Outage Probability Based Power and Time Optimization for Relay Networks

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Abstract

In the context of cooperative wireless networks that convey data on slow fading channels, outage probability $P_{\rm o}$ is the relevant performance index from the point of view of information theory. Derivation and minimization of this probability with respect to the relaying protocol parameters is of central importance. However, it is often hard to derive its expression, let alone to find its exact minimum for all possible values of the Signal to Noise Ratio (SNR). This problem can be simplified by studying the behavior of $P_{\rm o}$ in the asymptotic regime where the SNR ρ converges to infinity. In this regime, usually $\rho^{N+1} P_{\rm o}$ converges to a constant ξ where N is the number of relays. In this paper, a simple and general method for deriving and minimizing ξ with respect to the power distribution between the source and the relays, and with respect to the durations of the slots specified by the relaying protocol, is developed. While the proposed approach is designed for the high SNR regime, simulations show that outage probability is reduced in a similar proportion at moderate SNR.
The method applies to a general class of radio channels that includes the Rayleigh and the Rice channels as particular cases. Convexity of ξ with respect to the design parameters is shown. Decode-and-Forward as well as Amplify-and-Forward protocols are considered in the half duplex mode.

Index Terms

Amplify-and-Forward, Cooperative Wireless Networks, Convex Minimization, Decode-and-Forward, Outage Probability, Relay Channel.

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I. INTRODUCTION

In digital wireless communications over fading channels, antenna diversity is an efficient means of mitigating the effect of channel fades. By using multiple antennas at the transmitter and/or at the receiver, the transmitted message reaches its destination through different weakly correlated channels. It is unlikely that all these channels are simultaneously subject to deep fades. Hence, the performance improves with respect to a communication using a single antenna at each side.

Antenna diversity in wireless communications has been first studied in the context of point to point communications where multiple antennas are present at the transmitter or at the receiver. Recently, a new means of providing this diversity has been considered: in the vicinity of the transmitter/receiver link, radio terminals in an idle state are likely to be present. By giving some of these terminals the ability to relay the transmitter's signal towards the receiver, one creates a virtual multiple antenna system which is capable of providing diversity [1]–[6].

More formally, in the slow channel fading context, the relevant performance measure from the information theoretic point of view is the so called outage probability, which is the probability that Shannon's mutual information lies beneath a given rate. In a N–relay network with single antenna terminals, the outage probability $P_{\rm o}$ usually satisfies $\lim_{\rho\to\infty}\left(\rho^{N+1}P_{\rm o}\right)=\xi$ where ρ is the Signal to Noise Ratio (SNR) and ξ is a non zero constant. This equation indicates in particular that the diversity order of our N–relay network is $N+1$, which is precisely the diversity order of a Multiple Input Single Output (resp. Single Input Multiple Output) point-to-point system with $N+1$ antennas at the transmitter (resp. at the receiver).

Different relaying protocols have been studied in the literature. One well known protocol is the Decode and Forward (DF) protocol [3]: assume for simplicity there is only one relay and divide the transmission frame into two slots. During the first slot, the relay listens to the source signal. At the end of this slot, it attempts to decode this message then it re-encodes it and transmits the new signal in the second slot. The Amplify-and-Forward (AF) protocols are simpler than DF protocols: here the relay just applies a gain (or more generally a linear precoding) to the signal received during the first slot before retransmitting it in the second slot [4].

Beyond the diversity considerations, it is of clear interest to minimize the outage probability of a given protocol with respect to the protocol parameters such as the powers and the slot durations. This will be the subject of this paper. Our general assumptions are the following: we consider half duplex relay networks, *i.e.*, a given terminal cannot receive and transmit at the same time and in the same frequency band. Slow fading channels are considered. These channels are assumed perfectly known at the receivers and unknown at the transmitters. DF and AF protocols with N relays will be considered where N is some integer. The parameters involved in the outage probability minimization are the slots relative durations and the powers given to the source and to the relays. This minimization is performed by some resource allocation unit which relies on a statistical knowledge of the source-relays, source-destination and relays-destination channels.

The exact minimization of the outage probability for any value of the SNR is known to be a difficult task. Mathematical problems appear even for systems as simple as point to point Multiple Input Single Output systems [7]. A means to circumvent this difficulty is to just minimize the constant ξ introduced above with respect to powers and slot durations. While strictly speaking this minimization concerns the high SNR regime, simulations will show that it reduces outage probability at all SNRs considered in practice in a similar proportion. In the space-time coding literature, the counterpart of this constant is called "coding gain" factor. In parallel with this denomination, here we call this constant "outage gain" factor. We show in particular that the outage gain factor is a convex function of the powers and the slot durations for the considered protocols. We do not make any assumption on the channels probability distributions except for the fact that the probability densities of the channels power gains do not vanish at zero. This assumption is satisfied in particular by the so-called Rayleigh and Rice channels. In order to perform the outage gain minimization, the resource allocation only needs the values at zero of the channel gain densities. This information can be sent from the different receivers to the resource allocation unit with a negligible cost.

Related Work and paper's contribution

The subject of outage probability derivation and minimization began to attract researchers attention in the context of multiple antenna point to point communications ([7]–[9] just to name these). In the context of wireless relay networks, the authors of [10] propose a power optimization method for a multi hop system. In [11], the authors optimize the relay powers for a DF protocol by working on an upper bound of the outage probability. In [12], an AF protocol with one relay is considered while in [13], AF and DF are studied, and the optimization is performed by minimizing the constant ξ. In [14], an AF protocol with multiple relays is considered. In all these contributions, the protocols are said orthogonal in the sense that the relays and the source do not transmit their signals at the same time. Other works explore the idea of outage optimization in the case where a certain amount of instantaneous channel state information is available through feedback. In this line of thought, let us cite without being exhaustive [15]–[17].

