practice, the search for L can be achieved by dichotomy, computing a_l and b_l only for a limited number of values of l. Then, β_1, β_2 can be obtained by solving the set of equations (5) and (9) with $l = L$.

4. MULTICELL RESOURCE ALLOCATION

Section 3 describes the optimal resource allocation for a single cell A, while the multicell interference is assumed fixed. This is equivalent to assume that the resource allocation in the interfering cell B is fixed. In the present Section, we propose an algorithm which allows to achieve the joint resource allocation in both cells A and B.

4.1. Multicell Interference Model

In order to introduce our multicell algorithm, it is now necessary to define more clearly the way interference levels $\sigma_1^2, \ldots, \sigma_K^2$ depend on the neighboring base station B. In a large number of OFDMA system models, it is straightforward to show that for a given user k of cell A, interference power σ_k^2 does not depends on the particular resource allocation in cell B but only on i) the position of user k and ii) the average power Q_1^B transmitted by base station B in the interference bandwidth $\mathcal{I}$. More precisely,

$$\sigma_k^2 = \mathbb{E}\left[|\tilde{H}_k(n,m)|^2\right] Q_1^B + \sigma^2 \tag{11}$$

where $\tilde{H}_k(n,m)$ represents the channel between base station B and user k of cell A at frequency n and OFDM block m.

4.2. Iterative Resource Allocation

1. **Initialization**: Base station B is shut down: $Q_1^A = 0$.
2. **Cell A**. Evaluate the multicell interference level σ_k^2 from (11) for each user k of Cell A. Compute the resource allocation parameters $\{\gamma_{k,1}, \gamma_{k,2}, P_{k,1}, P_{k,2}\}_k$ using the single cell algorithm of Section3. Evaluate the average power Q_1^A transmitted in $\mathcal{I}$ and transmit its value to base station B.
3. **Cell B**. Evaluate the multicell interference level for each user k of Cell B. Compute the resource allocation parameters in Cell B. Transmit the value of Q_1^B to base station A.
4. **Iterate**. Go back to step 2.

This power algorithm converges. Indeed, it is clear that the total power used by both base stations increases from one iteration to the next. Moreover, any of these two powers is bounded by the power obtained with the naive procedure consisting in only transmitting in the protected bands $\mathcal{P}_A$ and $\mathcal{P}_B$.

5. ASYMPTOTIC ANALYSIS

The optimal value of the pivot-distance, derived in the previous sections, turned out, as one expects, to be dependent on the resource allocation parameters of all the users: the (exact) position of each user in the cell and the user's (exact) data rate requirement. From an implementation point of view, it would be convenient if the optimal pivot position and the optimal reuse factor were calculated using only a "global" characterization of users' allocation parameters rather than the exact values of these parameters for each user in the cell. One possible approach to achieve this goal is the asymptotic analysis proposed in [2] which consists in letting the number of users K grow to infinity. In this Section we will follow this approach and we will investigate the asymptotic expression of the average transmit power when the number of users K grows to infinity. The interest of doing so is that the results in the asymptotic regime are more tractable and can be given in a form that does not depend on the exact resource allocation parameters of each user but rather on the global distribution of these parameters in the cell.

5.1. Basic Assumptions and Asymptotic Regime

In the sequel, we denote by B the total bandwidth of the system in Hz and by $r_k = BR_k$ the data rate requirement of user k in bit/s. We consider the asymptotic regime where the number K of users in each cell tends to infinity. As K tends to infinity, note that the total rate $\sum_{k=1}^K r_k$ which should be delivered by the base station tends to infinity as well. Thus, we need then to let the bandwidth B grow to infinity in order to satisfy the growing data rate requirement. In fact, the asymptotic regime will be characterized by $K \to \infty, B \to \infty$ and $K/B \to c$ where c is a positive constant (in the sequel, we will even write $K = cB$ with slight abuse, for the sake of simplicity). Recall that coefficient $\gamma_{k,1}$ (resp. $\gamma_{k,2}$) is defined as the ratio between the part of the interference bandwidth $\mathcal{I}$ (resp. protected bandwidth $\mathcal{P}_A$) and the total bandwidth. Thus, $\gamma_{k,1}$ and $\gamma_{k,2}$ tend to zero as the total bandwidth B tends to infinity for each k.
We denote by x_k the position of each user k *i.e.* the distance between the user and the base station. The channel variance ρ_k of user k will be written as $\rho_k = \rho(x_k)$ where $\rho(x)$ models the path loss. Typically, function $\rho(x)$ has the form $\rho(x) = \lambda x^{-\alpha}$ where λ is a certain gain and where α is the path-loss coefficient, $\alpha > 2$. In the sequel, we denote by $g_2(x) = \frac{\rho(x)}{\sigma^2}$ the received signal to noise ratio in the protected bandwidth, for a user at position x. For a particular user k, $g_2(x_k) = g_{k,2}$. Similarly, we define $g_1(x)$ as the signal-to-noise ratio received in the interference bandwidth, for a user at position x. For a particular user k, $g_1(x_k) = g_{k,1}$. Functions $g_1(x)$ and $g_2(x)$ are assumed to be continuous functions of x, independent of K.
In view of the resource allocation problem described in Sections 3 and 4 above, user k is commpletely characterized by his rate R_k and his channel GNR $g_{k,1}$ and $g_{k,2}$. With the assumptions introduced above, user k is equivalently characterized by the couple (r_k, x_k). We introduce the following measure $\nu^{(K)}$ defined on the Borel sets of $\mathbb{R}_+ \times \mathbb{R}_+$ as follows

