HARMONIC RETRIEVAL IN NON-CIRCULAR COMPLEX-VALUED MULTIPLICATIVE NOISE : CRAMER-RAO BOUND

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ABSTRACT

We address the problem of harmonic retrieval in the presence of multiplicative and additive noise. We derive the finite-sample Cramér Rao Bound (CRB) as well as the asymptotic (large sample) CRB when the multiplicative noise is complex-valued and *non-circular*. These bounds are then analyzed with respect to the signal parameters. Finally, we prove that the Square-Power based frequency estimate, which is equivalent to the so-called Nonlinear Least Square estimate, is asymptotically efficient when the multiplicative noise is white.

1. INTRODUCTION

In many applications, such as non-data-aided frequency synchronization in digital communications and direction of arrival (DOA) estimation, the receiver needs to estimate a harmonic in the presence of multiplicative and additive noise sources. A well accepted model for the discrete-time received signal in such a scenario is

$$y(n) = a(n)e^{2i\pi(\phi_0+\phi_1 n)} + b(n) \tag{1}$$

where ϕ_0 is the phase and ϕ_1 is the frequency shift to be estimated. The random process $a(n)$ represents, for example in digital communications scheme, the convolution of the symbol stream with the transmit/receive filters and physical channel. The random process $b(n)$ is an additive noise.

A considerable work has already been carried out on the derivation of the Cramér-Rao bound (CRB) for the above estimation issue. Making the assumption that both $a(n)$ and $b(n)$ are Gaussian, expressions for the CRB were developed in [1, 2, 3, 4, 5, 6]. However, these expressions are valid when either i) $a(n)$ is real-valued, ([2, 3, 4, 5]), ii) or when $a(n)$ is complex-valued and circular ([6]).

Furthermore, most of the above cited papers derived the finite-sample (or exact) CRB only ([1, 2, 3, 4]). The expressions for the exact CRB are often not interpretable and only their numerical evaluation can give some insights into their behavior with respect to the relevant signal parameters. In order to overcome this drawback, a few papers developed closed-form expressions for the asymptotic CRB (i.e., in large sample context) ([5, 6]).

To the best of our knowledge, the CRB in the case of complex-valued *non-circular* multiplicative noise has not been derived in the literature. The objective of the paper is to fill this gap. Both the exact CRB (section 2) and the asymptotic CRB (section 3) will be investigated. Non-circular multiplicative noise is of interest because it occurs, for example, in digital communications when the complex envelope of the received signal is the convolution of real-valued symbols with the propagation channel ([7]).

The rest of the paper is organized as follows. In section 4, the asymptotic CRB is compared with the asymptotic performance of the Square-Power (SP)-based estimators ([8, 4, 5, 7]). Finally, section 5 presents numerical simulations results which are found to agree with the theory.

2. EXACT CRAMER-RAO BOUND

Throughout the paper, the model given in Eq. (1) is considered under the following assumptions :

$-$ $a(n)$ is Gaussian complex-valued non-circular stationary process with zero-mean, correlation $r_a(\tau) = \mathbb{E}[a(n+\tau)\overline{a(n)}]$, and conjugate correlation $u_a(\tau) = \mathbb{E}[a(n+\tau)a(n)]$ where the overline stands for complex conjugate. The spectrum and conjugate spectrum are denoted respectively as follows

$$s_a(e^{2i\pi f}) = \sum_{\tau\in\mathbb{Z}} r_a(\tau)e^{-2i\pi f\tau}$$

and

$$c_a(e^{2i\pi f}) = \sum_{\tau\in\mathbb{Z}} u_a(\tau)e^{-2i\pi f\tau}.$$

By construction, one can remark that $c_a(e^{2i\pi f}) = c_a(e^{-2i\pi f})$.

$-$ The entire statistics $\{r_a(\tau), u_a(\tau)\}_{\tau\in\mathbb{Z}}$ of $a(n)$ only depend on a finite number K of real-valued unknown parameters denoted by $\{a_k\}_{k=1,\ldots,K}$.

$-$ The additive noise $b(n)$ is a Gaussian complex-valued and circular stationary process with zero-mean and unknown variance $\sigma^2 = \mathbb{E}[|b(n)|^2]$.