The contributions of this paper can be summarized as follows:

- A general and novel method for deriving the outage gain $\xi = \lim_{\rho\to\infty} \rho^{N+1} P_o$ is proposed. The derivation of the outage gain is a key-issue in digital communications as far as performance evaluation or system optimization is concerned. As in [10]–[14], it is assumed that the information fed back to the resource allocation unit is only statistical. Our method is generic in its essence and can thus be applied to a wide range of relaying protocols, either orthogonal or nonorthogonal. To the best of our's knowledge, no other method in the literature allows to compute the outage gain of relaying protocols under such a general context. Most of the existing works (*e.g.* [10]–[14]) restrict the study to the orthogonal cases which are much simpler to analyze from the outage gain stand point. The protocols considered in this paper as study cases are non orthogonal. While it is perfectly possible to make a comparison between *e.g.* an orthogonal protocol with a non orthogonal one from the stand point of the outage gain, we do not undertake this task for lack of space. The primary concern of this paper is rather to optimize the parameters of a given protocol. We note that comparisons between orthogonal and non orthogonal protocols has been made in the literature via the so-called Diversity Multiplexing Tradeoff (DMT). From the point of view of the DMT, non-orthogonal schemes outperform orthogonal ones (see for instance [23]).
- Most of the existing works (*e.g.* [10]–[14]) only consider Rayleigh channels, while the Rayleigh assumption is not required in this paper. Indeed it is only assumed that the densities of the channel power gains are right continuous and non zero at zero. This is the minimum assumption that guarantees the required diversity order.
- Most of the existing works on optimization of relaying protocols only focus on the power distribution. The time slot durations are never taken into account in the references provided above.
- It is proven here that ξ is convex with respect to durations *and* powers, which is a new result. As is well known, convexity is a strong and sought after result in optimization problems.

General notations and channel assumptions

In this paper, scalar and vector random variables are represented by upper case letters. The probability density of a scalar random variable X will be denoted $f_X(x)$. We also denote by $\mathcal{CN}(a, \sigma^2)$ the complex circular Gaussian distribution with mean a and variance σ^2. Given two events $\mathcal{E}_1$ and $\mathcal{E}_2$, *i.e.*, measurable subsets of a probability space Ω, we denote by $\mathbb{P}[\mathcal{E}_1]$ the probability measure of $\mathcal{E}_1$ and by $\mathbb{P}[\mathcal{E}_1 \| \mathcal{E}_2]$ the probability of $\mathcal{E}_1$ conditional to $\mathcal{E}_2$.

Let $g : \mathbb{R}^d \to \mathbb{R}$ be a real function and $\mathcal{A}$ be a subset of $\mathbb{R}$. We denote by $[g \in \mathcal{A}]$ the subset $\{\mathbf{x} \in \mathbb{R}^d \ :$

$g(\mathbf{x}) \in \mathcal{A}\}$. The notation $\mathbf{1}\{g(\mathbf{x}) \in \mathcal{A}\}(\mathbf{x})$ or more concisely $\mathbf{1}\{g \in \mathcal{A}\}$ refers to the indicator function of the set $[g \in \mathcal{A}]$.

We denote by N the number of relays in the network. Node 0 will coincide with the source, nodes 1 to N are the relays and node $N+1$ is the destination. As the transmitted data frame is divided into slots, we shall denote by X_{in} the random vector that represents the message transmitted by node i during slot n. The signal received by node i during slot n will be denoted Y_{in}. Moreover, during slot n, node i is corrupted by an Additive White Gaussian Noise (AWGN) vector V_{in} with unit variance elements.

We denote by H_{ij} the complex random variable representing the Single Input Single Output radio channel that conveys data from node i to node j. The power gain of this channel will be $G_{ij} = |H_{ij}|^2$. All random variables G_{ij} are assumed to have densities $f_{G_{ij}}(x)$ which are right continuous at zero. We denote by c_{ij} the limit $c_{ij} = f_{G_{ij}}(0^+)$ and we assume that all these limits are positive. In particular, in the Rayleigh case, H_{ij} is complex Gaussian with mean 0 and variance σ_{ij}^2. In this case, G_{ij} has the exponential distribution $f_{G_{ij}}(x) = \sigma_{ij}^{-2} \exp(-x/\sigma_{ij}^2)\mathbf{1}\{x \geq 0\}$, and in particular $c_{ij} = \sigma_{ij}^{-2}$. More generally, in the Ricean case $H_{ij} \sim \mathcal{CN}(a_{ij}, \sigma_{ij}^2)$ where the mean a_{ij} is not necessarily zero, the density $f_{G_{ij}}$ is given by

$$f_{G_{ij}}(x) = \frac{1}{\sigma_{ij}^2} e^{-\frac{|a_{ij}|^2+x}{\sigma_{ij}^2}} I_0\left(2\sqrt{x}\frac{|a_{ij}|}{\sigma_{ij}^2}\right)\mathbf{1}\{x \geq 0\}$$

where I_0 is the modified zero order Bessel function of the first kind [18]. As $I_0(0^+) = 1$, we have in this case

$$c_{ij} = \frac{1}{\sigma_{ij}^2} e^{-\frac{|a_{ij}|^2}{\sigma_{ij}^2}} .$$

In the paper, all channels H_{ij} within a network are assumed independent and available at the receivers only. Furthermore, the constants $\{c_{0,i}\}_{i=1,\ldots,N+1}$ and $\{c_{i,N+1}\}_{i=1,\ldots,N}$ are assumed to be available to the resource allocation unit.

Paper Organization

In Section II, the outage gain factor is studied for a class of DF protocols. The AF case is considered in Section III. In both Sections II and III, we begin with the single relay case, then we extend the results to the N-relay case. Section IV is devoted to some numerical illustrations of the obtained results and to some simulations. A number of mathematical proofs are put in an appendix.

II. THE DF PROTOCOL

This section is devoted to the outage probability derivation and minimization in the DF case. For clarity, we begin by treating the single relay case. The N-relay case will follow.

A. Outage Probability in the Single Relay case

In this section, we study the following protocol already considered in [19], [20]: the source (node 0) needs to send information at a rate of R nats per channel use towards the destination (node 2). To this end, the source has as its disposal a frame of length T and a dictionary of $\lfloor e^{RT} \rfloor$ Gaussian independent vectors with independent $\mathcal{CN}(0,1)$ elements each. Call X_0 the $T \times 1$ vector (dictionary element) transmitted by the source. The relay (node 1) listens to the source message for a duration of $t_0 T$ channel uses where t_0 is a fixed parameter. At the end of this period of time that we refer to as slot 0, the relay attempts to decode the source message. In case of success, the relay searches in its own dictionary the word corresponding to the source's message and it transmits it during the remainder of the frame (slot 1) to the destination. The dictionaries of the source and the relay are independent and identically distributed. Let us partition the word X_0 transmitted by the source as $X_0 = [X_{00}^{\mathrm{T}}, X_{01}^{\mathrm{T}}]^{\mathrm{T}}$ where the lengths of X_{00} and X_{01} are $t_0 T$ and $t_1 T$ respectively with $t_1 = 1 - t_0$. The signal of size $t_0 T$ received by the relay during slot 0 writes

