Resource Allocation for the Downlink of OFDMA Cellular Networks and Optimization of the Reuse Factor

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Abstract

In this paper, we investigate the issue of power control and sub-carrier assignment in a downlink OFDMA system. We assume that a certain part of the available bandwidth is likely to be reused by the different base stations (and is thus subject to multicell interference) and that the other part of the bandwidth is shared in an orthogonal way between them (and is thus protected from multicell interference). Although this problem of multicell resource allocation is non-convex, we provide, in the limit of large number of users, the general form of its global solution. As a byproduct, we characterize the optimal value of the frequency reuse factor.

1. Introduction

In this paper, we consider the problem of resource allocation in the downlink of cellular OFDMA system with imperfect channel state information at the Base Station (BS) side. In the literature, most works on resource allocation for OFDMA adress the single cell case, while fewer works address the more complicated multicell case (*e.g* [1], [2], [3] and [4]). In this paper, our aim is to characterize the resource allocation strategy allowing to satisfy all users' rate requirements while spending the least power at the transmitters' side. Similarly to [1], we investigate the case where the transmitter CSI is limited to some channel statistics. However, contrary to [1], our model assumes that a certain part of the available bandwidth is shared orthogonally between the adjacent base stations (and is thus "protected" from multicell interference) while the remaining part is reused by different base stations (and is thus subject to multicell interference). Note that this so-called *fractional frequency reuse* is recommended in a number of standards *e.g.* in [5] for IEEE 802.16 (WiMax). We also assume that each user is likely to modulate in each of these two parts of the bandwidth. Thus, we stress the fact that *i)* no user is forced to modulate in a single frequency band, *ii)* we do not assume *a priori* a geographical separation of users modulating in the two different bands (as was the case in [2] and [3]). On the opposite, we shall *demonstrate* that such a geographical separation is actually optimal w.r.t. our allocation problem in the limit of large number of users.

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2. Contributions

The main contributions of this work are: **1.** Providing the form of the optimal solution to the non-convex OFDMA joint resource allocation problem in the context of our system model. **2.** Characterizing the asymptotic behavior of the optimal resource allocation when the number of users tends to infinity and proving the asymptotic optimality of the fractional frequency reuse scheme. **3.** Providing a method to calculate optimal (in a certain relevant sense) reuse factors that can be used during the network design process.

3. System Model

We consider three adjacent $120\,^\circ$ sectors from three adjacent circular cells, say Cells A, B and C for example, as is illustrated by Fig. 1. For each Cell c ($c \in \{A, B, C\}$), we denote by K^c the number of users. The total number of available subcarriers is denoted by N. For a given user $k \in 1, 2, \ldots, K^c$ in Cell c, we denote by r_k the distance that separates him/her from BS c, and by $\mathcal{N}_k$ the set of indices corresponding to the subcarriers modulated by k ($\mathcal{N}_k$ is a subset of $\{0, 1, \ldots, N-1\}$). The signal received by user k at the nth subcarrier ($n \in \mathcal{N}_k$) and at the mth OFDM block is given by