The purpose of this section is to derive the exact CRB, or equivalently the exact Fisher information matrix $\mathbf{F}$, for the deterministic parameter vector $\theta = [a_1, \ldots, a_K, \sigma^2, \phi_0, \phi_1]$ when N samples of $y(n)$ are available. Let $Y_N = [y(0), \ldots, y(N-1)]^{\mathrm{T}}$ where the superscript $^{\mathrm{T}}$ stands for transposition.

In order to use well-known results on the Fisher information matrix [9], we work with real-valued processes. We consider $\breve{Y}_N = [\Re[Y_N], \Im[Y_N]]^{\mathrm{T}}$ which is a multi-variate Gaussian variable with zero-mean and covariance matrix $\breve{\mathbf{R}}_{Y_N}$.

Due to frequency shift, $y(n)$ is stationary with respect to its correlation *but* cyclostationary with respect to its conjugate correlation [7]. Thus $\breve{\mathbf{R}}_{Y_N}$ is symmetric but not block-Toeplitz. However formula (5.2.1) in [9] holds true as long as the covariance matrix is symmetric. This leads to

$$\mathbf{F}_{k,l} = \frac{1}{2}\mathrm{Tr}\left(\frac{\partial\breve{\mathbf{R}}_{Y_N}}{\partial\theta_k}\breve{\mathbf{R}}_{Y_N}^{-1}\frac{\partial\breve{\mathbf{R}}_{Y_N}}{\partial\theta_l}\breve{\mathbf{R}}_{Y_N}^{-1}\right)$$

where $\mathbf{F}_{k,l}$ corresponds to the joint Fisher information for parameters (θ_k, θ_l) and where $\mathrm{Tr}(.)$ is the trace operator.

After straightforward algebraic manipulations, we show that

$$\mathbf{F}_{k,l} = \frac{1}{2}\mathrm{Tr}\left(\frac{\partial \widetilde{\mathbf{R}}_{Y_N}}{\partial \theta_k}\widetilde{\mathbf{R}}_{Y_N}^{-1}\frac{\partial \widetilde{\mathbf{R}}_{Y_N}}{\partial \theta_l}\widetilde{\mathbf{R}}_{Y_N}^{-1}\right)$$

where $\widetilde{\mathbf{R}}_{Y_N}$ is the covariance matrix of the random vector $\widetilde{Y}_N = [Y_N^{\mathrm{T}}, Y_N^{\mathrm{H}}]^{\mathrm{T}}$, and takes the following form

$$\widetilde{\mathbf{R}}_{Y_N} = \begin{bmatrix} \mathbf{R}_{Y_N} & \mathbf{U}_{Y_N} \\ \overline{\mathbf{U}}_{Y_N} & \overline{\mathbf{R}}_{Y_N} \end{bmatrix}$$

with $\mathbf{R}_{Y_N} = \mathbf{E}[Y_N Y_N^{\mathrm{H}}]$ and $\mathbf{U}_{Y_N} = \mathbf{E}[Y_N Y_N^{\mathrm{T}}]$. Superscript $^{\mathrm{H}}$ stands for the complex conjugate transposition.

Model (1) can also be written as follows

$$\mathbf{Y}_N = \mathbf{\Gamma}\mathbf{A}_N + \mathbf{B}_N$$

where $\mathbf{A}_N$ and $\mathbf{B}_N$ are defined in a similar way as $\mathbf{Y}_N$, and $\mathbf{\Gamma} = \mathrm{diag}(e^{2i\pi(\phi_0+\phi_1 n)}, n = 0, \ldots, N-1)$. Consequently, we have that

$$\widetilde{\mathbf{R}}_{Y_N} = \widetilde{\mathbf{\Gamma}}\widetilde{\mathbf{R}}_{X_N}\widetilde{\mathbf{\Gamma}}^{\mathrm{H}} \quad \text{and}$$

where $\widetilde{\mathbf{\Gamma}} = [\mathbf{\Gamma}, \mathbf{0}_{N,N}; \mathbf{0}_{N,N}, \overline{\mathbf{\Gamma}}]$, and $\widetilde{\mathbf{R}}_{X_N} = \widetilde{\mathbf{R}}_{A_N} + \sigma^2\mathbf{I}_{2N}$ with $\widetilde{\mathbf{R}}_{A_N}$ defined as $\widetilde{\mathbf{R}}_{Y_N}$. Notice that $\widetilde{\mathbf{R}}_{X_N}$ does not depend on the phase parameters. Therefore, we obtain the following expressions for Fisher information matrix