$$Y_{1,0} = \sqrt{\alpha_0 \rho} H_{01} X_{0,0} + V_{1,0}$$

The parameter ρ will represent the total power spent by the source and the relay to transmit the message as we shall see in a moment. The gain $\sqrt{\alpha_0}$ is an amplitude gain applied by the source. Recall that the random vector $V_{1,0}$ represents the unit variance AWGN received by the relay. Assuming that the relay has a perfect knowledge of the channel H_{01}, it will be able to decode the source message if the event $\mathcal{E}_{\{1\}} = \{\omega \ : \ t_0 \log(1 + \alpha_0 \rho G_{01}(\omega)) > R\}$ is realized. In case $\mathcal{E}_{\{1\}}$ is realized, the relay will transmit during slot 1 the signal $\sqrt{\alpha_1 \rho} X_{11}$ of length $t_1 T$ where $\sqrt{\alpha_1}$ is the amplitude gain of the relay. In that case, the destination receives the signal $Y_2 = [Y_{20}^{\mathrm{T}}, Y_{21}^{\mathrm{T}}]^{\mathrm{T}}$ given by the equation

$$Y_2 = \begin{bmatrix} Y_{20} \\ Y_{21} \end{bmatrix} = \sqrt{\rho} \underbrace{\begin{bmatrix} \sqrt{\alpha_0} H_{02} \mathbf{I}_{t_0 T} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \sqrt{\alpha_0} H_{02} \mathbf{I}_{t_1 T} & \sqrt{\alpha_1} H_{12} \mathbf{I}_{t_1 T} \end{bmatrix}}_{\mathbf{H}_{\mathcal{E}_1}} \begin{bmatrix} X_{00} \\ X_{01} \\ X_{11} \end{bmatrix} + \begin{bmatrix} V_{20} \\ V_{21} \end{bmatrix}$$

where $V_2 = [V_{20}^{\mathrm{T}}, V_{21}^{\mathrm{T}}]^{\mathrm{T}}$ is the unit variance AWGN received by the destination. Notice that the probability distribution of the vector $[X_{00}^{\mathrm{T}}, X_{01}^{\mathrm{T}} X_{11}^{\mathrm{T}}]^{\mathrm{T}}$ is $\mathcal{CN}(\mathbf{0}, \mathbf{I}_{(1+t_1)T})$. Conditionally to the event $\mathcal{E}_{\{1\}}$, the outage probability $P_{\mathrm{o},1}$ for the destination is therefore

$$\begin{aligned} P_{\mathrm{o},1} &= \mathbb{P}\left[\log\det(\rho \mathbf{H}_{\mathcal{E}_1} \mathbf{H}_{\mathcal{E}_1}^* + \mathbf{I}) \leq RT \ \| \ \mathcal{E}_1\right] \\ &= \mathbb{P}\left[t_0 \log(1 + \alpha_0 \rho G_{02}) + t_1 \log(1 + \alpha_0 \rho G_{02} + \alpha_1 \rho G_{12}) \leq R\right] \ . \end{aligned}$$

In case the relay does not succeed in decoding the source message, which corresponds to the complementary event $\overline{\mathcal{E}}_{\{1\}}$, the destination simply receives

$$Y_2 = \begin{bmatrix} Y_{2,0} \\ Y_{2,1} \end{bmatrix} = \sqrt{\rho} \begin{bmatrix} \sqrt{\alpha_0} H_{02} \mathbf{I}_{t_0 T} & 0 \\ 0 & \sqrt{\alpha_0} H_{02} \mathbf{I}_{t_1 T} \end{bmatrix} \begin{bmatrix} X_{0,0} \\ X_{0,1} \end{bmatrix} + \begin{bmatrix} V_{2,0} \\ V_{2,1} \end{bmatrix} .$$

Therefore, conditionally to $\overline{\mathcal{E}}_{\{1\}}$, the outage probability $P_{\mathrm{o},2}$ is

$$P_{\mathrm{o},2} = \mathbb{P}\left[\log(1 + \alpha_0 \rho G_{02}) \leq R\right] .$$

In conclusion, the outage probability P_{o} associated with this protocol is

$$P_{\mathrm{o}} = P_{\mathrm{o},1} \mathbb{P}[\mathcal{E}_{\{1\}}] + P_{\mathrm{o},2} \mathbb{P}[\overline{\mathcal{E}}_{\{1\}}] = P_{\mathrm{o},1}(1 - P_{\mathrm{or}}) + P_{\mathrm{o},2} P_{\mathrm{or}} \quad (1)$$

where $P_{\mathrm{or}} = \mathbb{P}\left[\overline{\mathcal{E}}_{\{1\}}\right] = \mathbb{P}\left[t_0 \log(1 + \alpha_0 \rho G_{01}) \leq R\right]$ is the relay's outage probability.

Before analyzing this outage probability, for more clarity we compare this DF protocol with the so-called Dynamic Decode and Forward (DDF) protocol introduced in [21]. A relay which operates according to the DDF protocol has the knowledge of the channel G_{01}. With this information at hand, the relay waits until the moment $t_{\mathrm{DDF}} < 1$ where the "instantaneous" mutual information $I_{01}(t_{\mathrm{DDF}}) = t_{\mathrm{DDF}} \log(1 + \alpha_0 \rho G_{01})$ outnumbers the rate R when this is is possible. At this moment, the relay decodes the information and sends it towards the destination during the $1 - t_{\mathrm{DDF}}$ remaining seconds. In the DF protocol studied in this paper, the moment t_0 depends on the channel statistics only while in the DDF protocol, the moment t_{DDF} depends on the source-relay channel realization. These two protocols are strongly different, both theoretically and practically.

Getting back to our subject, we need to show that $\rho^2 P_{\mathrm{o}}$ converges as $\rho \to \infty$ and to derive the outage gain factor ξ_{DF} given by $\xi_{\mathrm{DF}} = \lim_{\rho \to \infty} \rho^2 P_{\mathrm{o}}$. We have

$$P_{\mathrm{o},1} = \int_{\mathbb{R}_+^2} \mathbf{1}\{t_0 \log(1 + \alpha_0 \rho x_0) + t_1 \log(1 + \alpha_0 \rho x_0 + \alpha_1 \rho x_1) \leq R\}(x_0, x_1) f_{G_{02}}(x_0) f_{G_{12}}(x_1) \, dx_0 \, dx_1$$

By making the changes of variables $u_0 = \alpha_0 \rho x_0$ and $u_1 = \alpha_1 \rho x_1$ we obtain

$$P_{\mathrm{o},1} = \frac{1}{\alpha_0 \alpha_1 \rho^2} \int_{\mathbb{R}_+^2} \mathbf{1}\{t_0 \log(1+u_0) + t_1 \log(1+u_0+u_1) \leq R\} \, f_{G_{02}}\left(\frac{u_0}{\alpha_0 \rho}\right) f_{G_{12}}\left(\frac{u_1}{\alpha_1 \rho}\right) \, du_0 \, du_1 \ . \quad (2)$$

