α-Repetition/Modulation and blind second-order identification

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Abstract

In the context of redundant filter-bank precoders, we investigate the so-called α-Repetition/Modulation, for $\alpha \in [0, \frac{1}{2}]$, with regard to the blind second-order identification. For many α's, an algebraic result shows that the unknown channel can be recovered from the second-order statistics irrespective of both the channel zero location and the model over-determination. Namely, a structured subspace method provides an estimate of the channel. However, in the context of band-limited communication channels, numerical degeneracies appear for certain choices of α. We therefore analyze the numerical behavior of this method resorting to spheroidal wave sequences. We show there exist α's less than a bound depending on the channel bandwidth making the estimation robust to noise. Simulations are also provided.

1 Introduction

Let $\{s_n\}_{n\in\mathbb{Z}}$ be a zero mean unit variance i.i.d symbol sequence to be transmitted through a linear channel at the baud rate $1/T_s$. In the noise free case, the continuous time received signal $\tilde{y}(t)$ can be written as :

$$\tilde{y}(t) = \sum_{n\in\mathbb{Z}} s_n h_a(t - nT_s)$$

where the filter $\{h_a(t)\}$ is a dispersive channel resulting from a pulse shaping filter (in general, a square-root raised cosine with roll-off ρ) and from the multipath effects. In this paper, we assume without restriction that $\tilde{h}(t)$ is causal, time limited and static in regard of the duration of the observations. Generally, $\{\tilde{h}(t)\}$ is unknown, and has therefore to be estimated in order to retrieve the symbols from the received signal. In most single carrier communication systems, the emitter sends periodically a training sequence known from the receiver, and which allows to estimate the unknown channel. However, when the channel variations are fast, the performances of this approach decrease. Therefore, a number of works have been devoted to the so-called blind equalization problem consisting in identifying the channel from the sole knowledge of the received signal $\tilde{y}(t)$.

Gardner [6] and Tong et al [10] were the first to remark that it is possible to use the cyclostationarity of $\tilde{y}(t)$ in order to identify the channel from the second order statistics of the observations. For this, they proposed to sample $\tilde{y}(t)$ at rate $2/T_s$. This approach is called fractional sampling (FS). The discrete time signal $y(n) = \tilde{y}(nT_s/2)$ can be written as

$$y(n) = \sum_{k=0}^{L} h_k u_{n-k} = [h(z)]u(n) \qquad (1)$$

where u_n is defined by $u_{2n} = s_n$ et $u_{2n+1} = 0$ and where $h(z) = \sum_{k=0}^{L} h_k z^{-k}$, $h_k = \tilde{h}(k\frac{T}{2})$. $\{y(n)\}$ is cyclostationary with cyclic frequencies 0 et $\frac{1}{2}$ and its corresponding cyclospectra are given by $S_y^0(e^{2i\pi f}) = \frac{1}{2}|h(e^{2i\pi f})|^2$ and $S_y^{\frac{1}{2}}(e^{2i\pi f}) = \frac{1}{2}h(e^{2i\pi f})h^*(e^{2i\pi(f-\frac{1}{2})})$. It is well established that if $h(z)$ et $h(-z)$ have no common zero, the knowledge of S^0 and $S^{\frac{1}{2}}$ allow to retrieve $h(z)$. Moreover, a number of estimation algorithms of $h(z)$ have been proposed recently, in particular, the subspace method introduced by [5].

The above mentioned identifiability condition is of course verified in most cases. However we note that generally, the bandwidth of the shaping filter is an interval $[-\beta/T_s, \beta/T_s]$ with $\beta = \frac{1+\rho}{2}$. This implies that

$$h(e^{2i\pi f}) \approx 0 \quad \text{if} \quad f \in [-1/2, -\beta/2] \cup [\beta/2, 1/2] \quad (2)$$

So the discrete channel $h(z)$ is band-limited. In this context, Van der Veen [11] and Ciblat [4] show that, the subspace method have undesirable numerical behaviors due to the band-limited character of the channel.

Indeed, the cyclostationarity statistics do not provide information enough, because the cyclospectrum at the cyclic frequency $\frac{1}{2}$ is close to 0, due to the small excess bandwidth factors used in most communication contexts.

An idea consists in increasing the strength of the cyclospectra at the receiver. This can be achieved by passing, at the emitter, the symbol sequence into a periodic precoder. The artificial cyclostationarity these precoders lead to is referred to as Transmitter Induced Cyclostationarity.