$$y_k(n,m) = H_k(n,m)x_k(n,m) + w_k(n,m), \qquad (1)$$

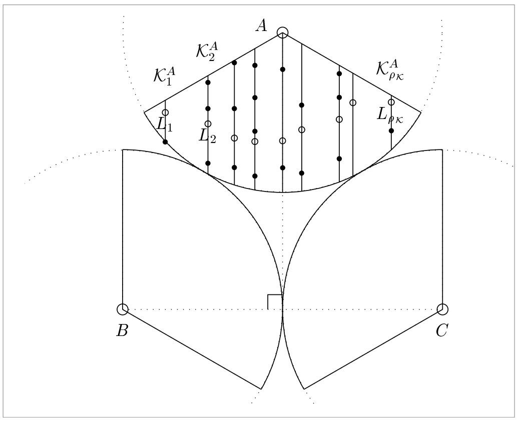

Figure 1: System Model and Definition of sets $\mathcal{K}_i^A$

where $x_k(n,m)$ represents the data symbol transmitted by BS c. Process $w_k(n,m)$ is an additive noise which encompasses the thermal noise and the possible multicell interference. Coefficient $H_k(n,m)$ is the frequency response of the channel at the subcarrier n and the OFDM block m. Random variables $H_k(n,m)$ For each user k in Cell c are assumed to be Rayleigh distributed with variance denoted in the sequel by $\rho_k^c = E[|H_k(n,m)|^2]$. For a given user k, $H_k(n,m)$ are identically distributed w.r.t. n, m, but are not supposed to be independent. Channel coefficients are supposed to be perfectly known at the receiver side, and unknown at the BS side. However, variances ρ_k are supposed to be known at the BS. As usual, we assume that ρ_k vanishes with r_k the distance between the BS and user k, based on a given path loss model. The set of available sub-carriers is partitioned into four subsets: $\mathcal{I}$ containing the reused subcarriers shared by the three sectors; $\mathcal{P}_A$, $\mathcal{P}_B$ and $\mathcal{P}_C$ containing the protected subcarriers only used by users in Cell A, B and C respectively. The *reuse factor* α is defined as the ratio between the number of reused subcarriers and the total number of subcarriers:

$$\alpha = \frac{\text{card}(\mathcal{I})}{N}$$

so that $\mathcal{I}$ contains αN subcarriers, and each one of the bands $\{\mathcal{P}_c\}_{c=A,B,C}$ contains $\frac{1-\alpha}{3}N$ subcarriers. If user k modulates a subcarrier $n \in \mathcal{I}$, the additive noise contains both thermal noise of variance σ^2 and interference. Therefore, the variance of this noise-plus-interference process depends on k and will be denoted by σ_k^2, and is crucially related to the position of user k. We thus define

$$\forall n \in \mathcal{I}, \mathbb{E}[|w_k(n,k)|^2] = \sigma_k^2 \,.$$

Note that we assumed σ_k^2 a constant w.r.t. the subcarrier index n. This assumption is valid in a large number of OFDMA multicell systems using frequency hopping or random subcarrier assignment. The remaining $(1-\alpha)N$ subcarriers are shared by the three cells A, B, C in an orthogonal way. If user k modulates such a subcarrier $n \in \mathcal{P}_c$, the additive noise $w_k(n,m)$ contains only thermal noise. In other words, subcarrier n does not suffer from multicell interference. Then we simply write $\mathbb{E}[|w_k(n,m)|^2] = \sigma^2$. The resource allocation parameters for user k are: $P_{k,1}^c$ the power transmitted on each of the subcarriers of the non-protected band $\mathcal{I}$ allocated to him, $\gamma_{k,1}^c$ his share of $\mathcal{I}$, $P_{k,2}^c$ the power transmitted on each of the subcarriers of the protected band $\mathcal{P}_c$ allocated to him and $\gamma_{k,2}^c$ his share of $\mathcal{P}_c$. In other words,

$$\gamma_{k,1}^c = \text{card}(\mathcal{I} \cap \mathcal{N}_k)/N \qquad \gamma_{k,2}^c = \text{card}(\mathcal{P}_c \cap \mathcal{N}_k)/N$$

Moreover, let $g_{k,1}$ (resp. $g_{k,2}$) be the channel Gain to Noise Ratio (GNR) in band $\mathcal{I}$ (resp. $\mathcal{P}_c$), namely $g_{k,1} = \rho_k/\sigma_k^2$ (resp. $g_{k,2} = \rho_k/\sigma^2$) and let $W_{k,i} = \gamma_{k,i}P_{k,i}$ be the average power transmitted to user k in $\mathcal{I}$ if $i = 1$ and in $\mathcal{P}_c$ if $i = 2$. "Setting a resource allocation for cell c" means setting a value for parameters $\{\gamma_{k,1}^c, \gamma_{k,2}^c, P_{k,1}^c, P_{k,2}^c\}_{k=1\ldots K^c}$, or equivalently for parameters $\{\gamma_{k,1}^c, \gamma_{k,2}^c, W_{k,1}^c, W_{k,2}^c\}_{k=1\ldots K^c}$.

4. Single Cell Resource Allocation

Before tackling the problem of optimal joint resource allocation in the three considered cells, it is useful to consider first the simpler single cell problem. The single cell formulation focuses on resource allocation in one cell (say Cell A), and assumes that the resource allocation parameters of users in the other cells are fixed.