The function $\varphi(u_0, u_1, \rho) = f_{G_{02}}\left(u_0/(\alpha_0 \rho)\right) f_{G_{12}}\left(u_1/(\alpha_1 \rho)\right)$ satisfies $\varphi(u_0, u_1, \rho) \xrightarrow[\rho \to \infty]{} c_{02} \, c_{12}$ by the assumptions of Section I. Assume $t_1 > 0$ and let $\mathcal{C}$ be the compact subset of $\mathbb{R}_+^2$ defined as $\mathcal{C} = \{(u_0, u_1) \in \mathbb{R}_+^2, t_0 \log(1 + u_0) + t_1 \log(1 + u_0 + u_1) \leq R\}$. As $f_{G_{02}}$ and $f_{G_{12}}$ are right continuous at zero, $\varphi(u_0, u_1, \rho)$ is bounded on $\mathcal{C}$ for ρ large enough. Therefore, we can apply Lebesgue's Dominated Convergence Theorem (DCT) to the integral in the right hand member of (2) to obtain

$$\lim_{\rho\to\infty} \rho^2 P_{\mathrm{o},1} = \frac{c_{02}c_{12}}{\alpha_0\alpha_1} \int_{\mathbb{R}_+^2} \mathbf{1}\{t_0 \log(1+u_0) + t_1 \log(1+u_0+u_1) \le R\}\; du_0\, du_1 \;. \tag{3}$$

In a similar manner, we also obtain

$$\lim_{\rho\to\infty} \rho P_{\mathrm{o},2} = \frac{c_{02}}{\alpha_0} \int_{\mathbb{R}_+} \mathbf{1}\{\log(1+u) \le R\}\, du \quad \text{and} \quad \lim_{\rho\to\infty} \rho P_{\mathrm{or}} = \frac{c_{01}}{\alpha_0} \int_{\mathbb{R}_+} \mathbf{1}\{t_0 \log(1+u) \le R\}\, du \;. \tag{4}$$

Plugging (3) and (4) into Equation (1) and using the fact that $t_0 + t_1 = 1$, we end up with

$$\begin{aligned} \xi_{\mathrm{DF}} \;=\; & \frac{c_{01}c_{02}}{\alpha_0^2} \left(\int_{\mathbb{R}_+} \mathbf{1}\{\log(1+u) \le R\}\, du \right) \left(\int_{\mathbb{R}_+} \mathbf{1}\{(1-t_1)\log(1+u) \le R\}\, du \right) \\ & + \frac{c_{02}c_{12}}{\alpha_0\alpha_1} \int_{\mathbb{R}_+^2} \mathbf{1}\{(1-t_1)\log(1+u_0) + t_1 \log(1+u_0+u_1) \le R\}\, du_0\, du_1 \end{aligned} \tag{5}$$

Our purpose is to minimize ξ_{DF} with respect to α_0, α_1 and t_1, this minimization being subject to $t_1 \in (0,1)$ and to a power constraint. Let us make explicit this constraint before going further. To this end, let us derive the total energy spent by the network to transmit a RT nat symbol. Whatever is the behavior of the relay, the source transmits the signal $(\sqrt{\alpha_0\rho}X_{00}, \sqrt{\alpha_0\rho}X_{01})$. Therefore, the energy E_0 spent by the source is $E_0 = \alpha_0\rho T$ Joules. The energy E_1 spent by the relay is $E_1 = \alpha_1\rho t_1 T \mathbb{P}[\mathcal{E}_{\{1\}}] = \alpha_1\rho t_1 T(1 - P_{\mathrm{or}})$. As $P_{\mathrm{or}} = \mathcal{O}(1/\rho)$ for large ρ by (4), the total energy E used to transmit one symbol satisfies $E = E_0 + E_1 = \rho T\left(\alpha_0 + \alpha_1 t_1(1 - P_{\mathrm{or}})\right) \approx \rho T\left(\alpha_0 + \alpha_1 t_1\right)$ for large ρ. Our power constraint for large SNR is therefore

$$\alpha_0 + \alpha_1 t_1 \le 1 \;. \tag{6}$$

Notice that this constraint becomes tight as $\rho \to \infty$ and is conservative for moderate values of ρ in the sense that it fixes a power threshold a little bit smaller than the affordable power for these values of ρ. Notice also that this constraint is not convex in α_0, α_1, t_1 because the function $g(\alpha_1, t_1) = \alpha_1 t_1$ is not convex. It will be convenient to replace it with a convex constraint by making the change of variables $\beta_0 = \alpha_0$ and $\beta_1 = \alpha_1 t_1$. With these new variables, the power constraint becomes

$$\beta_0 + \beta_1 \le 1 \;. \tag{7}$$

We have the following proposition:

Proposition 1: With respect to the parameters t_1, β_0 and β_1, the outage gain factor $\xi_{\mathrm{DF}}(t_1, \beta_0, \beta_1)$ for the single relay DF protocol described above is given by

$$\begin{aligned} \xi_{\mathrm{DF}}(t_1, \beta_0, \beta_1) \;=\; & \frac{c_{01}c_{02}}{\beta_0^2} \left(\exp(R) - 1\right) \left(\exp\left(\frac{R}{1-t_1} \right) - 1 \right) \\ & + \frac{c_{02}c_{12}t_1}{\beta_0\beta_1} \left(\frac{1}{4t_1 - 2} \exp(2R) - \frac{t_1}{2t_1 - 1} \exp\left(\frac{R}{t_1} \right) + \frac{1}{2} \right) \;. \end{aligned} \tag{8}$$

Moreover, the function $\xi_{\rm DF}(t_1,\beta_0,\beta_1)$ is convex in the domain $(t_1,\beta_0,\beta_1)\in(0,1)\times(0,\infty)^2$.

Equation (8) can be obtained by expanding the integrals in the right hand member of (5) and by replacing α_i with β_i for all i. We shall provide details about these derivations and prove convexity directly in the general N-relay case. Indeed, Proposition 1 is a particular case of Proposition 2 below. We remark that the outage probability minimization reduces to minimizing the right hand of (8) given the constraint (7). This reduces to minimizing $\xi_{\rm DF}$ on the line segment of $\mathbb{R}_+^2$ defined by $\beta_0+\beta_1=1$, *i.e.*, the constraint (7) is met with equality. The function $\xi_{\rm DF}(t_1,\beta_0,1-\beta_0)$ defined on the open square $(0,1)^2$ is convex as it coincides with the restriction of $\xi_{\rm DF}(t_1,\beta_0,\beta_1)$ to that line segment. Furthermore, it is clear that $\xi_{\rm DF}(t_1,\beta_0,1-\beta_0)$ goes to infinity on the frontier of $(0,1)^2$. Therefore, the minimum is in the interior of $(0,1)^2$, and can be obtained easily, for instance by a suitable descent method [22].