In [7], a general formalism lying on redundant filterbank is introduced. The blind second-order identification methods these precoders are associated with are shown to be robust to the channel in the sense that for each precoder, an algebraic result states that the channel can be recovered.

The so-called α-Repetition/Modulation precoder (α-RM) is a particular redundant precoder combining repetition and modulation, and was introduced in [1]. It is shown in [1] that the channel can be estimated by a structured subspace method. Moreover, under certain conditions on α, it is stated that the subspace method makes possible the identification of the channel, whatever this one. However, in the context of band-limited channel, which is to be considered since all communication channels share this band-limited property, poor performance can be noted for certain choices of α, as for the FS case. The purpose of this paper is therefore the study of the performance of the blind channel estimate in terms of parameter α in the case of band-limited channels. For this, we use the same heuristic arguments introduced in [4]. We show precisely why taking α less than a bound depending on the channel bandwidth may prevent the algorithms from having poor statistical performance.

This paper is organized as follows. The α-RM transmitter is described in Section 2. Section 3 is concerned with a blind second-order based estimate of the channel lying on a subspace method. The extraction of the one dimensional kernel of a certain matrix Q_α is shown to provide the channel up to a constant. In section 5, it is proved that, for certain values of α, the matrix Q_α exhibits a "numerical" kernel of dimension more than one, which naturally prevents the estimation of the filter. In order to justify our analysis, we study the asymptotic covariance matrix of the structured subspace estimate in Section 6. Section 7 is devoted to numerical tests.

2 Description of α-RM

The TIC approach proposes to transmit a transformed version $\{v_n\}$, which is called the pseudo-symbol sequence, of the symbol sequence $\{s_n\}$. In α-RM case, the transformation correspond to transmit a burst of $2M$ pseudo-symbols $\{v_n\}$ which arises from a burst of M symbols $\{s_n\}$ as the following. A burst of $2M$ pseudo-symbols v_n is split into two consecutive bursts of length M, the first one being the burst of symbols $\{s_n\}$, the second other a modulated version of this latter at the frequency α. We emit the pseudo-symbol sequence, at the baud rate $2/T_s$, through the same shaping filter designed for transmission at T_s. The required bandwidth therefore is the same as in the FS case. The continuous time received signal is sampled at $\frac{T_s}{2}$ and we obtain the following numerical model

$$y(n) = \sum_{k=0}^{L} h_k v_{n-k} = [h(z)].v_n$$

where $h(z)$ is the same as in FS case. This equation is the same as (1), but we exchange u_n by v_n.

We can interpret the α-RM by considering the bivariate process $\{Y_n\}$ given by $Y_n = \left[y_n, y_{n+M}e^{-2i\pi n\alpha}\right]^T$. At lags $n = L, ..., M-1$, $\{Y_n\}$ can be seen as

$$Y_n = \sum_{k=0}^{L} H_k^{(\alpha)} s_{n-k} = [H_\alpha(z)]s_n \tag{3}$$

where

$$H_\alpha(z) = \frac{1}{\sqrt{2}} \begin{pmatrix} h(z) \\ h(ze^{2i\pi\alpha}) \end{pmatrix}$$

In the sequel, the length of a burst, M, is supposed to be big as compared to L, which allows to neglect the border effects. For example, in European GSM system, the length M $(= 146)$ of a burst is greater than L. Therefore, there is no restriction to just exploit vectors Y_n for $n = L, ..., M-1$.

The model (3) states that the α-RM artificially creates a kind of diversity controlled by the values of α.

In the following sections, the blind second-order identification of $H_\alpha(z)$, hence of $h(z)$, is addressed. The bivariate model (3) is similar, up to the structure, to the one encountered in time or spatial diversity contexts. Therefore, it is natural to investigate the identification problem adapting the famous subspace method of [5].

