Performance bounds for harmonic retrieval in multiplicative noise

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Abstract— **In this communication, we address the performance lower bounds for harmonic retrieval in multiplicative noise. The new results are twofold : we derive the asymptotic (large sample) Gaussian Cramer-Rao bound as well as Barankin bound when the multiplicative noise is complex-valued and non-circular. The theoretical closed-form expressions are then analyzed with respect to design parameters.**

I. INTRODUCTION

We consider the following discrete-time process $y(n)$

$$y(n) = a(n)e^{2i\pi(\phi_0+\phi_1 n)} + b(n) \quad n = 1, \cdots, N \tag{1}$$

where ϕ_0 and ϕ_1 are the parameters of interest. This model holds for many applications, especially, for blind frequency synchronization in digital communications. In such a case, $a(n)$ represents the convolution of the information symbols with physical propagation channel. The random process $b(n)$ refers to noise and is assumed to be Gaussian complex-valued circular stationary with zero-mean and variance $\sigma^2 = \mathbb{E}[|b(n)|^2]$.

The literature about Cramer-Rao bound for harmonic retrieval in multiplicative noise is prolific. As $\mathbf{a} = [a(1), \cdots, a(N)]$ corresponds to parameters of nuisance, one can introduced several types of Cramer-Rao bounds [1] :

- The True/Unconditional CRB which is the standard shape of Cramer-Rao bound is defined as follows

$$\mathrm{UCRB} = \frac{1}{\mathbb{E}_{\mathbf{y}}\left[\left|\frac{\partial}{\partial\phi}\ln\mathbb{E}_{\mathbf{a}}[\Lambda(\phi,\mathbf{a})]\right|^2\right]}$$

with the likelihood

$$\Lambda(\phi,\mathbf{a}) \propto \exp\left\{-\frac{1}{\sigma^2}\sum_{n=0}^{N-1}\left|y(n) - a(n)e^{-2i\pi(\phi_0+\phi_1 n)}\right|^2\right\}.$$

In most practical problems, the UCRB is not tractable in closed-form expressions. To overcome this difficulty, other Cramer-Rao bounds have been studied.

- The average Conditional CRB takes the following form

$$\mathrm{CCRB} = \frac{1}{\mathbb{E}_{\mathbf{y},\mathbf{a}}\left[\left|\frac{\partial}{\partial\phi}\ln\Lambda(\phi,\hat{\mathbf{a}}_\phi)\right|^2\right]} \quad \text{with} \quad \left.\frac{\partial\Lambda(\phi,\mathbf{a})}{\partial\mathbf{a}}\right|_{\hat{\mathbf{a}}_\phi} = 0$$

The parameters of nuisance are thus viewed as deterministic ones and estimated jointly with the parameters of interest.

- The modified CRB is given by

$$\mathrm{MCRB} = \frac{1}{\mathbb{E}_{\mathbf{y},\mathbf{a}}\left[\left|\frac{\partial}{\partial\phi}\ln\Lambda(\phi,\mathbf{a})\right|^2\right]}$$

The MCRB is less tight than the UCRB [2]. Nevertheless this bound is of great interest since the derivations for providing its closed-form expressions are much easier.

- Finally, the so-called Gaussian CRB (GCRB) is equal to the UCRB by assuming (even if it is not accurate) that $\mathbf{a} = [a(0, \cdots, a(N-1)]$ is a Gaussian vector.

Actually the MCRB is mostly used by the Digital Communications community (e.g., synchronization issue). In contrast, the GCRB is rather spread in the Signal Processing community (e.g., Radar and DOA issues).

We hereafter only focus on the GCRB while, in the conference talk, we also overview the various kinds of CRB more deeply. To obtain interpretable expression, it is worth working with asymptotic (large sample) GCRB instead of with exact GCRB. ([3], [4]). In previously-mentioned papers, $a(n)$ were considered either real-valued or circular complex-valued. Our first contribution consists of providing asymptotic GCRB when the multiplicative noise $a(n)$ is assumed to be complex-valued and *non-circular*. Such multiplicative noise can be encountered in digital communications when received signal corresponds to the filtering of real-valued symbols with propagation channel ([5]) or when offset modulations are employed ([6]).

