Distributed estimation of the maximum value over a Wireless Sensor Network

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***Abstract*—This paper analyzes two algorithms for the maximum value estimation over a Wireless Sensor Network: the RANDOM GOSSIP relying on pairwise exchanges between the nodes, and the BROADCAST in which each sensor sends its value to all its neighbors. We prove the convergence of these algorithms, and we provide tight bounds for their convergence speed.**

I. INTRODUCTION

DISTRIBUTED algorithms over Wireless Sensors Networks (WSN) have been widely studied since the seminal work of Tsitsiklis [1]; in particular, a lot of results have been shown for the problem of averaging [2], [3], [4]. However various applications such as stock management or distributed computing could need the maximum value of the network. To share the maximum value over the entire network, the standard Random (Pairwise) Gossip approach can be used, but it does not take benefit of the broadcasting facilities of the wireless communications. Therefore, the Broadcast Gossip could be an interesting alternative. The purpose of this paper is to give convergence proofs and convergence speed bounds for these two approaches regarding the maximum value estimation.

II. PROPOSED ALGORITHMS

A. Model

We consider a network of N sensors modeled by an unweighted undirected graph $\mathcal{G} = (V, E)$ where V is the set of vertices/sensors ($|V| = N$) and E is the set of edges/links between the sensors. Therefore, each sensor i can exchange data with its neighborhood $\mathcal{N}_i = \{j \in V | (i,j) \in E\}$ and $d_{max} = \max_{i \in V} \ |\mathcal{N}_i|$. For practical reasons, we suppose that $\mathcal{G}$ is connected and that each sensor is equipped with an independent Poisson clock of parameter λ for its activation which is equivalent to a global clock of parameter $N\lambda$ and uniform selection of the awaking sensor. We will note t the instant of the t-th ring of the global clock.

Each sensor i has an initial value $x_i(0)$; we introduce $x_{max} = \max_{i \in V} \ x_i(0)$ and $\mathbf{x}(t) = [x_1(t) \dots x_N(t)]^T$ where $x_i(t)$ is the value of the i-th sensor at time t. It is therefore obvious that we wish $\mathbf{x}(t)$ to converge to $x_{max}\mathbf{1}$ (with $\mathbf{1}$ being the size N vector of ones) as t goes to infinity. Hence, the goal of the algorithms will be to achieve **max-consensus** in **finite time** τ, *i.e.*:

$$\forall \mathbf{x}(0) \in \mathbb{R}^N, \exists \tau_{max} | \ \forall t > \tau_{max}, \mathbf{x}(t) = x_{max}\mathbf{1} \quad (1)$$

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B. RANDOM GOSSIP *and* BROADCAST *algorithms*

- The RANDOM GOSSIP comes from the classical algorithm for average estimation over WSNs [2]: at each iteration, two connected nodes update their values according to the following rule.

  Algorithm 1. RANDOM GOSSIP *(RG)*
  When the sensor i wakes up (at global time t) :
  - *The sensor i chooses uniformly a neighbor $j \in \mathcal{N}_i$*
  - *i and j exchange their value*
  - *i and j update as follows*
    $x_i(t+1) = x_j(t+1) = \max(x_i(t), x_j(t))$
- The BROADCAST algorithm uses inherent broadcast property of a wireless channel: at each iteration, all the nodes connected to a chosen node update according to the broadcasted value of the chosen one (*broadcast* refers here to the iteration-wise update scheme like in [5] and not to the goal of information propagation like in the computer scientific literature [6]).

  Algorithm 2. BROADCAST *(BC)*
  When the sensor i wakes up (at global time t) :
  - *The sensor i broadcasts its value to all its neighbors*
  - *The sensors of the neighborhood $\mathcal{N}_i$ update as follows*
    $\forall j \in \mathcal{N}_i, \ x_j(t+1) = \max(x_i(t), x_j(t))$

III. PROOFS OF CONVERGENCE

Let $H_t = \{i \in V | x_i(t) = x_{max}\}$ be the set of vertices that have the maximum value at time t and $\bar{H}_t$ its complementary set. The first time to reach the max-consensus is $\tau_N = \arg\min_t\{|H_t| = N\}$ and we can easily notice that as soon as the consensus is reached, it is stable : $\forall t > \tau_N, \ |H_t| = N$. So, proving the convergence of an algorithm is equivalent to proving that $\mathbb{E}[\tau_N] < \infty$.