3 Structured subspace method

Let $\hat{L}$ an estimate of L, the channel order. In the sequel, it is assumed that $\hat{L} \geq L$, i.e. the model is possibly over-determined. We consider a scalar $N < M - \hat{L}$. We put $\mathcal{Y}_N(n) = \left[Y_n^T, Y_{n-1}^T, ..., Y_{n-N}^T\right]^T$ for the lags $n = N + \hat{L}, ..., M-1$. It is readily seen that

$$\mathcal{Y}_n = \mathcal{T}_N(H_\alpha)\mathcal{S}_{N+L}(n)$$

where $\mathcal{T}_N(H_\alpha)$ is the $2(N+1) \times (L+N+1)$ Sylvester matrix associated with the filter $H_\alpha(z)$. The covariance matrix of $\{\mathcal{Y}_N(n)\}$ is simply

$$\mathcal{R}_N = \mathcal{T}_N(H_\alpha)\mathcal{T}_N(H_\alpha)^*$$

We choose $N > \hat{L}$. Then the matrix $\mathcal{T}_N(H_\alpha)$ is tall. This implies that $\mathcal{R}_N$ is a singular matrix. Denote by Π_N the orthogonal projector onto its kernel. Take $f(z) = \sum_{k=0}^{\hat{L}} f_k z^{-k}$ a generic $\hat{L}$-order polynomial, and denote by $F_\alpha(z) = \frac{1}{\sqrt{2}}\left[f(z), f(ze^{2i\pi\alpha})\right]^T$ its associated bivariate structured polynomial. Moreover we define $\mathbf{f} = [f_0, f_1, ..., f_{\hat{L}}]^T$. The subspace method consists in investigating the quadratic mapping

$$f \mapsto \text{Trace}\left(\Pi_N \mathcal{T}(F_\alpha)\mathcal{T}_N(F_\alpha)^*\Pi_N^*\right) = \mathbf{f}^* Q_\alpha \mathbf{f}$$

It is easy to check that,

$$Q_\alpha = \mathcal{P}^* D_\Pi^* D_\Pi \mathcal{P} \quad (4)$$

where $\mathcal{P}$ is a block-diagonal structure matrix, the k^{th} 2×1 block given by $(\mathcal{P})_k = \left[1, e^{-2i\pi k\alpha}\right]^T$, and where

$$D_\Pi = \int_0^1 \overline{D_{N+\hat{L}}(e^{2i\pi f})} D_{\hat{L}}^T(e^{2i\pi f}) \otimes \Pi_N(e^{2i\pi f}) df \quad (5)$$

with $D_k(e^{2i\pi f}) = \left[1, e^{-2i\pi f}, \ldots, e^{-2i\pi k f}\right]^T$ and $\Pi_N(z) = \sum_{k=0}^{N} \Pi_{k,N} z^{-k}$, where $\Pi_N = [\Pi_{0,N}, \ldots, \Pi_{N,N}]$ and each matrix Π_k is $2(N+1)\times 2$.

The theorem below gives a condition on α for the channel to be retrieved up to constant from Q_α.

Theorem 1 (identifiability) *Suppose α is rational, $\alpha = \frac{p}{q}$ with p and q coprime.*

If $q \geq \hat{L}$, the kernel of the matrix Q_α is a one-dimensional subspace generated by the channel vector $\mathbf{h} = [h_0, h_1, ..., h_L, 0_{1,\hat{L}-L}]^T$

In others words, the structured noise subspace method allows to identify the filter $h(z)$ up to a scalar.

This algebraic result is proved in [8] and [2]. Notice that, in contrast with the classical non-structured subspace method, the unknown filter is recovered irrespective of the channel zero location and whatever the over-determination factor. For sake of clarity, we hence forward suppose that $\hat{L} = L$.

In practice, a finite number T of data is collected and the received signal is corrupted by noise. Therefore Q_α is unknown and must be estimated by consistent $\hat{Q}_{\alpha,T}$. The eigenvector associated to the smallest eigenvalue of $\hat{Q}_{\alpha,T}$ represents a consistent estimate of $\mathbf{h}$ denoting by $\hat{\mathbf{h}}_T$.

It is easy to understand that, if Q_α is ill-conditioned, i.e., has quite small eigenvalues apart 0, it will be difficult to separate the exact kernel of $\hat{Q}_{\alpha,T}$ and its numerical kernel generated by the eigenvectors associated to the non zero small eigenvalues. The estimate $\hat{\mathbf{h}}_T$ is likely to belong to a corrupted version of the numerical kernel of $\hat{Q}_{\alpha,T}$, especially when $\hat{Q}_{\alpha,T}$ is far from Q_α, i.e., when the noise is important or/and T is small.