B. Outage Probability in the N–Relay case

In this paragraph we turn to the study of a DF protocol in the N–relay case. The protocol we shall consider is illustrated by Figure 1. We have $N+1$ slots numbered from 0 to N, slot n having the duration t_nT. The source transmits during all the frame. Relay n transmits during slot n if it succeeds in decoding the signals sent in slots 0 to $n-1$ by the source and by those active relays among relays 1 to $n-1$. Source and relays dictionaries are independent.

Let $\mathcal{R}$ be a subset of $\{1,\ldots,N\}$ and $\overline{\mathcal{R}}$ be its complement in $\{1,\ldots,N\}$. Denote by $\mathcal{E}_{\mathcal{R}}$ the event that we define in a somehow informal manner as "the relays that belong to $\mathcal{R}$ decode successfully the source message and the relays that belong to $\overline{\mathcal{R}}$ fail to decode this message". The collection of the 2^N events $\mathcal{E}_{\mathcal{R}}$ where $\mathcal{R}$ spans the subsets of $\{1,\ldots,N\}$ is a partition of the probability space Ω. By consequence, the outage probability $P_{\rm o}$ can be written as

$$P_{\rm o}=\sum_{\mathcal{R}\subset\{1,\ldots,N\}}\mathbb{P}\left[\text{outage}\|\mathcal{E}_{\mathcal{R}}\right]\mathbb{P}\left[\mathcal{E}_{\mathcal{R}}\right]\ . \tag{9}$$

where $\mathbb{P}\left[\text{outage}\|\mathcal{E}_{\mathcal{R}}\right]$ is the probability of the outage event conditional to the event $\mathcal{E}_{\mathcal{R}}$. Let us give the expression of this conditional probability. Write $\mathcal{R}=\{n_1,n_2,\ldots,n_{|\mathcal{R}|}\}$ where $|\mathcal{R}|$ is the number of elements of $\mathcal{R}$. Generalizing the case $N=1$, we shall assume that when a relay n belongs to $\mathcal{R}$, it will transmit the signal $\sqrt{\alpha_n\rho}X_{nn}$ during slot n. Let $X=[X_{0,0},X_{0,1},\ldots,X_{0,N+1},X_{n_1,n_1},\ldots,X_{n_{|\mathcal{R}|},n_{|\mathcal{R}|}}]^{\rm T}$ and $V_{N+1}=[V_{N+1,0}^{\rm T},\ldots,V_{N+1,N}^{\rm T}]^{\rm T}$. As a result, when $\omega\in\mathcal{E}_{\mathcal{R}}$ the signal $Y_{N+1}=[Y_{N+1,0}^{\rm T},\ldots,Y_{N+1,N}^{\rm T}]^{\rm T}$

received by the destination can be written as

$$Y_{N+1} = \sqrt{\rho}\begin{bmatrix} \sqrt{\alpha_0}H_{0,N+1}\mathbf{I}_{t_0T} & & & \\ & \sqrt{\alpha_0}H_{0,N+1}\mathbf{I}_{t_1T} & \sqrt{\alpha_{n_1}}H_{n_1,N+1}\mathbf{I}_{t_{n_1}T} & \\ & \ddots & & \sqrt{\alpha_{n_{|\mathcal{R}|}}}H_{n_{|\mathcal{R}|},N+1}\mathbf{I}_{t_{n_{|\mathcal{R}|}}T} \\ & \sqrt{\alpha_0}H_{0,N+1}\mathbf{I}_{t_NT} & & \end{bmatrix} X + V_{N+1}$$

During slot 0 and slots $m \in \overline{\mathcal{R}}$, the destination receives a signal from the source only. During a slot $i \in \mathcal{R}$, the destination receives a signal from the source plus an independent signal sent by relay i. With these conditions, it is not difficult to verify that the conditional probability $\mathbb{P}\left[\text{outage}\|\mathcal{E}_{\mathcal{R}}\right]$ has the expression

$$\mathbb{P}\left[\text{outage}\|\mathcal{E}_{\mathcal{R}}\right] = \mathbb{P}\left[\left(t_0 + \sum_{m\in\overline{\mathcal{R}}} t_m\right)\log\left(1+\alpha_0\rho G_{0,N+1}\right) + \sum_{i\in\mathcal{R}} t_i \log\left(1+\alpha_0\rho G_{0,N+1} + \alpha_i\rho G_{i,N+1}\right) \le R\right] . \quad (10)$$

We now turn to the derivation of $\mathbb{P}[\mathcal{E}_{\mathcal{R}}]$. Let us denote by $\mathcal{D}_m^{(\mathcal{R})}$ the event "relay m decodes successfully" or the event "relay m fails to decode" according to whether $m \in \mathcal{R}$ or $m \notin \mathcal{R}$ respectively. We clearly have $\mathcal{E}_{\mathcal{R}} = \mathcal{D}_1^{(\mathcal{R})} \cap \mathcal{D}_2^{(\mathcal{R})} \cap \cdots \cap \mathcal{D}_N^{(\mathcal{R})}$ and therefore

$$\mathbb{P}\left[\mathcal{E}_{\mathcal{R}}\right] = \mathbb{P}\left[\mathcal{D}_1^{(\mathcal{R})}\right]\mathbb{P}\left[\mathcal{D}_2^{(\mathcal{R})}\|\mathcal{D}_1^{(\mathcal{R})}\right]\cdots\mathbb{P}\left[\mathcal{D}_N^{(\mathcal{R})}\|\mathcal{D}_1^{(\mathcal{R})}\cap\cdots\cap\mathcal{D}_{N-1}^{(\mathcal{R})}\right] . \quad (11)$$

Recall that relay m relies on the signal transmitted by the source during slots $0,\dots,m-1$ and the signal transmitted by those among relays $1,\dots,m-1$ which are active. One can therefore verify that the conditional probability $\mathbb{P}\left[\mathcal{D}_m^{(\mathcal{R})}\|\mathcal{D}_1^{(\mathcal{R})}\cap\cdots\cap\mathcal{D}_{m-1}^{(\mathcal{R})}\right]$ is written as

$$\mathbb{P}\left[\mathcal{D}_m^{(\mathcal{R})}\|\mathcal{D}_1^{(\mathcal{R})}\cap\cdots\cap\mathcal{D}_{m-1}^{(\mathcal{R})}\right] = \mathbb{P}\left[\left(t_0 + \sum_{k\in\overline{\mathcal{R}},k<m} t_k\right)\log\left(1+\alpha_0\rho G_{0,m}\right) + \sum_{i\in\mathcal{R},i<m} t_i\log\left(1+\alpha_0\rho G_{0,m}+\alpha_i\rho G_{i,m}\right) \underset{m\in\mathcal{R}}{\overset{m\in\overline{\mathcal{R}}}{\lesseqgtr}} R\right] . \quad (12)$$