$$\begin{cases} \mathbf{F}_{a_k,a_l} = \frac{1}{2}\mathrm{Tr}\left(\frac{\partial \widetilde{\mathbf{R}}_{A_N}}{\partial a_k}\widetilde{\mathbf{R}}_{X_N}^{-1}\frac{\partial \widetilde{\mathbf{R}}_{A_N}}{\partial a_l}\widetilde{\mathbf{R}}_{X_N}^{-1}\right) \\ \mathbf{F}_{\sigma^2,\sigma^2} = \frac{1}{2}\mathrm{Tr}\left(\widetilde{\mathbf{R}}_{X_N}^{-2}\right) \\ \mathbf{F}_{a_k,\sigma^2} = \frac{1}{2}\mathrm{Tr}\left(\frac{\partial \widetilde{\mathbf{R}}_{A_N}}{\partial a_k}\widetilde{\mathbf{R}}_{X_N}^{-2}\right) \\ \mathbf{F}_{\phi_k,\phi_l} = 2\pi^2\mathrm{Tr}\left(\mathbf{D}_k\widetilde{\mathbf{R}}_{X_N}\mathbf{D}_l\widetilde{\mathbf{R}}_{X_N}^{-1} + \mathbf{D}_l\widetilde{\mathbf{R}}_{X_N}\mathbf{D}_k\widetilde{\mathbf{R}}_{X_N}^{-1}\right. \\ \qquad\qquad \left. -2\mathbf{D}_k\mathbf{D}_l\right) \\ \mathbf{F}_{a_k,\phi_k} = i\pi\mathrm{Tr}\left(\frac{\partial \widetilde{\mathbf{R}}_{A_N}}{\partial a_k}[\widetilde{\mathbf{R}}_{X_N}^{-1}\mathbf{D}_k - \mathbf{D}_k\widetilde{\mathbf{R}}_{X_N}^{-1}]\right) \\ \mathbf{F}_{\sigma^2,\phi_k} = 0 \end{cases}$$

where $\mathbf{D}_k = [\mathbf{d}_k, \mathbf{0}_{N,N}; \mathbf{0}_{N,N}, -\mathbf{d}_k]$ with $\mathbf{d}_0 = \mathbf{I}_N$ and $\mathbf{d}_1 = \mathrm{diag}([0, \cdots, N-1])$. The above expressions are similar to the ones introduced in [6], because we have not used the specific form of $\widetilde{\mathbf{R}}_{X_N}$ yet. Nevertheless we already observe differences: for instance, the phase is identifiable (i.e., $\mathbf{F}_{\phi_0,\phi_0} \neq 0$); whereas in the case of zero-mean circular multiplicative noise the Fisher information for the phase is zero [6].

3. ASYMPTOTIC CRAMÉR-RAO BOUND

We now focus on the asymptotic behavior of the Fisher information matrix $\mathbf{F}$ and the CRB when N becomes large.

Unlike [6], here we can not apply Whittle's formula [10] for obtaining simple asymptotic expressions for the Fisher information matrix because $y(n)$ is cyclostationary. In the sequel, our derivations rely on theorems dealing with the inversion of (large) Toeplitz matrices ([11, 12]).

Let $\mathbf{t}_N = (t_{l-k})_{-N<k,l<N}$ be a Toeplitz matrix. Without loss of generality, we assume that the sequence $\{t_k; k = 0, \pm 1, \cdots\}$ is absolutely summable. Then

$$s(e^{2i\pi f}) = \sum_{k\in\mathbb{Z}} t_k e^{-2i\pi fk} \Leftrightarrow t_k = \int_0^1 s(e^{2i\pi f})e^{2i\pi fk}df$$

Matrix $\mathbf{t}_N$ can thus be entirely captured by $f \mapsto s(e^{2i\pi f})$ which justifies the following notation :

$$\mathbf{t}_N = \mathcal{T}_N(s).$$

Let A_N and B_N be two $N \times N$ bounded matrices. $|A_N|$ stands for $(\frac{1}{N}\mathrm{Tr}(A_N A_N^{\mathrm{H}}))^{1/2}$. A_N and B_N are said asymptotically equivalent (denoted by $\sim$) iff $|A_N - B_N| \to 0$ as $N \to \infty$.