4.1. Optimization problem

Assume that each user k has a rate requirement of R_k nats/s/Hz. Our aim is to optimize the resource allocation for Cell A which i) allows to satisfy all target rates R_k of all users, and ii) minimizes the power used by BS A in order to achieve these rates. Considering a fast fading context (i.e. channel coefficients $H_k(n,m)$ vary w.r.t. m all along the code word), we assume as usual that successful transmission at rate R_k is possible provided that $R_k < C_k$, where C_k denotes the ergodic capacity associated with user k. Unfortunately, the exact expression of the ergodic capacity is difficult to obtain in our context due to the fact that the noise-plus-interference process $(w_k(n,m))_{n,m}$ is not a Gaussian process in general. Nonetheless, if we endow the input symbols $x_k(n,m)$ with Gaussian distribution, the mutual information between $x_k(n,m)$ and the received signal $y_k(n,m)$ in Eq. (1) is minimum when the interference-plus-noise $w_k(n,m)$ is Gaussian distributed. Therefore, approximating the multicell interference noise by a Gaussian noise with the same variance provides a lower bound on the true capacity. For a given user k in Cell A, one can easily verify that the ergodic capacity C_k is equal to the sum of the ergodic capacities corresponding to both bands $\mathcal{I}$ and $\mathcal{P}_A$:

$$C_k = \sum_{i=1,2} \gamma_{k,i}^A \mathbb{E}\left[\log\left(1 + g_{k,i}\frac{W_{k,i}^A}{\gamma_{k,i}^A}X\right)\right] \quad (2)$$

where X represents a standard Chi-Square distributed random variable with two degrees of freedom. The quantity $Q^{A,(K)}$ defined by

$$Q^{A,(K)} = \sum_{k=1}^{K^A}(W_{k,1}^A + W_{k,2}^A) \quad (3)$$

denotes the average power spent by BS A during one OFDM block. The optimal resource allocation problem for cell A consists in characterizing the optimal value of $\{\gamma_{k,1}^A, \gamma_{k,2}^A, W_{k,1}^A, W_{k,2}^A\}_{k=1\dots K^A}$ allowing to satisfy the rate requirements of all users ($R_k < C_k$) so that the power $Q^{A,(K)}$ to be spent is minimum. Finally, the single cell optimization problem for Cell c ($c \in \{A,B,C\}$) can be formulated as follows.

Problem 1. *Minimize* $Q^{c,(K)}$ *w.r.t.* $(\gamma_{k,i}^c, W_{k,i}^c)_{\substack{i\in\{1,2\}\\ k\in\{1\dots K^c\}}}$ *under the following constraints.*

$$\mathbf{C1}: \forall k, R_k \le C_k \qquad \mathbf{C4}: \gamma_{k,1}^c \ge 0, \gamma_{k,2}^c \ge 0$$

$$\mathbf{C2}: \sum_{k=1}^{K^c}\gamma_{k,1}^c = \alpha \qquad \mathbf{C5}: W_{k,1}^c \ge 0, W_{k,2}^c \ge 0.$$

$$\mathbf{C3}: \sum_{k=1}^{K^c}\gamma_{k,2}^c = \frac{1-\alpha}{3}.$$

Here, **C1** is the rate constraint, **C2-C3** are the bandwidth constraints, **C4-C5** are the positivity constraints. We expressed the resource allocation problem in terms of parameters $\gamma_{k,i}^c, W_{k,i}^c$ ($i=1,2$) instead of $\gamma_{k,i}^c, P_{k,i}^c$ because the ergodic capacity given by Eq. (2) is a convex function of $\gamma_{k,i}^c, W_{k,i}^c$. As a consequence, Problem 1 is convex in $\{\gamma_{k,1}^c, \gamma_{k,2}^c, W_{k,1}^c, W_{k,2}^c\}_k$.

4.2. Optimal single cell resource allocation

Since Problem 1 is convex, one can use the Lagrange Karush-Kuhn-Tucker (KKT) conditions to solve it. In order to present the results of solving the KKT conditions, it turns out useful to regroup the users of Cell c into a number $\rho_{\mathcal{K}}$ of subsets $\{\mathcal{K}_i^c\}_{1\le i\le\rho_{\mathcal{K}}}$: each subset $\mathcal{K}_i^c$ is composed of the users situated on a certain line perpendicular to the axis passing through the two adjacent BS, as illustrated by Fig. 1. For example, subsets $\mathcal{K}_i^A$ are constructed as follows. First, a line perpendicular to axis BC is drawn passing through each user k in Cell A. If more than one user happen to be on the same line, then all of these users will be part of the same subset $\mathcal{K}_i^A$. Otherwise, the corresponding subset will contain a single element. Note that, by construction, $\rho_{\mathcal{K}^A}$ the number of subsets satisfies $\rho_{\mathcal{K}} \le K^A$, and that $\forall i \in \{1,\dots,\rho_{\mathcal{K}}\}$, $\mathcal{K}_i^A \ne \emptyset$.
We present now the result of simplifying the KKT conditions associated with Problem 1. For that sake, define the following functions on $\mathbb{R}_+$