$$\nu^{(K)}(I,J) = \frac{1}{K}\sum_{k=1}^{K} \delta_{(r_k,x_k)}(I,J)$$

where I and J are any intervals of $\mathbb{R}_+$ and where $\delta_{(r_k,x_k)}$ is the Dirac measure at point (r_k, x_k). In other words, $\nu^{(K)}(I,J)$ is simply equal to

$$\nu^{(K)}(I,J) = \frac{\text{number of users located in } J \text{ and requiring a rate in } I}{\text{total number of users}}.$$

Thus, measure $\nu^{(K)}$ can be interpreted as the distribution of the set of couples (r_k, x_k). As $\nu^{(K)}$ is a positive measure which integrates to one, it is a probability measure. We make the following assumptions:

A 1. *As K varies, the supports of all measures $\nu^{(K)}$ are included in $[0, r_{\max}] \times [\varepsilon, D]$ where $r_{\max}$ is an upper bound on all required data rates, $\varepsilon > 0$ is a minimum distance from the BS (which circumvents problems due to singularity of $\rho(x)$ at $x = 0$), and D is the cell radius.*

A 2. *As K tends to infinity, the sequence of measures $\nu^{(K)}$ converges weakly to a measure ν. This limit measure satisfies $d\nu(r,x) = d\zeta(r) \times d\lambda(x)$ where ζ is the limit distribution of rates and λ is the limit distribution of the users locations. Here $\times$ denotes the product of measures.*

We refer to [1] for the material on the convergence of measures. Let us provide more insights on measure $\nu = \zeta \times \lambda$. The fact that ν is a product measure is motivated by the observation that in practice, the rate requirement of a given user is usually independent of the position x_k of the user in the cell. Here λ describes the users geographic distribution in the cell. For instance, if we assume that λ has a density, say $p(x)$: $d\lambda(x) = p(x)dx$. Then, $p(x)$ is simply equal to the density of users around position x in the cell. Similarly, ζ corresponds to the distribution of the data rate requirements in interval $[0, r_{\max}]$.

5.2. Iterative Resource Allocation : Asymptotic Analysis

The aim of this section is to analyze the asymptotic behaviour of the average power per Hertz $Q^{(K)} = \sum_k (\gamma_{k,1}P_{k,1} + \gamma_{k,2}P_{k,2})$ spent by a base station after the convergence of the iterative algorithm described in Section 4.2. The following Theorem (given without proof) states the convergence of $Q^{(K)}$ and provides the expression of its limit Q_∞. This limit is the same for all cells because we assumed that the limit distribution ν is the same for all cells.

Theorem 2. *Assume* **A1** *and* **A2**. *Let $\bar{r}$ be the average rate requirement per user $\bar{r} = c\int_0^{r_{max}} r\, d\zeta(r)$. Then the power $Q^{(K)}$ delivered by any base station converges to $Q_\infty = Q_{1,\infty} + Q_{2,\infty}$ where $Q_{1,\infty}$ and $Q_{2,\infty}$ are the solutions of the following implicit equations:*

$$Q_{1,\infty} = \bar{r}\int_\epsilon^P \frac{f^{-1}(g_1(x,Q_{1,\infty})\beta_1)}{g_1(x,Q_{1,\infty})C(g_1(x,Q_{1,\infty})\beta_1)}\, d\lambda(x) \quad (12)$$

$$Q_{2,\infty} = \bar{r}\int_P^D \frac{f^{-1}(g_2(x)\beta_2)}{g_2(x)C(g_2(x)\beta_2)}\, d\lambda(x)\,, \quad (13)$$