One can remark that $\mathbf{R}_{A_N}$ and $\mathbf{U}_{A_N}$ are Toeplitz matrices and can be written as follows

$$\mathbf{R}_{A_N} = \mathcal{T}_N(s_a) \quad \text{and} \quad \mathbf{U}_{A_N} = \mathcal{T}_N(c_a).$$

This implies that

$$\mathbf{R}_{X_N} = \mathcal{T}_N(s) \quad \text{and} \quad \mathbf{U}_{X_N} = \mathcal{T}_N(c) \tag{2}$$

with $s(e^{2i\pi f}) = s_a(e^{2i\pi f}) + \sigma^2$ and $c(e^{2i\pi f}) = c_a(e^{2i\pi f})$. Furthermore, we get

$$\overline{\mathbf{R}}_{X_N} = \mathcal{T}_N(\underline{s}) \quad \text{and} \quad \overline{\mathbf{U}}_{X_N} = \mathcal{T}_N(\underline{c})$$

with $\underline{s}(e^{2i\pi f}) = \overline{s(e^{-2i\pi f})}$ and $\underline{c}(e^{2i\pi f}) = \overline{c(e^{-2i\pi f})}$.

To obtain the asymptotic value of $\mathbf{F}$, we firstly need an asymptotic equivalent for $\widetilde{\mathbf{R}}_{X_N}^{-1}$. According to Schur's lemma, we get

$$\widetilde{\mathbf{R}}_{X_N}^{-1} = \left[\begin{array}{c|c} \mathbf{R}_{X_N}^{-1} + \mathbf{R}_{X_N}^{-1}\mathbf{U}_{X_N}\mathbf{\Delta}^{-1}\overline{\mathbf{U}}_{X_N}\mathbf{R}_{X_N}^{-1} & -\mathbf{R}_{X_N}^{-1}\mathbf{U}_{X_N}\mathbf{\Delta}^{-1} \\ \hline -\mathbf{\Delta}^{-1}\overline{\mathbf{U}}_{X_N}\mathbf{R}_{X_N}^{-1} & \mathbf{\Delta}^{-1} \end{array}\right]$$

with

$$\mathbf{\Delta} = \overline{\mathbf{R}}_{X_N} - \overline{\mathbf{U}}_{X_N}\mathbf{R}_{X_N}^{-1}\mathbf{U}_{X_N}.$$

Thanks to Eq. (2), we have

$$\mathbf{\Delta} = \mathcal{T}_N(\underline{s}) - \mathcal{T}_N(\underline{c})\mathcal{T}_N(s)^{-1}\mathcal{T}_N(c).$$

Since s is real-valued and does not admit zero over the interval [0,1), we get $\mathcal{T}_N(s)^{-1} \sim \mathcal{T}_N(s^{-1})$ for large N ([11, 12]). Then

$$\mathbf{\Delta} \sim \mathcal{T}_N(\underline{s}) - \mathcal{T}_N(\underline{c})\mathcal{T}_N(s^{-1})\mathcal{T}_N(c)$$

Since s^{-1} and c are bounded over $[0, 1)$, we have

$$\mathbf{\Delta} \sim \mathcal{T}_N([s\underline{s} - c\underline{c}]/s).$$

Let

$$\mathcal{X}(e^{2i\pi f}) = s(e^{2i\pi f})\underline{s}(e^{2i\pi f}) - c(e^{2i\pi f})\underline{c}(e^{2i\pi f}).$$

One can see that $\mathcal{X}$ is real-valued and positive. This leads to

$$\mathbf{\Delta}^{-1} \sim \mathcal{T}_N(s/\mathcal{X}).$$

After straightforward manipulations, we conclude that

$$\widetilde{\mathbf{R}}_{X_N}^{-1} \sim \begin{bmatrix} \mathcal{T}_N(\underline{s}/\mathcal{X}) & -\mathcal{T}_N(c/\mathcal{X}) \\ -\mathcal{T}_N(\underline{c}/\mathcal{X}) & \mathcal{T}_N(s/\mathcal{X}) \end{bmatrix}.$$