$$f(x) = \frac{\mathbb{E}\left[\log(1+xX)\right]}{\mathbb{E}\left[\frac{X}{1+xX}\right]} - x, \qquad F(x) = \mathbb{E}\left[\frac{X}{1+f^{-1}(x)X}\right],$$

where X represents a standard Chi-square distributed random variable with two degrees of freedom. Recall that r_k denotes the distance separating user k from the BS.

Theorem 1. *Any global solution* $\{\gamma_{k,1}^c, \gamma_{k,2}^c, W_{k,1}^c, W_{k,2}^c\}_{k=1\dots K^c}$ *to Problem 1 has the following property. There exist two nonnegative numbers* β_1^c, β_2^c*, and there exists an integer* $L_{\mathcal{K}_i^c} \in \mathcal{K}_i^c$ *for each* $i \in \{1,\dots,\rho_{\mathcal{K}}\}$*, such that*
1. For each $k \in \mathcal{K}_i^c$ *such that* $r_k < r_{L_i^c}$*,*

$$\left.\begin{array}{l} P_{k,1}^c = g_{k,1}^{-1} f^{-1}\left(g_{k,1}\beta_1^c\right) \\ \gamma_{k,1}^c = \dfrac{R_k}{\mathbb{E}\left[\log\left(1+g_{k,1}P_{k,1}^c X\right)\right]} \end{array}\right| \begin{array}{l} P_{k,2}^c = 0 \\ \gamma_{k,2}^c = 0 \end{array} \quad (4)$$

2. For each $k \in \mathcal{K}_i^c$ *such that* $r_k > r_{L_i^c}$*,*

$$\left.\begin{array}{l} P_{k,1}^c = 0 \\ \gamma_{k,1}^c = 0 \end{array}\right| \begin{array}{l} P_{k,2}^c = g_{k,2}^{-1} f^{-1}(g_{k,2}\beta_2^c) \\ \gamma_{k,2}^c = \dfrac{R_k}{\mathbb{E}\left[\log\left(1+g_{k,2}P_{k,2}^c X\right)\right]} \end{array} \quad (5)$$

3.For $k = L_i^c$

$$\begin{array}{l} P_{k,1}^c = g_{k,1}^{-1} f^{-1}\left(g_{k,1}\beta_1^c\right) \\ P_{k,2}^c = g_{k,2}^{-1} f^{-1}(g_{k,2}\beta_2^c) \\ g_{k,1}F\left(g_{k,1}\beta_1^c\right) = g_{k,2}F\left(g_{k,2}\beta_2^c\right) \end{array} \quad (6)$$

where β_1^c*,* β_2^c *are the Lagrange multipliers associated with constraints* **C2** *and* **C3** *respectively.*

Comments on Theorem 1: Theorem 1 states that any global solution to Problem 1 is "binary" along each line perpendicular to the axis passing through the two adjacent BS: Except for at most one user L_i^c in each subset $\mathcal{K}_i^c$, any user k of $\mathcal{K}_i^c$ modulates either in the protected band $\mathcal{P}_c$ (and thus $\gamma_{k,2}^c > 0$), or in the non-protected band $\mathcal{I}$ (and thus $\gamma_{k,1}^c > 0$), but not in both.

Note that in practice, most of the subsets $\mathcal{K}_i^c$ would contain only one user, and therefore $\rho_{\mathcal{K}} = K^c$. In this case, Theorem 1 will not be of real help in computing the optimal value of the resource allocation parameters of users of Cell c. Indeed, determining the value of parameters $\beta_1^c, \beta_2^c, \{L_i^c\}_i$ will require in this case the use of prohibitively computational complex methods. Nonetheless, when the number of users grows to infinity in the way described in Section 6, the number of users in each of the subsets $\mathcal{K}_i^c$ will also grow to infinity, and Theorem 1 will prove very useful in this case for characterizing the optimal resource allocation.

