HARMONIC RETRIEVAL IN NON-CIRCULAR COMPLEX-VALUED MULTIPLICATIVE NOISE : BARANKIN BOUND

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ABSTRACT

We focus on harmonic retrieval in multiplicative and additive noise. At low SNR, Maximum-Likelihood based estimate does not reach the Cramer-Rao bound. Actually, at low SNR, the Cramer-Rao bound is not a tight bound anymore and has to be replaced with the so-called Barankin bound which is tighter but more complicate. In this paper, we derive the Barankin bound when the multiplicative noise is complex-valued and non-circular. We observe that the Barankin bound is much more greater than the Cramer-Rao bound, especially when the multiplicative noise is not non-circular enough.

1. INTRODUCTION

Estimating harmonic corrupted by multiplicative and additive noise has received increasing interest during the last decade ([1, 2], [3, 4, 5]). Indeed such concern may occur in non-data-aided carrier frequency offset synchronization or in direction of arrival (DOA) estimation. Nevertheless the most of attention deals with Cramer-Rao Bound (CRB) derivations ([1, 2, 5] and references therein). However, as SNR is low and/or when the number of available samples is small, the Cramer-Rao bound does not predict well the performance bound. Indeed one can observe the so-called outliers effect, that is to say that, the mean-square error for any unbiased estimate is strongly larger than the Cramer-Rao bound when the SNR or the number of available samples is below a certain threshold. Actually the tightest lower bound is provided by the so-called Barankin bound. Therefore this previous bound has been already introduced for analyzing the outliers effect ([6, 7, 8], [9, 10, 11], [12, 13]). For harmonic retrieval in additive noise, several works focus on Barankin bound ([9, 12, 13] and references therein). Conversely, derivations for Barankin bound for harmonic embedded in multiplicative noise is seldom.

In the literature, only two main works addressed Barankin bound in presence of multiplicative noise ([10, 11]). These papers focus on DOA estimation which boils down to estimate exponential frequency disturbed by multiplicative noise described by a complex-valued and circular process. Furthermore, even if their main derivations hold for any complex-valued and circular multiplicative noise, their comments and their numerical illustrations are strongly connected to DOA estimation field.

Therefore this paper focuses on the derivations of Barankin Bound (BB) when the multiplicative noise is assumed to be complex-valued and *non-circular*. We will see further that our derivations correspond to an extension of those introduced in [10, 11] and can be done by using similar tools. Notice that such multiplicative noise can be encountered in digital communications context when the complex envelope of the receive signal corresponds to the filtering of real-valued symbols with propagation channel ([14]) or when offset modulation are employed ([15]).

This paper is organized as follows : in section 2, we provide closed-form expressions for Barankin Bound. In section 3, we recall Square-Power based estimate which is close to Maximum-Likelihood based estimate and also exhibits SNR threshold ([3, 4]). Finally section 4 is devoted to the analyze of the influence of various design parameters by means of numerical computations of the Barankin bound. We especially remark that more the multiplicative noise is non-circular, less the outliers effect degrades the optimal performance provided by the Barankin bound.

2. BARANKIN BOUND

We consider to receive the following discrete-time process $y(n)$

$$y(n) = a(n)e^{2i\pi(\phi_0+\phi_1 n)} + b(n) \tag{1}$$

where the phase ϕ_0 and the frequency ϕ_1 are the parameters of interest. Multiplicative noise $a(n)$ is complex-valued and non-circular stationary process. It is assumed to be Gaussian with zero-mean, correlation components $r_a(\tau) = \mathbb{E}[a(n+\tau)\overline{a(n)}]$, and conjugate correlation components $u_a(\tau) = \mathbb{E}[a(n+\tau)a(n)]$ where the overline stands for complex conjugate. Additive noise $b(n)$ is Gaussian complex-valued circular stationary process with zero-mean and variance $\sigma^2 = \mathbb{E}[|b(n)|^2]$.

For sake of simplicity, we assume that noise statistics, i.e., $\{r_a(\tau), u_a(\tau)\}_{\tau\in\mathbb{Z}}$ and σ^2, are known at the receiver. This assumption is usually done in [10] or partially done [11] for deriving Barankin bound because the computational and analytical complexities are too high if not. Moreover, notice that the Cramer-Rao bound for the frequency estimator is insensitive to the knowledge of noise statistics as soon as the number of samples is large enough [16]. We thus can expect that the error induced by neglecting the estimation step of noise statistics may be sufficiently small to guess that our further conclusions (especially about SNR threshold) still hold in case of unknown noise statistics.