and $(P,\beta_1,\beta_2) \in [\epsilon, D] \times \mathbb{R}_+ \times \mathbb{R}_+$ is the unique solution of the following system of equations

$$g_1(P,Q_{1,\infty})F(g_1(P,Q_{1,\infty})\beta_1) = g_2(P)F(g_2(P)\beta_2) \quad (14)$$

$$\bar{r}\int_\epsilon^P \frac{d\lambda(x)}{C(g_1(x,Q_{1,\infty})\beta_1)} = \alpha \quad (15)$$

$$\bar{r}\int_P^D \frac{d\lambda(x)}{C\,(g_2(x)\beta_2)} = \frac{1-\alpha}{2}\,. \quad (16)$$

The limit values $Q_{1,\infty}$ and $Q_{2,\infty}$ are the asymptotic powers spent in regions $\mathcal{I}$ and $\mathcal{P}_{A,B}$ respectively, and the real number P is the position of the pivot user in the asymptotic regime. They can be determined by classical root search methods. More interestingly, the solution $Q_\infty = Q_{1,\infty} + Q_{2,\infty}$ is a function of the desired reuse factor α. It is therefore possible to characterize the optimal reuse factor α minimizing Q_∞ as the outcome of a classical minimum search algorithm. The most immediate approach is to solve the system of equations given by Theorem 2 for each value of α on a grid in interval $[0,1]$. Once the optimum α has been obtained, it is straight forward to evaluate the optimum pivot distance P, by simple application of Theorem 2.

Thanks to this asymptotic analysis, this optimum value can be fixed when designing the cellular network.

6. SIMULATIONS

In our simulations, we considered a Free Space Loss model (FSL) characterized by a path loss exponent $s = 2$ along with Okumura-Hata (O-H) model for open areas ([6]). Path loss in dB is given by $g_{dB}(x) = 20\log_{10}(x) + 97.5$, where x is the distance in kilometers between the BS and the user. The signal bandwidth is equal to 5MHz and the thermal noise power spectral density is equal to $N_0 = -170$ dBm/Hz. Each cell has a radius $D = 500$ and contains $K = 20$ equally spaced users having identical target rates. Figure 1 represents the ratio $Q_\infty(\alpha)/Q_\infty(\alpha_{opt})$ in dB as a function of the reuse factor α, for an average data rate requirement of $r = 4.33$ bit/s/Hz/km and $r = 5.77$ bit/s/Hz/km respectively (α_{opt} is the value of the reuse factor α that minimizes the transmit power). Power gains are considerable compared to the extreme cases $\alpha = 0$ and $\alpha = 1$. In figure 2 the results of Section 5 are illustrated. The curve show the asymptotically optimum reuse factor and pivot distance in terms of the mean rate $\bar{r}$ obtained by Theorem 2. We notice that α and P are both decreasing functions of $\bar{r}$. This is expected result, given that higher values of $\bar{r}$ will lead to higher transmit power in order to satisfy the users requirements, and consequently to higher levels of interference. More users will need then to be "protected" from the higher interference.

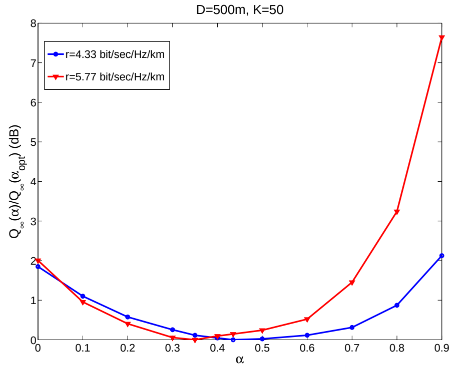

Fig. 1. Power vs. α for $D = 500$ m

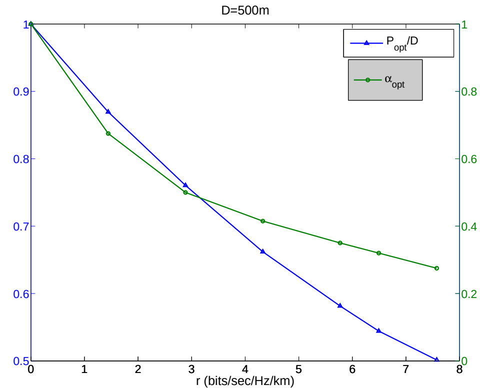

Fig. 2. Optimal reuse factor and pivot distance vs. mean rate, $D = 500$ m

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