Therefore, we now investigate the conditioning of the matrix Q_α. As α is big, it appears that the α-RM becomes (quasi) equivalent to the FS. Indeed, the spectral matrix of the bi-variate process $Y(n)$ using in FS and $\frac{1}{2}$-RM cases provide the same statistical information. In the FS case, the conditioning of Q_{fs} which is obtained by a standard subspace method, is bad. We therefore want to show that α should probably be bounded, with a bound depending on the excess bandwidth, so as to ensure a good conditioning of Q_α.

4 Spheroidal wave sequences

The order $L+1$ spheroidal wave sequences [9] on an interval $\mathcal{I} \subseteq [\frac{1}{2}, \frac{1}{2}]$ are the (unit norm) eigenvectors $\{\mathbf{k}_{j,\mathcal{I}}\}_{j=0,L}$ associated with the eigenvalues $\lambda_0 \leq \ldots \leq \lambda_L$ of the positive $(L+1)\times(L+1)$ Toeplitz matrix $\mathcal{K}_{L,\mathcal{I}}$ defined as

$$\mathcal{K}_{L,\mathcal{I}} = \int_{\mathcal{I}} \overline{D_L(e^{2i\pi f})} D_L^T(e^{2i\pi f}) df$$

The matrix $\mathcal{K}_{L,\mathcal{I}}$ is known to be ill conditioned, and its "numerical" rank is equal to int$((L+1)|\mathcal{I}|)$, where int(.) stands for the integer part of (.) and where $|\mathcal{I}|$ represents the size of $\mathcal{I}$. In the following, we denote by s the dimension of the numerical kernel of $\mathcal{K}_{L,\mathcal{I}}$. For a given $\mathbf{k}_{j,\mathcal{I}} = [k_{j,0}, ..., k_{j,L}]^T$, we let the associated transfer function $k_{j,\mathcal{I}}(z) = \sum_{l=0}^{L} k_{j,l} z^{-l}$. We obtain

$$\mathbf{k}_{j,\mathcal{I}}^* \mathcal{K}_{L,\mathcal{I}} \mathbf{k}_{j,\mathcal{I}} = \int_{\mathcal{I}} |k_{j,\mathcal{I}}(e^{2i\pi f})|^2 df.$$

The existence of a numerical kernel of $\mathcal{K}_{L,\mathcal{I}}$ implies

$$\forall j < s, \quad k_{j,\mathcal{I}}(e^{2i\pi f}) \approx 0 \quad \text{if} \quad f \in \mathcal{I}$$

In others words, the FIR filters $k_{j,\mathcal{I}}(e^{i2\pi f})$ for $j = 0, s-1$ associated with the s so-called "smallest" spheroidal wave sequences of $\mathcal{I}$ are nearly band-limited and their support coincide with $\mathcal{I}^c$ [1].

5 The numerical kernel of Q_α

In this section, we show (see [3]) that Q_α may exhibit, if α is great, a undesired numerical kernel in addition to its natural one dimensional kernel.

Claim 1 *If $\alpha > \frac{\beta}{2}$, Q_α exhibits a numerical kernel of dimension more than* 1. *Actually,*

$$\dim Ker(Q_\alpha) \approx 1 + s$$

[1] $\mathcal{I}^c$ denotes the complementary set of $\mathcal{I}$ in $[-\frac{1}{2}, \frac{1}{2}]$.

with $s = int((L+1)(2\alpha - \beta))$

In this case, denote by $\mathcal{W}$ *the interval*

$$\mathcal{W} = \left[\frac{\beta}{2} - \alpha \ , \ \alpha - \frac{\beta}{2}\right]$$

The linear space spanned by the true channel $\mathbf{h}$ *and the smallest spheroidal wave sequences of length* $L+1$ *of the interval* $\mathcal{W}^c$ *belong to the numerical kernel of* Q_α.

The claim implies that the subspace method is subject to severe numerical problems as soon as α is beyond $\beta/2$. The extraction of the eigenvector associated to "the" smallest eigenvalue of Q_α is an ill-posed problem since there are at least $s+1$ almost null eigenvalues. Take one vector $\mathbf{l}$ of the numerical kernel of Q_α. Then $\mathbf{l}$ is likely to be a linear combination of $\mathbf{h}$ and some of the s smallest spheroidal wave sequences of the band $\mathcal{W}^c$. In the frequency domain, this implies

$$l(e^{i2\pi f}) = \sum_{j=0}^{s-1} r_j k_{j,\mathcal{W}^c}(e^{i2\pi f}) + r_s h(e^{i2\pi f})$$

for some r_j's. $k_{j,\mathcal{W}^c}(e^{i2\pi f}) \approx 0$ when $f \in \mathcal{W}^c$, but this is not the case if $f \in \mathcal{W}$. Hence, $l(e^{i2\pi f})$ cannot be a reliable estimate of $h(e^{i2\pi f})$ (up to a constant) in the band of frequencies $\mathcal{W}$.