Our purpose now is to derive Barankin Bound for unknown deterministic vector $\phi = [\phi_0, \phi_1]^{\mathrm{T}}$. We assume that N samples of $y(n)$ are available and are stacked into the following vector $\mathbf{y}_N = [y(0), \ldots, y(N-1)]^{\mathrm{T}}$ where the superscript $^{\mathrm{T}}$ stands for transposition.

Before going further, we define the following set of the so-called "test-points"$\{\psi(k) = [\psi_0(k), \psi_1(k)]^{\mathrm{T}}\}_{1\leq k\leq n}$. We are now able to define the Barankin bound of order n as follows :

$$\mathrm{BB}_n(\phi_0, \phi_1) = \sup_{\mathscr{E}} S_n(\mathscr{E})$$

where

$$S_n(\mathscr{E}) = \mathscr{E}(\mathbf{B}(\mathscr{E}) - \mathbf{1}_n\mathbf{1}_n^{\mathrm{T}})^{-1}\mathscr{E}^{\mathrm{T}}$$

with $\mathscr{E} = [\psi(1)-\phi, \ldots, \psi(n)-\phi]$, and $\mathbf{1}_n = \mathrm{ones}(n,1)$. Furthermore $\mathbf{B} = (B_{k,l})_{1\leq k,l\leq n}$ is the following $n\times n$ matrix

$$B_{k,l} = \mathbb{E}[L(\mathbf{y}_N, \phi, \psi(k))L(\mathbf{y}_N, \phi, \psi(l))]$$

with

$$L(\mathbf{y}_N, \phi, \psi(k)) = \frac{p(\mathbf{y}_N|\psi(k))}{p(\mathbf{y}_N|\phi)}$$

and $p(\mathbf{y}_N|\theta)$ means the likelihood of phase parameters θ.

The mean square error of any unbiased estimator is greater than Barankin bound of any order ([17]). In an asymptotic point of view (as $n \to \infty$), the Barankin bound is even the tightest lower bound that one can found ([6, 11]). As for the choice of the test-points, it is usual to consider the following ones ([12, 11])

$$\mathscr{E} = \begin{bmatrix} \psi_0 - \phi_0 & 0 \\ 0 & \psi_1 - \phi_1 \end{bmatrix} = \mathrm{diag}(\varepsilon_0, \varepsilon_1). \tag{2}$$

Our main concern hereafter is to derive in closed-form expression the matrix $\mathbf{B}$ for such previous test-points.

Since $\mathbf{y}_N$ is complex-valued and non-circular, we need to introduce the following process $\widetilde{\mathbf{y}}_N = [\mathbf{y}_N^{\mathrm{T}}, \mathbf{y}_N^{\mathrm{H}}]^{\mathrm{T}}$ in order to encompass all the second-order statistics of $\mathbf{y}_N$. The superscript $^{\mathrm{H}}$ stands for complex conjugate transposition.

Let us now introduce some notations. The covariance matrix $\widetilde{\mathbf{R}}_\phi$ of multivariate process $\widetilde{\mathbf{y}}_N$ can be written as follows

$$\widetilde{\mathbf{R}}_\phi = \widetilde{\Gamma}_\phi \left(\widetilde{\mathbf{R}}_a + \sigma^2 \mathbf{Id}_{2N} \right) \widetilde{\Gamma}_\phi^{\mathrm{H}} \tag{3}$$

where $\widetilde{\mathbf{R}}_a$ is the covariance matrix of $\widetilde{\mathbf{a}}_N = [\mathbf{a}_N^{\mathrm{T}}, \mathbf{a}_N^{\mathrm{H}}]^{\mathrm{T}}$ with $\mathbf{a}_N = [a(0), \ldots, a(N-1)]^{\mathrm{T}}$ and where $\mathbf{Id}_{2N}$ is the $2N \times 2N$ identity matrix. Moreover we get

$$\widetilde{\Gamma}_\phi = \begin{bmatrix} \Gamma_\phi & \mathbf{0}_{N,N} \\ \mathbf{0}_{N,N} & \overline{\Gamma}_\phi \end{bmatrix}$$

with

$$\Gamma_\phi = \mathrm{diag}(e^{2i\pi(\phi_0 + \phi_1 n)}, n = 0, \ldots, N-1).$$