Moreover, in harmonic retrieval ([7]), outliers effect occurs for which the performance of any estimate is far away from the CRB at low SNR. Actually the so-called Barankin bound which is tighter than the CRB at low SNR can be introduced to analysis such a phenomenon. The second contribution of this communication is to derive the Barankin bound for harmonic retrieval whatever the nature of the multiplicative noise (real-valued/complex-valued, circular/non-circular).

II. ASYMPTOTIC GAUSSIAN CRAMER-RAO BOUND

Due to the lack of space, we only put the main results. For more details, the reader may refer to submitted papers in ICASSP'2004 and EUSIPCO'2004.

Throughout the paper, we consider that $a(n)$ is Gaussian complex-valued non-circular stationary process with zero-mean, correlation $r(\tau) = \mathbb{E}[a(n+\tau)\overline{a(n)}]$, and conjugate correlation $u(\tau) = \mathbb{E}[a(n+\tau)a(n)]$ where the overline stands for complex conjugate. The spectrum and conjugate spectrum are denoted respectively as follows

$$s(e^{2i\pi f}) = \sum_{\tau\in\mathbb{Z}} r(\tau)e^{-2i\pi f\tau} \text{ and } c(e^{2i\pi f}) = \sum_{\tau\in\mathbb{Z}} u(\tau)e^{-2i\pi f\tau}.$$

The entire statistics $\{r(\tau), u(\tau)\}_{\tau\in\mathbb{Z}}$ of $a(n)$ only depend on a finite number K of real-valued unknown parameters denoted by $\{\theta_k\}_{k=1,\ldots,K}$.

To analyze the asymptotic GCRB, we proceed into two steps :

- The first step corresponds to express in closed-form the Fisher information matrix for parameters $[\phi_0, \phi_1, \theta_1, \cdots, \theta_K]$.

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• The second step relies on following theorem dealing with the inversion of (large) Toeplitz matrix [8]. Let $\mathbf{t}_N = (t_{l-k})_{-N<k,l<N}$ be a Toeplitz matrix entirely described by

$$s(e^{2i\pi f}) = \sum_{k\in\mathbb{Z}} t_k e^{-2i\pi fk} \Leftrightarrow t_k = \int_0^1 s(e^{2i\pi f})e^{2i\pi fk}df$$

which justifies the following notation : $\mathbf{t}_N = \mathcal{T}_N(s)$. Under mild conditions on $\{t_k; k = 0, \pm 1, \cdots\}$, we get for N large that

$$\mathcal{T}_N(s)^{-1} \sim \mathcal{T}_N(s^{-1}). \quad (2)$$

According to expressions obtained in the first step and result (2), straightforward but tedious calculations leads to

$$\mathrm{CRB}(\phi_1) \sim \frac{3}{4\pi^2 \xi N^3}$$

where

$$\xi = \int_0^1 \frac{c(e^{2i\pi f})\overline{c(e^{-2i\pi f})}}{\mathcal{X}(e^{2i\pi f})}df$$

with

$$\begin{aligned}\mathcal{X}(e^{2i\pi f}) &= (s(e^{2i\pi f}) + \sigma^2)(\overline{s(e^{-2i\pi f})} + \sigma^2) \\ &- c(e^{2i\pi f})\overline{c(e^{-2i\pi f})}.\end{aligned}$$

Previous expressions enable us to yield following comments :
i) The convergence rate of frequency estimation are $1/N^3$ regardless the colorness of multiplicative noise.
ii) The frequency estimation performance depends only on ξ. Herein ξ refers to an information rate provided by the non-circularity. Indeed larger is ξ, and better is performance.
iii) In noiseless case, we observe a floor effect (i.e., $\mathrm{CRB} \neq 0$ when $\sigma^2 = 0$). This effect vanishes iff $s(e^{2i\pi f})\overline{s(e^{-2i\pi f})} = c(e^{2i\pi f})\overline{c(e^{-2i\pi f})}$. For instance, later condition is at least fulfilled when the multiplicative noise is real-valued.

III. Barankin bound

For sake of simplicity, we assume that noise statistics, i.e., $\{r(\tau), u(\tau)\}_{\tau\in\mathbb{Z}}$ and σ^2, are known at the receiver. This assumption is usually done in [9] or partially done [10].