Equations (9–12) provide the expression of the outage probability for every value of ρ. Our purpose is to derive the expression of the outage gain factor ξ_{DF} given by $\xi_{\mathrm{DF}} = \lim_{\rho\to\infty}\rho^{N+1}P_{\mathrm{o}}$ and to minimize it with respect to the powers $(\alpha_i)_{i=0,\ldots,N+1}$ and the slot durations $(t_i)_{i=0,\ldots,N+1}$. The constraints on these parameters are the positivity constraints, the time constraint

$$t_1 + \cdots + t_N < 1 \tag{13}$$

where we put $t_0 = 1-(t_1+\cdots+t_N)$, and the power constraint at high SNR $\alpha_0+\alpha_1 t_1+\alpha_2 t_2+\cdots+\alpha_N t_N \leq 1$ which generalizes (6). Similarly to the single relay case, we make this last constraint convex by putting $\beta_0 = \alpha_0$ and $\beta_n = \alpha_n t_n$ for $n = 1, \ldots, N$. The power constraint becomes then

$$\beta_0 + \beta_1 + \cdots + \beta_N \leq 1 \ . \tag{14}$$

Let us write the outage gain factor as $\xi_{\rm DF} = \xi_{\rm DF}(t_1, \ldots, t_N, \beta_0, \ldots, \beta_N)$. It is given by the following proposition, which is the main result of this section:

Proposition 2: The outage gain factor $\xi_{\rm DF}(t_1, \ldots, t_N, \beta_0, \ldots, \beta_N)$ for the DF protocol described in this section is given by

$$\xi_{\rm DF}(t_1, \ldots, t_N, \beta_0, \ldots, \beta_N) = c_{0,N+1} \sum_{n=1}^{N+1} \frac{\prod_{m=1}^{n-1} c_{0,m}}{\beta_0^n} \left(\prod_{m=n}^{N} \frac{c_{m,N+1}}{\beta_m} \right) \left(\prod_{m=1}^{n-1} \left(\exp\left(\frac{R}{1-\sum_{k=m}^N t_k} \right) - 1 \right) \right) \mathcal{I}_n \tag{15}$$

with

$$\mathcal{I}_n = \int_{\mathbb{R}_+^{N-n+2}} \mathbf{1}\left\{ \sum_{m=n}^{N+1} v_m \leq R \right\} \exp\left(\frac{v_n}{t_n} + \cdots + \frac{v_N}{t_N} + (N-n+2) v_{N+1} \right) \prod_{m=n}^{N+1} dv_m \ . \tag{16}$$

The function $\xi_{\rm DF}(t_1, \ldots, t_N, \beta_0, \ldots, \beta_N)$ is convex in the convex set $\mathcal{S}_N \times (0,\infty)^{N+1}$ where $\mathcal{S}_N$ is the subset of $(0,\infty)^N$ delineated by the constraint (13).

The proof of Proposition 2 is drawn in Appendix II-A. Notice that the result and the proof of Proposition 2 can be rather easily modified and adapted to DF protocols other than the one described here such as the so called repetition or the space-time protocols considered in [3].

In order to obtain $\xi_{\rm DF}$ in practice, one has to compute the integrals $\mathcal{I}_n$ given by Equation (16). To that end, one can use the following lemma:

Lemma 1: Let $J_K(a_0, \ldots, a_K, R) : \mathbb{R}^{K+1} \times \mathbb{R}_+ \to \mathbb{R}_+$ be the function defined as

$$J_K(a_0, \ldots, a_K, R) = \int_{\mathbb{R}_+^{K+1}} \mathbf{1}\{x_0 + \cdots + x_K \leq R\} \exp(a_0 x_0 + \cdots + a_K x_K) \prod_{k=0}^{K} dx_k. \tag{17}$$

When parameters $a_0, \cdots, a_K$ are all distinct, $J_K(a_0, \dots, a_K, R)$ is given by

$$J_K(a_0, \dots, a_K, R) = \sum_{k=0}^{K} \frac{\eta(k, K)}{a_k} \left(\exp(a_k R) - 1\right) \tag{18}$$

where $(\eta(0, i), \dots, \eta(i, i))_{i=0,\dots,K}$ is a triangular array of real numbers given by the following recurrence: $\eta(0, 0) = 1$, $\eta(k, i) = \eta(k, i-1)/(a_k - a_i)$ for $k = 0, \dots, i-1$, and $\eta(i, i) = -\sum_{k=0}^{i-1} \eta(k, i)$.

The proof of this lemma is given in Appendix I. As $\mathcal{I}_n = J_{N+1-n}(t_N^{-1}, \dots, t_n^{-1}, N-n+2, R)$, Lemma 1 provides an easy way to compute the expression of ξ_{DF}. The result of Lemma 1 is formally suited to the case where parameters $a_0, \dots, a_K$ of the function J_K are all different. Notice that expressions for the cases where at least two of them are equal can be obtained by continuous extension. However, in those cases it is easier to work out directly the expression (16).

Generalizing the single relay case, at the minimum of ξ_{DF} the β_i belong to the hyper plane $\beta_0 + \cdots + \beta_N = 1$. By consequence, the problem reduces to minimizing the convex function with $2N$ parameters $\xi_{\mathrm{DF}}(t_1, \dots, t_N, \beta_0, \dots, \beta_{N-1}, 1 - \sum_{i=0}^{N-1} \beta_i)$ on the constraint set $\sum_{i=1}^{N} t_i < 1$ and $\sum_{i=0}^{N-1} \beta_i < 1$. The function ξ_{DF} goes to infinity at the frontier of this set. The minimum is in its interior and can be found by a descent method [22].

III. The AF Protocol

Similarly to the DF case, we first consider an AF protocol with a single relay in order to help the reader to get some insight on the proposed approach.