The probability density of $\mathbf{y}_N$ writes as follows

$$p(\mathbf{y}_N|\psi) = \frac{1}{\pi^N \left(\det(\widetilde{\mathbf{R}}_\psi) \right)^{1/2}} \exp\left\{ -\frac{1}{2} \widetilde{\mathbf{y}}_N^{\mathrm{H}} \widetilde{\mathbf{R}}_\psi^{-1} \widetilde{\mathbf{y}}_N \right\}$$

According to Eq. (3), one can see that $\det(\widetilde{\mathbf{R}}_\psi)$ is independent of ψ. This implies that

$$L(\mathbf{y}_N, \phi, \psi(k)) = \exp\left\{ -\frac{1}{2} \widetilde{\mathbf{y}}_N^{\mathrm{H}} \left(\widetilde{\mathbf{R}}_{\psi(k)}^{-1} - \widetilde{\mathbf{R}}_\phi^{-1} \right) \widetilde{\mathbf{y}}_N \right\}$$

Finally we wish to derive the following term

$$B_{k,l} = \mathbb{E}\left[\exp\left\{ -\frac{1}{2} \widetilde{\mathbf{y}}_N^{\mathrm{H}} \mathbf{W}_{k,l} \widetilde{\mathbf{y}}_N \right\} \right]$$

with

$$\mathbf{W}_{k,l} = \widetilde{\mathbf{R}}_{\psi(k)}^{-1} + \widetilde{\mathbf{R}}_{\psi(l)}^{-1} - 2\widetilde{\mathbf{R}}_\phi^{-1}.$$

In order to calculate properly the previous term, we first rewrite it in terms of $\check{\mathbf{y}}_N = [\Re[\mathbf{y}_N]^{\mathrm{T}}, \Im[\mathbf{y}_N]^{\mathrm{T}}]^{\mathrm{T}}$. We obtain that

$$B_{k,l} = \mathbb{E}\left[\exp\left\{ -\frac{1}{2} \check{\mathbf{y}}_N^{\mathrm{T}} \check{\mathbf{W}}_{k,l} \check{\mathbf{y}}_N \right\} \right]$$

where $\check{\mathbf{W}}_{k,l} = \mathbf{P}^{\mathrm{H}} \mathbf{W}_{k,l} \mathbf{P}$ with $\mathbf{P} = [\mathbf{Id}_N, i\mathbf{Id}_N; \mathbf{Id}_N, -i\mathbf{Id}_N]$. Let $\check{\mathbf{R}}_\phi = \mathbf{E}[\check{\mathbf{y}}_N \check{\mathbf{y}}_N^{\mathrm{T}}]$ be the covariance matrix of the real-valued process $\check{\mathbf{y}}_N$. Since $\check{\mathbf{R}}_\phi$ is symmetric, it can be diagonalized as follows $\check{\mathbf{R}}_\phi = \mathbf{D}_x^{\mathrm{T}} \Lambda_x \mathbf{D}_x$ where $\mathbf{D}_x$ is the orthogonal matrix composed by the eigenvectors and where Λ_x is the diagonal matrix composed by the eigenvalues. Let $\mathbf{x} = \Lambda_x^{-1/2} \mathbf{D}_x \check{\mathbf{y}}_N$. By construction, vector $\mathbf{x}$ is still Gaussian with covariance matrix $\mathbf{Id}_{2N}$. Thus each component of $\mathbf{x}$ is independent of another one. Then

$$B_{k,l} = \mathbb{E}\left[\exp\left\{ -\frac{1}{2} \mathbf{x}^{\mathrm{T}} \mathbf{V}_{k,l} \mathbf{x} \right\} \right]$$

with $\mathbf{V}_{k,l} = \Lambda_x^{1/2} \mathbf{D}_x \check{\mathbf{W}}_{k,l} \mathbf{D}_x^{\mathrm{T}} \Lambda_x^{1/2}$. Once again, as matrix $\mathbf{V}_{k,l}$ is symmetric, it can be decomposed as follows $\mathbf{V}_{k,l} = \mathbf{D}^{\mathrm{T}} \Lambda \mathbf{D}$ where $\mathbf{D}$ is the orthogonal matrix composed by the eigenvectors and where $\Lambda = \mathrm{diag}([\lambda_0, \cdots, \lambda_{2N-1}])$ is the diagonal matrix composed by the eigenvalues $\{\lambda_m\}_{0 \le m \le 2N-1}$. Let $\mathbf{z} = [z_0, \cdots, z_{2N-1}]^{\mathrm{T}} = \mathbf{D}\mathbf{x}$. Vector $\mathbf{z}$ is still Gaussian with covariance matrix equal to the identity matrix, i.e., with independent components. Therefore we get