Theorem 1. *The* RANDOM GOSSIP *algorithm reaches max-consensus in finite time τ_{RG}.*

Proof: While max-consensus is not reached, there is at least one vertex in H_t that is connected to a vertex in $\bar{H}_t$. The probability of choosing one of these two vertices at time t is $2/N$ and the probability that they exchange with each other is at least $1/d_{max}$. Hence : $\mathbb{P}[|H_{t+1}| = |H_t|+1] \geq 2/(Nd_{max})$. Considering the Bernouilli variable $b_t = |H_{t+1}| - |H_t|$ of parameter $p \geq 2/(Nd_{max})$, we can conclude that H_t follows a geometrical distribution of parameter p so $\mathbb{E}[\tau_{RG}] = Np \leq d_{max}N^2/2$ which concludes the proof. ■

Corollary 1. *The* Broadcast *algorithm reaches max-consensus in finite time* τ_{BC}.

Proof: Similarly, the probability of incrementing the number of max-informed sensors is probability of choosing an informed node connected to a uninformed one which is greater than $1/N$. Hence, $\mathbb{E}[\tau_{BC}] \leq N^2$. ∎

IV. Convergence Speed bounds

Simple convergence speed bounds have already been obtained in previous Section but they do not depend much on the graph; yet, simulations show that the max-consensus time changes significantly according to the underlying graph. Therefore we propose here graph-dependent bounds.

A. Bound for the Random Gossip

Result 1. $\mathbb{E}[\tau_{RG}] \leq \frac{N d_{max}}{2\lambda_2^L} \sum_{k=1}^{N-1} \frac{N}{k(N-k)} \sim \frac{N d_{max} \ln(N-1)}{\lambda_2^L}$ *where* λ_2^L *is the second greatest eigenvalue of the Laplacian of the graph.*

Proof: Let S be a subset of V and $\partial S = \{e = (i,j) \in E | i \in S, j \in \bar{S}\}$ the edge frontier of S, we know from Cheeger [7], [8] that $|\partial S| \geq \lambda_2^L |S|(1 - |S|/|V|)$. Applying this inequality we have $\mathbb{P}[\text{The exchange is done over an edge of } \partial H_t] \geq 2|\partial H_t|/(N d_{max}) \geq 2\lambda_2^L |H_t|(|V| - |H_t|)/(d_{max}|V|^2) = p'$. Noting that $|V| = N$ and replacing the Bernouilli parameter of the proof of theorem 1 by p' concludes the proof. ∎

B. Bound for the Broadcast

Result 2. $\mathbb{E}[\tau_{BC}] \leq \bar{\epsilon} N + (\bar{\epsilon} - 1) N \ln \left(\frac{N-1}{\bar{\epsilon}-1} \right)$ *where* $\bar{\epsilon}$ *is the mean eccentricity of the graph.*

Proof: First, like the works of Feige, Frieze and Grimmett [6], [9], let us consider the spanning tree subgraph $\mathcal{G}'$ of $\mathcal{G}$ rooted on a[1] sensor with maximal value at time 0. It is evident that the Broadcast algorithm will reach max-consensus less quickly on $\mathcal{G}'$ than it does on $\mathcal{G}$. Let us denote by $\mathcal{L}^{(i)}$ the set of nodes (or layer) of distance i from the root node, as this transformation of graph keeps the smallest distance between the nodes, the number of layers is the eccentricity of the root node $\epsilon_r = \epsilon(root)$. The time for all the nodes of the layer $\mathcal{L}^{(j+1)}$ to be informed is $\tau^{(j+1)}$. $\mathbb{E}[\tau^{(j+1)} | \mathcal{L}^{(j)} \text{ informed}] = N \sum_{i=1}^{|\mathcal{L}^{(j)}|} \frac{1}{i}$ (this formula, that can be seen like a generalization of the coupon collector problem, is not proved here by lack of space).
So, $\mathbb{E}[\tau_{BC}] \leq \sum_{i=0}^{\epsilon_r - 1} \mathbb{E}[\tau^{(j+1)} | \mathcal{L}^{(j)} \text{ informed}] = N \sum_{j=1}^{\epsilon_r - 1} \sum_{k=1}^{|\mathcal{L}^{(j)}|} \frac{1}{k} \leq \epsilon_r N + (\epsilon_r - 1) N \ln \left(\frac{N-1}{\epsilon_r - 1} \right)$. Taking the expectation over all the vertices for the root node concludes the proof. ∎