After simple but tedious calculations, the Fisher information matrix is found to be

$$\begin{cases} \lim_{N\to\infty} \frac{1}{N}\mathbf{F}_{a_k,a_l} & = & \frac{1}{2}\alpha_{k,l} \\ \lim_{N\to\infty} \frac{1}{N}\mathbf{F}_{a_k,\sigma^2} & = & \frac{1}{2}\beta_k \\ \lim_{N\to\infty} \frac{1}{N}\mathbf{F}_{a_k,\phi_0} & = & 4\pi\delta_k \\ \lim_{N\to\infty} \frac{1}{N^2}\mathbf{F}_{a_k,\phi_1} & = & 2\pi\delta_k \\ \lim_{N\to\infty} \frac{1}{N}\mathbf{F}_{\sigma^2,\sigma^2} & = & \frac{1}{2}\gamma \\ \lim_{N\to\infty} \frac{1}{N}\mathbf{F}_{\phi_0,\phi_0} & = & 16\pi^2\xi \\ \lim_{N\to\infty} \frac{1}{N^2}\mathbf{F}_{\phi_0,\phi_1} & = & 8\pi^2\xi \\ \lim_{N\to\infty} \frac{1}{N^3}\mathbf{F}_{\phi_1,\phi_1} & = & \frac{16\pi^2}{3}\xi \end{cases}$$

where

$$
\begin{aligned}
\alpha_{k,l} &= \int_0^1 \frac{\frac{\partial \mathcal{X}(e^{2i\pi f})}{\partial a_k}\frac{\partial \mathcal{X}(e^{2i\pi f})}{\partial a_l}}{\mathcal{X}(e^{2i\pi f})^2} + \frac{\mathcal{V}_{k,l}^{(c)}(e^{2i\pi f})-\mathcal{V}_{k,l}^{(s)}(e^{2i\pi f})}{\mathcal{X}(e^{2i\pi f})} df \\
\beta_k &= \int_0^1 \frac{1}{\mathcal{X}(e^{2i\pi f})}\frac{\partial \mathcal{X}(e^{2i\pi f})}{\partial a_k} df \\
\gamma &= \int_0^1 \frac{s(e^{2i\pi f})^2+\underline{s}(e^{2i\pi f})^2+2\underline{c}(e^{2i\pi f})c(e^{2i\pi f})}{\mathcal{X}(e^{2i\pi f})^2} df \\
\delta_k &= \Im[\int_0^1 \frac{\partial c(e^{2i\pi f})}{\partial a_k}\frac{\underline{c}(e^{2i\pi f})}{\mathcal{X}(e^{2i\pi f})} df] \\
\xi &= \int_0^1 \frac{c(e^{2i\pi f})\underline{c}(e^{2i\pi f})}{\mathcal{X}(e^{2i\pi f})} df
\end{aligned}
$$

with the following mapping

$$\mathcal{V}_{k,l}^{(\nu)}(e^{2i\pi f}) = \frac{\partial \nu(e^{2i\pi f})}{\partial a_k}\frac{\partial \underline{\nu}(e^{2i\pi f})}{\partial a_l} + \frac{\partial \underline{\nu}(e^{2i\pi f})}{\partial a_k}\frac{\partial \nu(e^{2i\pi f})}{\partial a_l}$$

Next we study different scenarios. Firstly, we consider the case where the receiver knows $\mathbf{a} = [a_1, \cdots, a_K]$ and σ^2, i.e., the statistics of multiplicative and additive noises. In this case, the CRBs are given by

$$\mathrm{CRB}(\phi_0)_{|(\mathbf{a},\,\sigma^2)\text{ known}} \sim \frac{1}{4\pi^2\xi N}$$

and

$$\mathrm{CRB}(\phi_1)_{|(\mathbf{a},\,\sigma^2)\text{ known}} \sim \frac{3}{4\pi^2\xi N^3}.$$