5. Joint Resource Allocation for Cells A, B, C

Our aim now is to jointly optimize the resource allocation for the three cells which i) allows to satisfy the target rates R_k of all users, and ii) minimizes the power used by the network in order to achieve these rates.

5.1. Optimization problem

The ergodic capacity associated with user k in Cell A is given by Eq. (2), where coefficient $g_{k,1}$ in that equation coincides with

$$g_{k,1}(Q_1^B, Q_1^C) = \frac{\rho_k}{\sum_{c=B,C} \mathbb{E}\left[|H_{c,k}(n,m)|^2\right] Q_1^c + \sigma^2}, \tag{7}$$

where $H_{c,k}(n,m)$ is the frequency response of the channel between BS c and user k at the subcarrier n and the OFDM block m. Note that $g_{k,1}(Q_1^B, Q_1^C)$ depends on the powers $Q_1^B = \sum_{k=1}^{K^B} W_{k,1}^B$ and $Q_1^C = \sum_{k=1}^{K^C} W_{k,1}^C$ transmitted respectively by BS B and C in band $\mathcal{I}$. The joint allocation problem can be formulated as follows.

Problem 2. *Minimize* $Q_T^{(K)} = \sum_{c=A,B,C} \sum_{k=1}^{K^c} (W_{k,1}^c + W_{k,2}^c)$ *the total power spent by the network w.r.t* $\{\gamma_{k,1}^c, \gamma_{k,2}^c, W_{k,1}^c, W_{k,2}^c\}_{\substack{c=A,B,C \\ k=1\ldots K^c}}$ *such that constraints* **C1**−**C5** *of Problem 1 are satisfied for* $c = A, B, C$.

Unfortunately, the ergodic capacity C_k is non-convex w.r.t the optimization variables. This is due to the fact that the gain-to-noise ratio $g_{k,1}(Q_1^B, Q_1^C)$ is a function of the resource allocation parameters of users belonging to the interfering cells. Therefore, Problem 2 cannot be solved by classical convex optimization methods.

5.2. Characterizing the optimal joint allocation

Even though Problem 2 is non-convex, we manage to characterize its solution. Indeed, we prove that this problem can be decomposed into three single cell problems similar to Problem 1.

For each cell $c \in \{A, B, C\}$, denote by $c^{'}$ and $c^{''}$ the two adjacent cells. ($A^{'} = B$ and $A^{''} = C$).

Theorem 2. *Any global solution to Problem 2 has the following property. For each Cell c there exist four non-negative numbers* β_1^c, β_2^c, $Q_1^{c^{'}}$, $Q_1^{c^{''}}$ *, and there exists an integer* $L_i^c \in \mathcal{K}_i^c$ *for each* $i \le \rho_{\mathcal{K}^c}$*, such that the resource allocation parameters* $\gamma_{k,1}^c$, $\gamma_{k,2}^c$, $P_{k,1}^c$, $P_{k,2}^c$ *are given by Eq.* (4), (5) *and* (6)*, where* $g_{k,1}$ *in these equations coincides with* $g_{k,1}(Q_1^{c^{'}}, Q_1^{c^{''}})$ *defined by* (7).

Comments on Theorem 2: Theorem 2 implies that the optimal allocation parameters of users in Cell A in the multicell case are solution to the single cell problem (Problem 1). Of course, similar result holds for Cells B and C. We conclude that in each cell, the binary property stated by Theorem 1 holds. It is worth mentioning that the proof of Theorem 2 is not a simple generalization of the single cell case. Indeed, proving Theorem 2 required solving the KKT conditions of a modified optimization problem derived from Problem 1.

6. Asymptotic Analysis

We characterize now the asymptotic behavior of the optimal resource allocation when the number of users in each cell tends to infinity. The interest of this analysis is threefold. First, the optimal resource allocation in the asymptotic regime can be fully characterized, in contrast to the case of finite number of users. Second, the asymptotic expressions do not depend on the exact position of each user in the cell and his exact rate requirement. Instead, they only depend on a “global” characterization of these parameters. This property is very useful to define an “optimal” reuse factor. Third, the above two aspects of the asymptotic regime inspires a simplified, yet relevant, method to perform resource allocation for finite number of users.