Conversely, the condition $\alpha \leq \frac{\beta}{2}$ does not ensure that the numerical kernel is reduced to the span of $\mathbf{h}$. However, the previous bandwidth considerations cannot prove the contrary. Of course, α should not be taken to close from 0 in order to ensure a good diversity between the 2 components of Y_n.

In the sequel, for numerical tests, we use a square-root raised cosine pulse with roll-off $\rho = 0.2$ (i.e. $\beta = 0.6$) and a certain multi-path realization. The complete impulse response is truncated such that 1% of the total energy is removed. This makes $h(z)$ be a degree $L = 11$ polynomial.

For $\alpha = 0.18$, our analysis ensures that Q_α does not exhibit a numerical kernel because $\alpha < \beta/2$. In practice, it is verified. In the contrary, for $\alpha = 0.42$, we expect the presence of a numerical kernel. In Table 1, we compute the dimension of the kernel and the values $\mathbf{k}_j^*(Q_\alpha)_{\alpha=0.42}\mathbf{k}_j$ for all the smallest spheroidal wave sequences $\mathbf{k}_j$ relative to the indeterminacy band $\mathcal{W}$ which is equal to $[-0.12, 0.12]$. We compare the α-RM with the FS too. Therefore the same quantities are computed for Q_{fs}. We can prove that, the numerical behavior is quite the same. In this case, as $\frac{1}{2}$-RM case, the associated indeterminacy interval [4] is equal to $[-0.2, 0.2]$.

The results perfectly corroborate our analysis, since the $(\mathbf{k}_{j,\mathcal{W}^c})_{j=0,s-1}$ belong to the numerical kernel.

(Values in dB)	FS	0.42-RM
Kernel dimension	5	3
$\{\mathbf{k}_j^* Q \mathbf{k}_j\}_{j=0,\ldots,s-1}$	-30 -25 -21 -12	-24 -16

Table 1: *The "spheroidal effect".*

6 Statistical analysis

$\hat{\mathbf{h}}_T$ is a consistent estimate of $\mathbf{h}$. By using some classical arguments about the subspace method, we can prove that, $\sqrt{T}(\hat{\mathbf{h}}_T - \mathbf{h})$ converge, in law, to a zero mean and C variance Gaussian law. The asymptotic covariance matrix

$$C = \lim_{T\to\infty} T\ \mathbb{E}\ \left[(\hat{\mathbf{h}}_\mathbf{T} - \mathbf{h})(\hat{\mathbf{h}}_\mathbf{T} - \mathbf{h})^*\right]$$

is given by

$$C = Q_\alpha^{\#}\mathcal{P}^* D_\Pi^* \Sigma D_\Pi \mathcal{P} Q_\alpha^{\#} \quad (6)$$

where $(.)^{\#}$ stands for the pseudo-inverse and Σ is a positive hermitian matrix (see [3], for more details).

If $\alpha > \frac{\beta}{2}$ holds, Q_α admits a numerical kernel of dimension more than one. The terms $Q_\alpha^{\#}$ in (6) make the covariance matrix be huge in the numerical kernel of Q_α. In particular, the asymptotic covariance matrix C is prone to exploding in the directions of the spheroidal wave sequences $(\mathbf{k}_{j,\mathcal{W}^c})_{j=0,s-1}$.

We now investigate the quantity

$$\mathcal{C}(e^{2i\pi f}) = D_L^T(e^{i2\pi f}) C \overline{D_L(e^{i2\pi f})}$$

which is equal to $\lim_{T\to\infty} T\mathbb{E}|\hat{h}_T(e^{i2\pi f}) - h(e^{i2\pi f})|^2$. It represents the localization in the frequency domain of the estimate error. When $\alpha > \frac{\beta}{2}$ is verified, as C may be explosive in the directions of the sequences $(\mathbf{k}_{j,\mathcal{W}})_{j=0,s-1}$, the estimate error is to be huge in the band of indetermination $\mathcal{W}$.