V. Simulations and remarks on the tightness

In Figure 1, we plot the communication cost for the Random Gossip and the Broadcast algorithm and the proposed bounds given in Section IV versus the number of sensors. As the power consumption is rather associated with the number of communications than the number of algorithms iterations, we advocate the use of the communication cost as a measure of algorithm performance. This cost can be deduced from τ_N (the *iteration cost*) by multiplying it by the number of communications by iteration, namely, 2 for the Random Gossip and 1 for the Broadcast. The underlying graph is a Random Geometric Graph (RGG) [10] of radius $r = 4\sqrt{\log(N)/N}$. We can see that the Broadcast algorithm is much better than Random Gossip and that the bounds tend to confirm this idea. It is also worth noticing that the proposed bounds are tight for the complete graph and the bound for the Broadcast is also tight for the path graph.

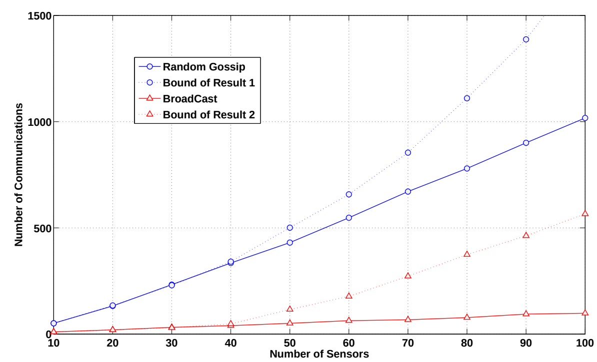

Fig. 1. Communication costs and proposed bounds for Random Gossip and Broadcast algorithms on a RGG.

[1]'the' if there is only one sensors with maximal value at time 0.

VI. Conclusion

We proved that the Random Gossip and the Broadcast algorithms dealing with the distributed maximum value estimation over a WSN converge to the max-consensus and we gave fair bounds for their convergence speed.

References

[1] J. Tsitsiklis, "Problems in decentralized decision making and computation," Ph.D. dissertation, M. I. T., Dept. of Electrical Engineering and Computer Science, 1984.

[2] S. Boyd, A. Ghosh, B. Prabhakar, and D. Shah, "Randomized gossip algorithms," *IEEE Transactions on Information Theory*, vol. 52, no. 6, pp. 2508–2530, 2006.

[3] A. D. G. Dimakis, A. D. Sarwate, and M. J. Wainwright, "Geographic Gossip: Efficient Averaging for Sensor Networks," *IEEE Transactions on Signal Processing*, vol. 56, no. 3, pp. 1205–1216, 2008.

[4] F. Benezit, A. G. Dimakis, P. Thiran, and M. Vetterli, "Order-Optimal Consensus Through Randomized Path Averaging," *IEEE Transactions on Information Theory*, vol. 56, no. 10, pp. 5150–5167, 2010.

[5] T. C. Aysal, M. E. Yildiz, A. D. Sarwate, and A. Scaglione, "Broadcast Gossip Algorithms for Consensus," *IEEE Transactions on Signal Processing*, vol. 57, no. 7, pp. 2748–2761, 2009.

[6] U. Feige, D. Peleg, P. Raghavan, and E. Upfal, "Randomized broadcast in networks," *Random Structures & Algorithms*, vol. 1, no. 4, pp. 447–460, 1990.

[7] D. Spielman, "Spectral Graph Theory," Lecture Notes, Yale University, 2009. [Online]. Available: http://www.cs.yale.edu/homes/spielman/561/

[8] J. Cheeger, "Spectral geometry of singular Riemannian spaces," *Journal of differential geometry*, vol. 18, no. 4, pp. 575–657, 1983.

[9] A. Frieze and G. Grimmett, "The shortest-path problem for graphs with random arc-lengths," *Discrete Applied Mathematics*, vol. 10, no. 1, pp. 57–77, 1985.

[10] M. Penrose, *Random geometric graphs*, O. S. in Probability, Ed. Oxford University Press, USA, 2003.