A. Outage Probability in the Single Relay case

One AF protocol frequently considered in the literature is the following [5], [21]: the source transmits its codeword during the whole frame of length T. The relay saves in its memory the signal it receives from the source during the first half of the frame. Then the relay applies a gain to this signal and transmits it during the second half of the frame. Here, we consider a slightly more general model: the relay does not necessarily consider the signal received from the source during the first $T/2$ channel uses. Instead, it just considers a section of this signal of length $t_1 T$ with $t_1 \leq 1/2$, and one of our purposes will be to find the value of t_1 that minimizes the outage gain factor. As is shown on figure 2 (with $N = 1$) , in general we now have three slots instead of two. The lengths of these slots are $t_0' T$, $t_1 T$ and $t_1 T$ respectively, with $t_0' + 2t_1 = 1$.

During slots 0 and 1, the destination receives Y_{20} and Y_{21} with dimensions $t_0' T$ and $t_1 T$ respectively. These signals are given by $Y_{2i} = \sqrt{\alpha_0 \rho} H_{02} X_{0i} + V_{2i}$ for $i = 0, 1$, where $\alpha_0 \rho$ is the power spent by the source as in the previous sections. During slot 1, the relay receives the signal Y_{11} with length t_1T given by the equation $Y_{11} = \sqrt{\alpha_0\rho}H_{01}X_{01} + V_{11}$. During slot 2, the relay transmits $\sqrt{\gamma_1}Y_{11}$ towards the destination where γ_1 is the power gain applied by the relay. We assume as above that $\alpha_1\rho$ is the power transmitted by the relay. As $\mathbb{E}[|Y_{11}|^2 \| H_{01}] = \alpha_0\rho G_{01} + 1$, the gain γ_1 is given by

$$\gamma_1 = \frac{\alpha_1\rho}{\alpha_0\rho G_{01} + 1} \ . \tag{19}$$

During slot 2, the source transmits X_{02} and the destination receives the signal $Y_{22} = \sqrt{\alpha_0\rho}H_{02}X_{02} + \sqrt{\alpha_0\gamma_1\rho}H_{01}H_{12}X_{01} + \sqrt{\gamma_1}H_{12}V_{11} + V_{22}$ with length t_1T. Putting the signal received by the destination in a matrix form, we obtain

$$\begin{bmatrix} Y_{20} \\ Y_{21} \\ Y_{22} \end{bmatrix} = \sqrt{\alpha_0\rho} \begin{bmatrix} H_{02}\mathbf{I}_{t_0'T} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & H_{02}\mathbf{I}_{t_1T} & \mathbf{0} \\ \mathbf{0} & \sqrt{\gamma_1}H_{01}H_{12}\mathbf{I}_{t_1T} & H_{02}\mathbf{I}_{t_1T} \end{bmatrix} \begin{bmatrix} X_{00} \\ X_{01} \\ X_{02} \end{bmatrix} + \begin{bmatrix} V_{20} \\ V_{21} \\ V_{22} + \sqrt{\gamma_1}H_{12}V_{11} \end{bmatrix} ,$$

an equation that we write compactly as $Y_2 = \mathbf{H}X_0 + V_2$. Recall that all noises V_{ij} are white with $\mathcal{CN}(0,1)$ elements. The mutual information conditional to the channels associated with this model is then given by $I = \log\det(\mathbf{R}_Y\mathbf{R}_V^{-1})$ where $\mathbf{R}_Y$ and $\mathbf{R}_V$ are the covariance matrices

$$\begin{aligned} \mathbf{R}_Y &= \mathbb{E}[Y_2Y_2^* \| H_{01}, H_{02}, H_{12}] = \mathbf{H}\mathbf{H}^* + \mathbf{R}_V \quad \text{and} \\ \mathbf{R}_V &= \mathbb{E}[V_2V_2^* \| H_{01}, H_{12}] = \begin{bmatrix} \mathbf{I}_{t_0'T} & & \\ & \mathbf{I}_{t_1T} & \\ & & (1+\gamma_1 G_{12})\mathbf{I}_{t_1T} \end{bmatrix} . \end{aligned}$$

By expanding these expressions and by replacing γ_1 with its value in (19), we obtain after some simple computations

$$I = t_1T\log\left(1 + \alpha_0\rho G_{02} + \frac{\alpha_0\rho G_{02}(\alpha_0\rho G_{02}+1)(\alpha_0\rho G_{01}+1) + \alpha_0\alpha_1\rho^2 G_{01}G_{12}}{1 + \alpha_0\rho G_{01} + \alpha_1\rho G_{12}}\right) + t_0'T\log(1+\alpha_0\rho G_{02}) \ . \tag{20}$$

Our purpose is to obtain the outage gain factor ξ_{AF} given by $\xi_{\text{AF}} = \lim_{\rho\to\infty}\rho^2\mathbb{P}[I \le RT]$ where R is the targeted data rate.

We shall make here a heuristic and non rigorous derivation of ξ_{AF}. The rigorous mathematical derivations will be made directly in the N-relay case below. Typically, the outage corresponds to the two exclusive events $\mathcal{E}_1$ and $\mathcal{E}_2$ that we describe roughly (for the moment) as

$\mathcal{E}_1$: $I \le RT$, gains G_{02} and G_{01} are small (of order $1/\rho$), and G_{12} is not small.

$\mathcal{E}_2$: $I \le RT$, gains G_{02} and G_{12} are small (of order $1/\rho$), and G_{01} is not small.

Typically, the network is in outage when the source-destination channel and *one* of the source-relay or relay-destination channels are small. The probability that the three channels are small is indeed negligible. As $\mathcal{E}_1 \cap \mathcal{E}_2 = \emptyset$, the outage probability P_o satisfies $P_\text{o} \approx \mathbb{P}[\mathcal{E}_1] + \mathbb{P}[\mathcal{E}_2]$.

Let us consider now the fractional expression at the RHS of (20). For $\omega \in \mathcal{E}_1$, we have for large ρ

$$\frac{\alpha_0\rho G_{02}(\alpha_0\rho G_{02}+1)(\alpha_0\rho G_{01}+1)+\alpha_0\alpha_1\rho^2 G_{01}G_{12}}{1+\alpha_0\rho G_{01}+\alpha_1\rho G_{12}} \approx \alpha_0\rho G_{01}$$

while for $\omega \in \mathcal{E}_2$

$$\frac{\alpha_0\rho G_{02}(\alpha_0\rho G_{02}+1)(\alpha_0\rho G_{01}+1)+\alpha_0\alpha_1\rho^2 G_{01}G_{12}}{1+\alpha_0\rho G_{01}+\alpha_1\rho G_{12}} \approx \alpha_0\rho G_{02}(\alpha_0\rho G_{02}+1)+\alpha_1\rho G_{12}\ .$$