This phenomenon is confirmed by simulations. For this, we consider the FS case and the α-RM for $\alpha \in (0.18, 0.42)$ cases. The SNR is equal to $30dB$.

In Figure 1, the function $\mathcal{C}(e^{2i\pi f})$ is plotted versus f, in dB. As foreseen by our analysis, we observe that, the function $\mathcal{C}(e^{2i\pi f})$ is huge in the band of frequencies $\mathcal{W}$, when this interval is not empty, i.e., for the FS ($\mathcal{W}_{FS} = [-0.2, 0.2]$) and α Repetition/Modulation with $\alpha = 0.42$ ($\mathcal{W}_{0.42\text{-RM}} = [-0.12\ ,\ 0.12]$) cases.

7 Simulation results

In Figure 2 are represented the experimental mean square error $\mathbb{E}\sum_{k=0}^{L}|\hat{h}_{T,k} - h_k|^2$ for FS and α-Repetition-/Modulation with $\alpha \in (0.18, 0.42)$. The number of observations is $T = 300$. The number of Monte Carlo trials is 100. The SNR varies from $5dB$ to $50dB$.

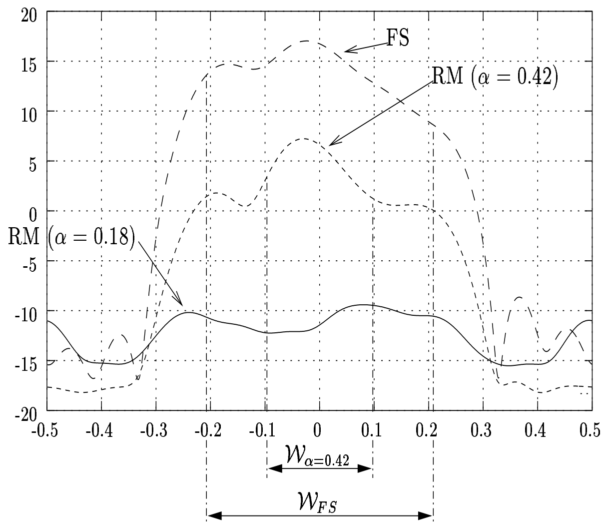

Figure 1: $\lim_{T\to\infty} T\mathbb{E}|\hat{h}_T(e^{2i\pi f}) - h(e^{2i\pi f})|^2$ *vs.* f

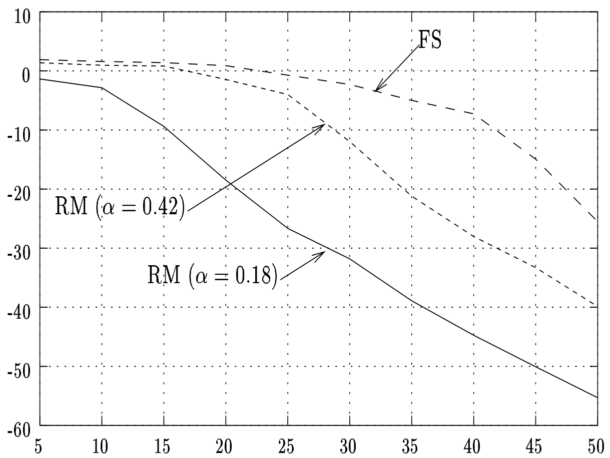

Figure 2: *Mean square error (dB) vs. SNR.*

We see that, the FS and α-RM, for $\alpha = 0.42$, have poor performances while Q_α is ill-conditioned On the contrary, α-Repetition/Modulation, for $\alpha = 0.18$, allows to identify the channel well, even if the SNR is small. This results confirm our analysis.

8 Conclusion

A novel precoder called α-Repetition/Modulation is investigated on the blind second-order identification point of view. We have proved that the estimate of the unknown channel, relying on a structured subspace method, can show poor performance if the channel is band-limited. More precisely, when $\alpha > \frac{\beta}{2}$, the estimate is likely to be the desired channel, superimposed with undesired band-limited spheroidal sequences. If the condition $\alpha \leq \frac{\beta}{2}$, no such band-limited effect can be exhibited and, indeed, α-Repetition/Modulation is then an appealing precoder in the sense it allows a high performance estimation of the unknown channel.

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