We define $\boldsymbol{\phi} = [\phi_0, \phi_1]^{\mathrm{T}}$ and the so-called "test-points" $\{\boldsymbol{\psi}(k) = [\psi_0(k), \psi_1(k)]^{\mathrm{T}}\}_{1\leq k\leq n}$ where the superscript $^{\mathrm{T}}$ stands for transposition. Then the Barankin bound of order n is defined as follows :

$$\mathrm{BB}_n(\phi_0, \phi_1) = \sup_{\mathcal{E}} S_n(\mathcal{E})$$

where

$$S_n(\mathcal{E}) = \mathcal{E}(\mathbf{B}(\mathcal{E}) - \mathbf{1}_n\mathbf{1}_n^{\mathrm{T}})^{-1}\mathcal{E}^{\mathrm{T}}$$

with $\mathcal{E} = [\boldsymbol{\psi}(1) - \boldsymbol{\phi}, \ldots, \boldsymbol{\psi}(n) - \boldsymbol{\phi}]$, and $\mathbf{1}_n = \mathrm{ones}(n, 1)$. Furthermore $\mathbf{B} = (B_{k,l})_{1\leq k,l\leq n}$ is the following $n \times n$ matrix

$$B_{k,l} = \mathbb{E}_{\mathbf{y}}\left[\frac{p(\mathbf{y}|\boldsymbol{\psi}(k))}{p(\mathbf{y}|\boldsymbol{\phi})}\frac{p(\mathbf{y}|\boldsymbol{\psi}(l))}{p(\mathbf{y}|\boldsymbol{\phi})}\right]$$

and $\mathbf{y} = [y(0), \ldots, y(N-1)]^{\mathrm{T}}$ the N available samples.

The mean square error of any unbiased estimator is greater than any Barankin bound of any order. As $n \to \infty$, the Barankin bound becomes the tightest lower bound ([11], [10]).

Since $\mathbf{y}$ is complex-valued and non-circular, we introduce the following process $\widetilde{\mathbf{y}} = [\mathbf{y}^{\mathrm{T}}, \mathbf{y}^{\mathrm{H}}]^{\mathrm{T}}$ and its associated covariance matrix $\widetilde{\mathbf{R}}_{\boldsymbol{\phi}}$ which depends on the phase parameters $\boldsymbol{\phi}$. The superscript $^{\mathrm{H}}$ stands for complex conjugate transposition.

After straightforward algebraic manipulations based on exponential moments and Wishart distribution ([12]), we get

$$B_{k,l} = \begin{cases} \frac{1}{\sqrt{\det(\mathbf{Q}_{k,l})}} & \text{if } \mathbf{Q}_{k,l} > 0 \\ +\infty & \text{otherwise} \end{cases},$$

with

$$\mathbf{Q}_{k,l} = (\widetilde{\mathbf{R}}_{\boldsymbol{\psi}(k)}^{-1} + \widetilde{\mathbf{R}}_{\boldsymbol{\psi}(l)}^{-1})\widetilde{\mathbf{R}}_{\boldsymbol{\phi}} - \mathbf{Id}_{2N}.$$

In the literature, the following test-points are usually considered ([10])

$$\mathcal{E} = \begin{bmatrix} \psi_0 - \phi_0 & 0 \\ 0 & \psi_1 - \phi_1 \end{bmatrix} = \mathrm{diag}(\varepsilon_0, \varepsilon_1).$$

Then the Barankin bound for frequency parameter ϕ_1 takes the following form

$$\mathrm{BB}_2(\phi_1) = \sup_{\varepsilon_0,\varepsilon_1} \frac{\varepsilon_1^2}{(\mathbf{B}_{1,1} - 1) - (\mathbf{B}_{0,1} - 1)^2/(\mathbf{B}_{0,0} - 1)}.$$

The term $(\mathbf{B}_{0,1} - 1)^2/(\mathbf{B}_{0,0} - 1)$ represents the loss in performance due to joint phase parameter estimation.

Due to the lack of space, we do not display numerical illustrations. However, we may establish that more colored is the multiplicative noise, smaller is the Barankin bound of order 2. In contrast the asymptotic GCRB is quite insensitive to the colorness of the multiplicative noise.

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