6.1. Asymptotic Regime

In the sequel, we denote by B the total bandwidth of the system in Hz, and by u_k the data rate of user k in nats/sec ($u_k = BR_k$, where R_k is the rate in nats/sec/Hz). We focus first on Cell A, and we consider the asymptotic regime where the number K^A of users in cell A tends to infinity. As K^A tends to infinity, note that the total rate $\sum_{k=1}^{K^A} u_k$ (in nats/sec) which should be delivered by the BS tends to infinity as well. Thus, we need to let the bandwidth B grow to infinity in order to satisfy the growing data rate requirement. In fact, the asymptotic regime will be characterized by $K^A \to \infty, B \to \infty$ and $K^A/B \to a^A$ where a^A is a positive constant. Let (r_k, θ_k) be the polar position of user k in Cell A (r_k is the distance previously defined). The channel variance ρ_k of user k will be written as $\rho_k = \rho(r_k)$ where $\rho(r)$ models the path loss and depends only on the distance separating user k from BS A. Typically, function $\rho(r)$ has the form $\rho(r) = \eta r^{-s}$ where η is a certain gain and where s is the path-loss coefficient, $s \geq 2$. In the sequel, we denote by $g_2(r) = \frac{\rho(r)}{\sigma^2}$ the received signal to noise ratio in the protected band, for a user at distance r. This way, $g_2(r_k) = g_{k,2}$. Similarly, we define $g_1(r, \theta, Q_1^B, Q_1^C)$ as the signal-to-noise ratio received in the non-protected band, for a user at position (r, θ) when the average power transmitted by BS B and C in the non-protected band $\mathcal{I}$ is equal to Q_1^B and Q_1^C respectively. For a particular user k, $g_1(r_k, \theta_k, Q_1^B, Q_1^C) = g_{k,1}$. Note that $g_{k,1}$ depends on θ_k the angular position of user k in the cell. This is because the distances separating user k from BS B and C are both functions of θ_k.

6.2. Statistical tools for the asymptotic regime

We characterize now the asymptotic behavior of $Q_T^{(K)}$ the total transmit power as $K \to \infty$. Recall that $Q_T^{(K)} = \sum_c \sum_k W_{k,1}^c + W_{k,2}^c$. One can expect that summation w.r.t k in the latter expression would be replaced by integration when $K \to \infty$. In order to obtain this integral expression, note that user k is completely characterized by the triple (r_k, θ_k, u_k) and define measure $\nu^{A,(K)}$ on the Borel sets of $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$ as

$$\nu^{A,(K)}(I, J, L) = \frac{1}{K} \sum_{k=1}^{K^A} \delta_{(r_k, \theta_k, u_k)}(I, J, L),$$

where I, J, L are intervals of $\mathbb{R}_+$, $\mathbb{R}$ and $\mathbb{R}_+$ respectively and where $\delta_{r_k, \theta_k, u_k}$ is the Dirac measure at point (r_k, θ_k, u_k). We assume in the sequel that $\forall k$, $r_k \in [\varepsilon, D]$, $\theta_k \in [-\frac{\pi}{3}, \frac{\pi}{3}]$ and $u_k \in [0, u_{\max}]$, where $u_{\max}$ is an upper bound on all required data rates and $\varepsilon > 0$ is a minimum distance from the BS.

Assumption 1. *As K^A tends to infinity, the sequence of measures $\nu^{A,(K)}$ converges weakly to a measure ν^A. This limit measure satisfies $d\nu^A(r, \theta, u) = d\lambda^A(r, \theta) \times d\zeta^A(u)$ where λ^A is the limit distribution of users' locations and ζ^A is the limit distribution of rates. Here $\times$ denotes the product of measures.*

The fact that ν^A is a product measure is motivated by the observation that the rate requirement of a given user is usually not related to the position (r_k, θ_k) of the user in the cell. Here λ^A describes users' geographic distribution in the cell. For instance, if we assume that λ has a density, say $p(r, \theta)$: $d\lambda^A(r, \theta) = p(r, \theta) r dr d\theta$. Then, $p(r, \theta)$ is simply equal to the density of users around position (r, θ) in the cell. Similarly, ζ^A corresponds to the distribution of rate requirements in $[0, u_{\max}]$.

6.3. Multicell allocation in the asymptotic regime

Define $C(x)$ on $\mathbb{R}_+$ as $C(x) = \mathbb{E}\left[\log(1 + f^{-1}(x))\right]$, and recall the notation $c^{'}$ and $c^{''}$ which designates the two adjacent cells for each cell $c \in \{A, B, C\}$.