Secondly, in the case when $\mathbf{a} = [a_1, \cdots, a_K]$ and σ^2 are unknown at the receiver, we obtain

$$\mathrm{CRB}(\phi_0)_{|(\mathbf{a},\,\sigma^2)\text{ unknown}} = \mathrm{CRB}(\phi_0)_{|(\mathbf{a},\,\sigma^2)\text{ known}} + \frac{\mu}{16\pi^2\xi^2 N}$$

where μ is a bounded scalar taking the following form

$$\mu = \boldsymbol{\delta}^{\mathrm{T}}\left(\boldsymbol{\alpha}/2 - \boldsymbol{\delta}\boldsymbol{\delta}^{\mathrm{T}}/\xi - \boldsymbol{\beta}\boldsymbol{\beta}^{\mathrm{T}}/(2\gamma)\right)^{-1}\boldsymbol{\delta},$$

where $\boldsymbol{\alpha} = (\alpha_{k,l})_{1\le k,l\le K}$, $\boldsymbol{\beta} = (\beta_k)_{1\le k\le K}$, $\boldsymbol{\delta} = (\delta_k)_{1\le k\le K}$. Lastly

$$\mathrm{CRB}(\phi_1)_{|(\mathbf{a},\,\sigma^2)\text{ unknown}} = \mathrm{CRB}(\phi_1)_{|(\mathbf{a},\,\sigma^2)\text{ known}}.$$

Using the previous expressions for the asymptotic CRB, we make the following comments :

− The convergence rates for the phase and frequency estimations are $1/N$ and $1/N^3$ respectively regardless of the color of the multiplicative noise. Such rates correspond to the ones obtained in the case of real-valued multiplicative noise [5]. Recall that for circular complex-valued processes, the phase is not identifiable and the frequency is identifiable only if the multiplicative noise is colored, with a convergence rate of $1/N$. Thus, the non-circular complex-valued case is closer (in terms of estimation performance) to the real-valued case than to the circular complex-valued case. Consequently, in terms of performance, the main cut-off is not complex/real but circular/non-circular [1].

− Surprisingly, the same frequency estimation performance is obtained whether the statistics of $a(n)$ are known or not.

− The frequency estimation performance depends only on ξ, which refers to an information rate provided by the non-circularity. Indeed, the performance improves when ξ increases.

− In the noiseless case, we observe a floor effect (i.e., $\mathrm{CRB} \neq 0$ when $\sigma^2 = 0$). This effect vanishes iff $s_a(e^{2i\pi f})\overline{s_a(e^{-2i\pi f})} = c_a(e^{2i\pi f})\overline{c_a(e^{-2i\pi f})}$. This condition is fulfilled for example when the multiplicative noise is real-valued.

[1] Notice that a real-valued process can be viewed as a specific case of a non-circular complex-valued process where the imaginary part is zero.

4. LINK WITH SQUARE-POWER BASED ESTIMATOR

If the multiplicative noise is non-circular, the following estimate, based on the Squaring loop [8], can be carried out

$$\hat{\phi}_1^{(N)} = \arg\max_{\phi\in[0,1/2)} \sum_{l=-L}^{L}\left|\frac{1}{N}\sum_{n=0}^{N-1} y(n)y(n+l)e^{-4i\pi\phi n}\right|^2.$$

In the real-valued case, this estimate with $L = 0$ is well-known ([4, 5] and references therein). Surprisingly, the choice $L = 0$ was always made even when the multiplicative process was colored. In [4], the CRB and estimation performance are derived for high SNR. In [5], the asymptotic CRB is compared with the asymptotic estimation performance for arbitrary SNRs. It was proven that the estimate is asymptotically efficient for high SNR.

In [7], the above estimate with any value of L was introduced and analyzed in the context of non-circular and real-valued multiplicative noise. The asymptotic covariance of the estimates was derived for any L and at any SNR. In [7], we show that the asymptotic covariance is minimum when $L = M$, where M is memory length of $a(n)$ which is assumed to be finite. According to Theorem 4 in [7], the asymptotic variance of the above frequency estimate with $L = M$ takes the following form

$$\gamma_f \sim \frac{3\eta}{4\pi^2 N^3}$$

with

$$\eta = \frac{\int_0^1 |c(e^{2i\pi f})|^2\mathcal{X}(e^{2i\pi f})df}{\left(\int_0^1 |c(e^{2i\pi f})|^2 df\right)^2}.$$

By using $c(e^{2i\pi f}) = c(e^{-2i\pi f})$ and Schwartz's inequality, we have $\eta \geq 1/\xi$. Equality holds only if mapping $f \mapsto \mathcal{X}(e^{2i\pi f})$ is constant. This implies that Square-Power estimate (assuming $L = M$) is at least asymptotically efficient for any SNR if the multiplicative noise is white.