$$\begin{aligned} B_{k,l} = & \ \mathbb{E}\left[\exp\left\{ -\frac{1}{2} \mathbf{z}^{\mathrm{T}} \Lambda \mathbf{z} \right\} \right] \\ = & \ \mathbb{E}\left[\exp\left\{ -\frac{1}{2} \sum_{m=0}^{2N-1} \lambda_m z_m^2 \right\} \right] \\ = & \ \prod_{m=0}^{2N-1} \mathbb{E}\left[\exp\left\{ -\frac{1}{2} \lambda_m z_m^2 \right\} \right] \end{aligned}$$

One can easily check that z_m^2 follows a Chi-square distribution with one degree of freedom. This leads to [17, 18]

$$B_{k,l} = \begin{cases} \prod_{m=0}^{2N-1} \frac{1}{\sqrt{1+\lambda_m}} & \text{if} \quad (1+\lambda_m) > 0, \forall m \\ +\infty & \text{otherwise} \end{cases}.$$

Last expression can be compacted as follows

$$B_{k,l} = \begin{cases} \frac{1}{\sqrt{\det(\mathbf{Id}_{2N} + \mathbf{V}_{k,l})}} & \text{if} \quad \mathbf{Id}_{2N} + \mathbf{V}_{k,l} > 0 \\ +\infty & \text{otherwise} \end{cases}.$$

After straightforward algebraic manipulations, we finally obtain that

$$B_{k,l} = \begin{cases} \frac{1}{\sqrt{\det(\mathbf{Q}_{k,l})}} & \text{if } \mathbf{Q}_{k,l} > 0 \\ +\infty & \text{otherwise} \end{cases},$$

with

$$\begin{aligned} \mathbf{Q}_{k,l} &= \mathbf{Id}_{2N} + \mathbf{W}_{k,l} \widetilde{\mathbf{R}}_\phi \\ &= (\widetilde{\mathbf{R}}_{\psi(k)}^{-1} + \widetilde{\mathbf{R}}_{\psi(l)}^{-1}) \widetilde{\mathbf{R}}_\phi - \mathbf{Id}_{2N} \end{aligned}$$

Obviously, we get

$$\widetilde{\mathbf{R}}_\phi = \begin{bmatrix} \mathbf{R}_\phi & \mathbf{U}_\phi \\ \overline{\mathbf{U}}_\phi & \overline{\mathbf{R}}_\phi \end{bmatrix}$$

with $\mathbf{R}_\phi = \mathbf{E}[\mathbf{y}_N \mathbf{y}_N^{\mathrm{H}}]$ and $\mathbf{U}_\phi = \mathbf{E}[\mathbf{y}_N \mathbf{y}_N^{\mathrm{T}}]$. Matrix $\mathbf{U}_\phi$ refers to conjugate correlation of the received signal and is non-null because of the non-circularity of the signal. In [10] and [11], the expression for $B_{k,l}$ is slightly different : the square root is removed and $\mathbf{Q}_{k,l}$ depends only on $\mathbf{R}_\phi$ instead of $\widetilde{\mathbf{R}}_\phi$. Actually, our expression is an extension of that one obtained in [10] and [11]. Indeed, by setting $\mathbf{U}_\phi = 0$, $\widetilde{\mathbf{R}}_\phi$ is block-diagonal and then our expression is equal to that one introduced in [10] and [11]. Notice that the difference between our expression and that one presented in aforementioned papers are exactly similar to the difference existing between Chi-square characteristic function with one degree of freedom (real-valued case) and with two degrees of freedom (circular complex-valued case).

In the sequel, we focus rather on the frequency parameter because we have observed numerically that the outliers effect is "stronger" for such a parameter, i.e., the difference (at low SNR) between Cramer-Rao bound and Barankin bound is larger for the frequency estimation issue. For standard test-points described in Eq. (2), the Barankin bound for ϕ_1 takes the following form [11]

$$\mathrm{BB}(\phi_1) = \sup_{\varepsilon_0,\varepsilon_1} \frac{\varepsilon_1^2}{(\mathbf{B}_{1,1}-1)-(\mathbf{B}_{0,1}-1)(\mathbf{B}_{0,0}-1)^{-1}(\mathbf{B}_{0,1}-1)}.$$

The term $(\mathbf{B}_{0,1}-1)(\mathbf{B}_{0,0}-1)^{-1}(\mathbf{B}_{0,1}-1)$ represents the loss in performance due to joint phase parameter estimation.