Theorem 3. *Assume that $K^c \to \infty$ for $c \in \{A, B, C\}$ such that $K^c/K \to a^c$. Let $\bar{\rho}^c$ be the average rate requirement per channel use $\bar{\rho}^c = a^c \int_0^{u_{\max}} u \, d\zeta^c(u)$. There exist 9 positive numbers $(\beta_1^c, \beta_2^c, Q_1^c)_{c \in \{A,B,C\}}$ such that the total power $Q_T^{(K)}$ spent by the network when users' resource allocation parameters are global solution to Problem 2 converges to $Q_T = \sum_c Q_1^c + Q_2^c$, where*

$$Q_1^c = \bar{\rho}^c \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \int_{\epsilon}^{d^c(\theta)} \frac{f^{-1}(g_1(r, \theta, Q_1^{c^{'}}, Q_1^{c^{''}}) \beta_1^c) d\lambda^c(r, \theta)}{g_1(r, \theta, Q_1^{c^{'}}, Q_1^{c^{''}}) C(g_1(r, \theta, Q_1^{c^{'}}, Q_1^{c^{''}}) \beta_1^c)} \tag{8}$$

$$Q_2^c = \bar{\rho}^c \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \int_{d^c(\theta)}^{D} \frac{f^{-1}(g_2(r) \beta_2^c) d\lambda^c(r, \theta)}{g_2(r) C(g_2(r) \beta_2^c)}, \tag{9}$$

where $(d^c(\theta), \beta_1^c, \beta_2^c) \in [\epsilon, D] \times \mathbb{R}_+ \times \mathbb{R}_+$ is the unique solution to the system $\mathcal{S}^c(Q_1^{c^{'}}, Q_1^{c^{''}})$ formed by:

$$g_1(d^c(\theta), \theta, Q_1^{c^{'}}, Q_1^{c^{''}}) F(g_1(d^c(\theta), \theta, Q_1^{c^{'}}, Q_1^{c^{''}}) \beta_1^c) = g_2(d^c(\theta)) F(g_2(d^c(\theta)) \beta_2^c) \tag{10}$$

$$\bar{\rho}^c \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \int_{\epsilon}^{d^c(\theta)} \frac{d\lambda^c(r, \theta)}{C(g_1(r, \theta, Q_1^{c^{'}}, Q_1^{c^{''}}) \beta_1^c)} = \alpha \tag{11}$$

$$\bar{\rho}^c \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \int_{d^c(\theta)}^{D} \frac{d\lambda^c(r, \theta)}{C(g_2(r) \beta_2^c)} = \frac{1 - \alpha}{3}. \tag{12}$$

Comments on Theorem 3: a) Denote by $Q_1^{c,(K)}$ (resp. $Q_2^{c,(K)}$) the power transmitted by BS c in the non-protected band $\mathcal{I}$ (resp. the protected band $\mathcal{P}_c$) when optimal allocation is used *i.e,* $Q_i^{c,(K)} = \sum_k W_{k,i}^A$ ($i = 1, 2$). The quantities Q_1^A and Q_2^A given by Eq. (8) and (9) can be considered as the limit of $Q_1^{c,(K)}$ and $Q_2^{c,(K)}$ respectively. Eq. (10) results from the third equality of Eq. (6) of Theorem 1. As for Eq. (11) and (12), they can be considered as the respective limit of constraints **C2** and **C3** of Problem 1.

b) A careful look at integration boundaries in Eq. (11) and (12) reveals that optimal resource allocation in the asymptotic regime is "binary": The protected band $\mathcal{P}_c$

is shared only between users k satisfying $r_k > d^c(\theta_k)$; and the non-protected band $\mathcal{I}$ is assigned only to users satisfying $r_k < d^c(\theta_k)$. This result proves that the fractional frequency reuse scheme [5] is asymptotically optimal with respect to our resource allocation problem.
c) Theorem 3 reduces the problem of characterizing the optimal resource allocation in the asymptotic regime into the problem of determining the value of a limited number of parameters: $(\beta_1^c, \beta_2^c, Q_1^c)_{c=A,B,C}$, which is addressed in the sequel.