5. NUMERICAL ILLUSTRATIONS

For sake of simplicity, the multiplicative noise is assumed to be an AR(1) process, i.e. $a(n) = s(n) + as(n-1)$ where $\{s(n)\}_{n\in\mathbb{Z}}$ is a white non-circular Gaussian process with $\rho = \mathbb{E}[s(n)^2]$. In each figure, we display four curves: dashed line corresponds to the empirical mean square error (MSE) for the Square-Power estimate (carried out with $L = 1$). Disk point represents the theoretical MSE of the estimate computed via γ_f. The CRB and asymptotic CRB are depicted using a solid line and circles respectively. We have that $\mathrm{SNR} = 10\log_{10}((1+a^2)/\sigma^2)$. Unless otherwise stated, we set $\mathrm{SNR} = 10\mathrm{dB}$, $N = 100$, $a = 0.75$, and $\rho = 0.75$.

Figure 1 displays the performance measures versus SNR. We observe that the CRB and asymptotic CRB are very close. Even if the multiplicative noise is colored; the performance of the Square-Power estimate almost reaches the CRB. The well-known outliers effect obviously occurs at low and medium SNR [13].

Figure 2 display the different results versus N. The outliers effect vanishes as soon as N is chosen large enough.

In Figure 3 depicts the results versus a. The performance depends slightly on a. However the more $a(n)$ is colored, the greater the gap between CRB and Square-power's estimate performance.

In Figure 4, the results are displayed versus ρ. One can notice that the more $a(n)$ is non-circular (i.e., larger values of ρ), the better the estimation performance. Furthermore, the outliers effect significantly degrades the performance if $a(n)$ is not non-circular enough.

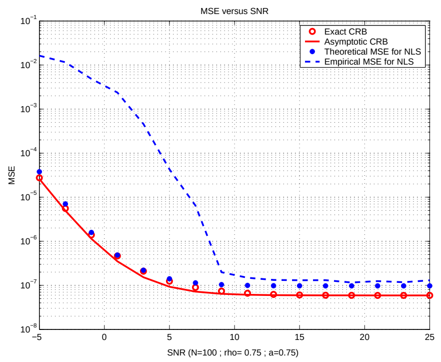

Fig. 1. MSE versus SNR

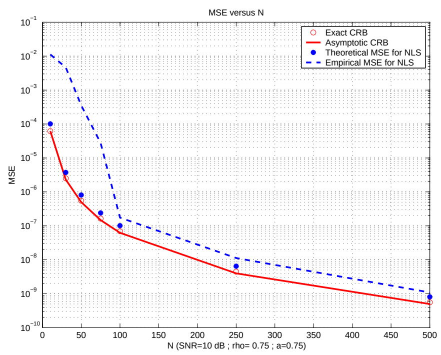

Fig. 2. MSE versus N

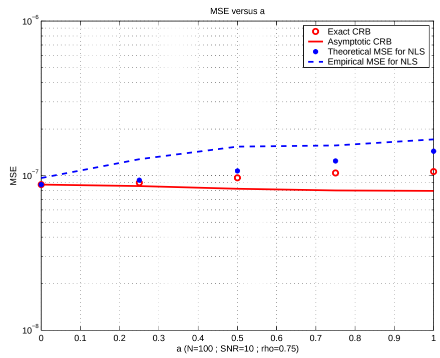

Fig. 3. MSE versus a

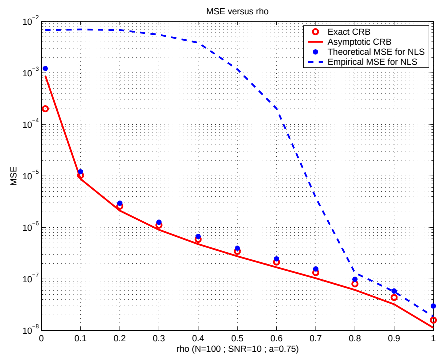

Fig. 4. MSE versus ρ

6. CONCLUSION

In this paper, a simple closed-form expression for the asymptotic CRB is derived. Simulations results show that outliers effect notably affects the performance. Therefore further investigations should concentrate on the analysis of such phenomenon (perhaps, by means of the Barankin bound).

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