The next step would be to derive closed form expression for SNR threshold, i.e, the SNR beyond which the Barankin bound and the Cramer-Rao bound are equal. In order to yield closed-form expression (by following approach given in [12]), parameter has to be considered unknown one by one. Obtaining such an expression requires further work and so is beyond scope of this communication. Nevertheless the corresponding complicated derivations should be drawn in the journal version paper.

3. REVIEW ON SQUARE-POWER BASED ESTIMATOR

In the section, we briefly review the well-known Square-Power estimate for frequency shift which is strongly related to the so-called Non-Linear Least Square (NLLS) estimate ([19, 3, 4, 5]). We herein introduce this estimate in order to compare it with the Barankin bound in the section devoted to numerical simulations. The comparison is not sound because this estimator is not unbiased but asymptotically unbiased whereas the Barankin bound makes sense for unbiased estimate only. However such a comparison is of interest as mentioned in [7] and [13].

The Square-power estimate can be defined as follows

$$\hat{\phi}_1 = \arg \max_{\phi\in[0,1/2)} \sum_{l=-L}^{L} \left| \frac{1}{N} \sum_{n=0}^{N-1} y(n)y(n+l)e^{-4i\pi\phi n} \right|^2. \qquad (4)$$

This estimate has been deeply analyzed in many papers when $L = 0$ and when the multiplicative noise was real-valued ([19, 3, 4, 5, 20] and references therein). The extension and the analysis for any L and for non-circular complex-valued multiplicative noise have been performed in [16]. The comparison between Square-Power estimate and the Cramer-Rao bound has led to a lot of papers ([19, 3, 4, 5, 16]). It is especially shown that Square-Power estimate is asymptotically efficient either for any SNR if the multiplicative noise is white or for high SNR whatever the color of noise. Moreover Square-Power based estimate is numerically close to Cramer-Rao Bound anyway except when outliers effect obviously occurs.

The cost function to be maximized in Eq. (4) admits numerous local maxima. In practice, in order to minimize the estimation error, L is fixed to the memory of the process $y(n)$ ([14]). However it has generally a particular shape which can be exploited. Indeed, the cost function usually can be depicted as a flat ground-level noise plus a peak around the true value ϕ_1. Therefore one can proceed into two steps to compute the maximization in Eq. (4) :

- the first step, also called *coarse* step, detects the main peak by means of FFT. If the estimate works well, this main peak is around the true value ϕ.
- the second step, also called *fine* step, refines the estimation around the detected peak by means of Gradient-descent algorithm initialized by the coarse estimate provided by the first step.

Generally the outliers effect corresponds to the failure of the coarse step [7].

4. NUMERICAL ILLUSTRATIONS

For sake of simplicity, the multiplicative noise is assumed to be AR(1) as follows : $a(n) = s(n) + as(n-1)$ where $\{s(n)\}_{n\in\mathbb{Z}}$ is white non-circular Gaussian process with $\rho = \mathbb{E}[s(n)^2]$. Notice that the colorness rate given by $|r_a(1)|/|r_a(0)|$ is maximum iff $a = 1$ and that ρ refers to a noncircularness rate. In each figure, we display four curves : dashed line corresponds to the empirical mean square error (MSE) for the Square-Power estimate. Dash line with disk point represents the theoretical MSE for the estimate ([14, 16]). Solid line without point and with circle point plot the Cramer-Rao Bound (see [16]) and Barankin Bound respectively. We get $\mathrm{SNR} = 10\log_{10}((1+a^2)/\sigma^2)$. Finally $L = 1$ since we treat a AR(1) process.

In Figure 1, we plot all the curves versus SNR with $N = 100$, $\rho = 0.75$ and $a = 0.75$. We notice that there is a large gap between

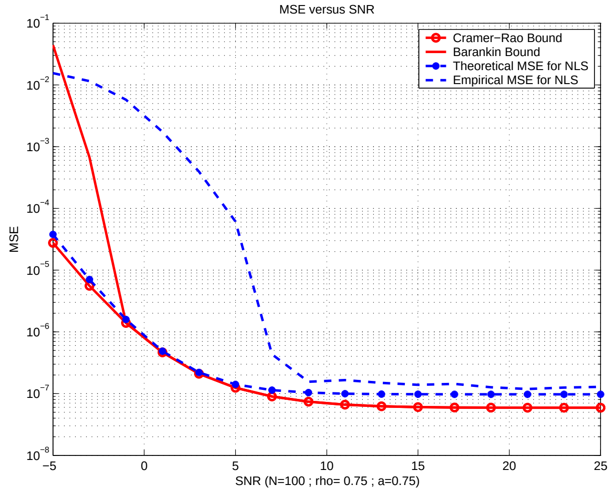

Figure 1: MSE versus SNR

the Barankin bound and the real performance of the Square-Power estimate.