Determination of $(\beta_1^c, \beta_2^c, Q_1^c)_{c=A,B,C}$: Consider first Cell A. Note that for a fixed value of Q_1^B and Q_1^C, (β_1^A, β_2^A) is defined as the unique solution to the system $\mathcal{S}^A(Q_1^B, Q_1^C)$ formed by Eq. (10)-(12). The issue of solving this system is addressed in the extended version of this paper. The difficulty resides in the determination of the optimal Q_1^A, Q_1^B, Q_1^C. Denote by $Q_1^c = J^c(Q_1^{c'}, Q_1^{c''})$ the value of Q_1^c given by Eq. (8). Define the vector function $\mathbf{J}(Q_1^A, Q_1^B, Q_1^C) = (J^A(Q_1^B, Q_1^C), J^B(Q_1^A, Q_1^C), J^C(Q_1^B, Q_1^A))$. Note that for any global solution, (Q_1^A, Q_1^B, Q_1^C) is a fixed point of $\mathbf{J}$:

$$(Q_1^A, Q_1^B, Q_1^C) = \mathbf{J}(Q_1^A, Q_1^B, Q_1^C) .$$

First consider the case when $\mathbf{J}$ admits a unique fixed point. In this case, a classic fixed point algorithm (the fixed point iteration) can be used to determine (Q_1^A, Q_1^B, Q_1^C). Unfortunately, we did not manage to analytically prove the uniqueness of the fixed point. Therefore, it might happen that the fixed point iteration does not converge or oscillates between different fixed points. In this case, the most immediate approach would consist in an exhaustive search with respect to two out of the three values Q_1^A, Q_1^B, Q_1^C.

Applications:

1. Selection of the optimal reuse factor α_{opt}: In practice, the reuse factor should be fixed prior to resource allocation and its value should be independent of the particular cells' configurations. Here we propose to select α_{opt} as $\alpha_{opt} = \arg\min_\alpha \lim_{K\to\infty} Q_T^{(K)}(\alpha)$, where $\lim_{K\to\infty} Q_T^{(K)}(\alpha)$ is given by Theorem 3.

2. Simplified resource allocation: The binary form of the optimal allocation in the asymptotic regime stated by Theorem 3 can inspire a simplified scheme to perform resource allocation for finite number of users. The exact description of this scheme and its performance will be provided in the extended version of this paper.

7. Simulations

We considered a free space loss model characterized by a path loss exponent $s = 2$ along with Okumura-Hata model for open areas. The signal bandwidth is equal to 5 MHz and the thermal noise power spectral density is equal to $N_0 = -170$ dBm/Hz. Each cell has a radius $D = 500$m and the same uniform distribution of users: $\lambda^A = \lambda^B = \lambda^C = \lambda$ and $d\lambda(r,\theta) = r dr d\theta/|\mathcal{C}|$, where $|\mathcal{C}|$ is the cell surface. The rate requirement is assumed the same for all the users in the three cells. For each value of the reuse factor α, denote by $Q_T(\alpha)$ the minimal total power spent by the network in the asymptotic regime as given by Theorem 3. Define $\bar{r}$ as the average rate per channel use per cell surface unit *i.e*, $\bar{r} = \bar{\rho}/|\mathcal{C}|$, where $\bar{\rho} = \bar{\rho}^A$ was defined in Theorem 3. Fig. 2 represents, for an average data rate requirement of $\bar{r} = 4$ and 6 bits/sec/Hz/km^2 respectively, the value of $Q_T(\alpha)$ normalized by its minimum value w.r.t α, *i.e* the ratio $Q_T(\alpha)/Q_T(\alpha_{opt})$ in dB (α_{opt} is the value of the reuse factor α that minimizes $Q_T(\alpha)$). Power gains are considerable compared to the extreme cases $\alpha = 0$ (the available bandwidth is shared in an orthogonal way between Cells A, B and C) and $\alpha = 1$ (all the available bandwidth is reused).

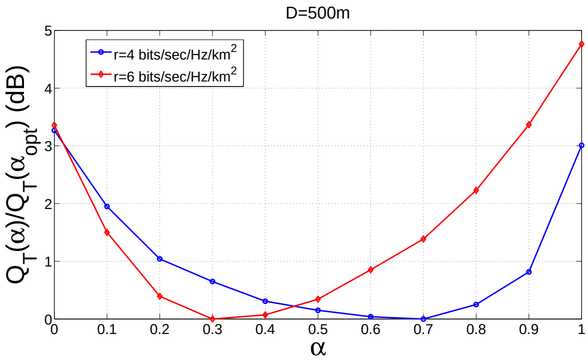

Figure 2: Total Power vs. α for $D = 500$ m

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