In Figure 2, we plot all the curves versus a with SNR = 5dB, $N = 32$, $\rho = 0.25$. The outliers effect is slightly more important when the signal becomes less colored ($a \neq 1$). Besides, we have also remarked that the influence of the color vanishes as soon as N is sufficiently large ($N \geq 128$ when $\rho = 0.25$).

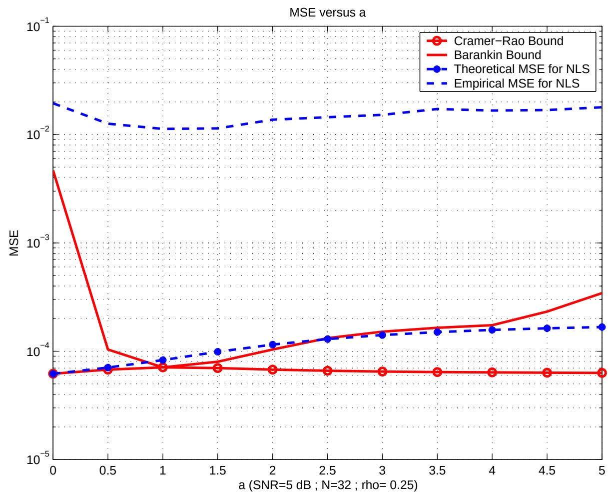

Figure 2: MSE versus a

In Figure 3, we plot all the curves versus ρ with SNR = 10dB, $N = 64$, $a = 0.75$. The signal being highly colored ($a = 0.75$), the Barankin bound is not affected by the outliers effect. In contrast, as the Square-Power estimate fails once again because the colorness information is not cleverly used.

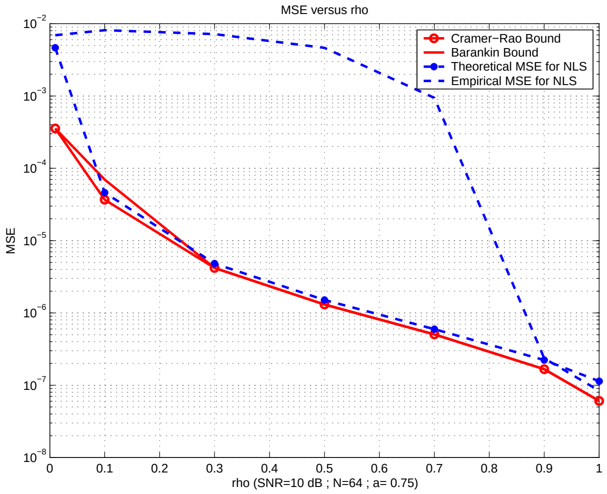

Figure 3: MSE versus ρ (high a)

In Figure 4, we plot all the curves versus ρ with SNR $=$ 10dB, $N = 64$, $a = 0.10$. The outliers effect is now slightly visible on the Barankin bound because the considered signal is not colored enough. Finally we also observe that more $a(n)$ is non-circular (i.e., ρ large), better is the performance.

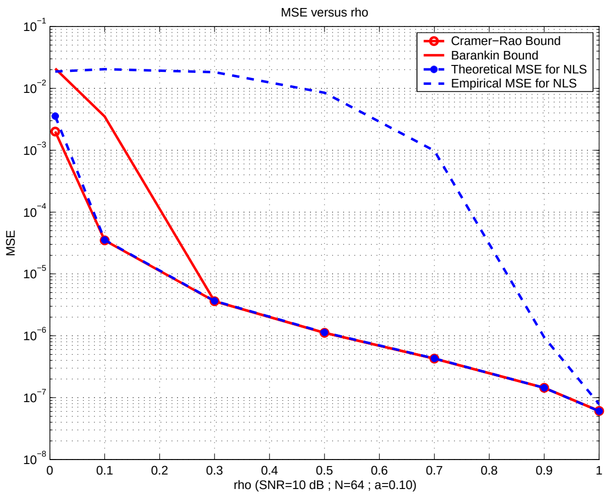

Figure 4: MSE versus ρ (low a)

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