We therefore have

$$\begin{aligned}
\mathbb{P}[\mathcal{E}_1] &\approx \mathbb{P}\left[t_1\log\left(1+\alpha_0\rho G_{02}+\alpha_0\rho G_{01}\right)+t_0'\log(1+\alpha_0\rho G_{02}) \le R\right] \\
&= \int_{\mathbb{R}_+^2} \mathbf{1}\{t_1\log\left(1+\alpha_0\rho x+\alpha_0\rho y\right)+t_0'\log(1+\alpha_0\rho x)\le R\} f_{G_{02}}(x)f_{G_{01}}(y)dx\,dy \\
&= \frac{1}{\alpha_0^2\rho^2}\int_{\mathbb{R}_+^2} \mathbf{1}\{t_1\log\left(1+u+v\right)+t_0'\log(1+u)\le R\} f_{G_{02}}\left(\frac{u}{\alpha_0\rho}\right) f_{G_{01}}\left(\frac{v}{\alpha_0\rho}\right) du\,dv \\
&\overset{\rho\to\infty}{\sim} \frac{c_{01}c_{02}}{\alpha_0^2\rho^2}\int_{\mathbb{R}_+^2} \mathbf{1}\{t_1\log\left(1+u+v\right)+t_0'\log(1+u)\le R\} du\,dv \qquad (21)
\end{aligned}$$

similarly to the DF case above (passage from (2) to (3)). We also have

$$\begin{aligned}
\mathbb{P}[\mathcal{E}_2] &\approx \mathbb{P}\left[t_1\log\left((1+\alpha_0\rho G_{02})^2+\alpha_1\rho G_{12}\right)+t_0'\log(1+\alpha_0\rho G_{02})\le R\right] \\
&\overset{\rho\to\infty}{\sim} \frac{c_{12}c_{02}}{\alpha_0\alpha_1\rho^2}\int_{\mathbb{R}_+^2} \mathbf{1}\{t_1\log\left((1+u)^2+v\right)+t_0'\log(1+u)\le R\} du\,dv\ . \qquad (22)
\end{aligned}$$

As a conclusion, ξ_AF is written

$$\begin{aligned}
\xi_\text{AF} &= \frac{c_{01}c_{02}}{\alpha_0^2}\int_{\mathbb{R}_+^2} \mathbf{1}\left\{t_1\log\left(1+u+v\right)+t_0'\log(1+u)\le R\right\} du\,dv \qquad (23)\\
&+ \frac{c_{12}c_{02}}{\alpha_0\alpha_1}\int_{\mathbb{R}_+^2} \mathbf{1}\left\{t_1\log\left((1+u)^2+v\right)+t_0'\log(1+u)\le R\right\} du\,dv
\end{aligned}$$

We need to develop the RHS of this expression and to minimize ξ_AF subject to the time constraints $t_1 \le 1/2$, $t_0'+2t_1=1$, and the power constraint $\alpha_0+\alpha_1 t_1 \le 1$. Similarly to the DF case, we make the power constraint convex by putting $\beta_0=\alpha_0$ and $\beta_1=\alpha_1 t_1$. We have the following proposition:

Proposition 3: The outage gain factor ξ_AF for the protocol described in this paragraph is

$$\begin{aligned}
\xi_\text{AF}(t_1,\beta_0,\beta_1) &= \frac{c_{01}c_{02}}{2\beta_0^2}\left(\frac{1-t_1}{3t_1-1}\exp\left(\frac{2R}{1-t_1}\right)-\frac{2t_1}{3t_1-1}\exp\left(\frac{R}{t_1}\right)+1\right) \\
&+\frac{c_{12}c_{02}t_1}{\beta_0\beta_1}\left(\frac{t_1}{3t_1-1}\left(\exp(3R)-\exp\left(\frac{R}{t_1}\right)\right)-\frac{1}{3}\left(\exp(3R)-1\right)\right) \qquad (24)
\end{aligned}$$

Moreover the function $\xi_{\rm AF}(t_1, \beta_0, \beta_1)$ is convex on $(0, 1/2] \times \mathbb{R}_+^2$.

By expanding the integrals in the RHS of (23) and replacing the α_i with the β_i, we obtain the expression (24). More generally, Proposition 3 is a particular case of Proposition 4 below.

B. Outage Probability in the N–Relay case

Generalizing the single relay protocol studied in the previous section, we consider now the N-relay protocol described by Figure 2. According to this protocol, the data frame of length T is divided into $2N + 1$ slots. Slot 0 has a length equal to $t'_0 T$. During this slot, the destination is the only node that listens to the source. Relay n (where $n = 1, \ldots, N$) listens to the source during slot $2n - 1$ which has the length $t_n T$. During slot $2n$ which has the same length $t_n T$, relay n transmits an amplified version $X_{n,2n} = \sqrt{\gamma_n} Y_{n,2n-1}$ of the signal $Y_{n,2n-1}$ received by that relay during slot $2n - 1$. Here γ_n is the power gain factor applied by relay n.

Before going further, let us note that a version of this protocol with $t'_0 = 0$ and $t_n = 1/(2N)$ has been considered in [21] from the point of view of the so called Diversity Multiplexing Tradeoff (DMT). More sophisticated AF protocols for the N-relay case have also been studied in [23] from that same point of view. The derivation of the outage gain factor for those protocols, or at least an upper bound on this gain, is under study.

By a derivation identical to the single relay case, the signal received by the destination during the couple of slots $(2n - 1, 2n)$ is given by

$$\begin{bmatrix} Y_{N+1,2n-1} \\ Y_{N+1,2n} \end{bmatrix} = \sqrt{\alpha_0 \rho} \begin{bmatrix} H_{0,N+1} \mathbf{I}_{t_n T} & \mathbf{0} \\ \sqrt{\gamma_n} H_{0n} \mathbf{I}_{t_n T} & H_{0,N+1} \mathbf{I}_{t_n T} \end{bmatrix} \begin{bmatrix} X_{0,2n-1} \\ X_{0,2n} \end{bmatrix} + \begin{bmatrix} V_{N+1,2n-1} \\ V_{N+1,2n} + \sqrt{\gamma_n} H_{n,N+1} V_{n,2n-1} \end{bmatrix}.$$

Due to the fact that the power transmitted by relay n is $\alpha_n \rho$, the gain factor γ_n satisfies $\gamma_n = \frac{\alpha_n \rho}{1 + \alpha_0 \rho G_{0,n}}$ as in (19). In these conditions, by a derivation similar to the single relay case, the mutual information $I_n = I\left((X_{0,2n-1}, X_{0,2n}); (Y_{N+1,2n-1}, Y_{N+1,2n})\right)$ between the source and the destination during slots $(2n - 1, 2n)$ is shown to be

$$I_n = t_n T \log\left(1 + \alpha_0 \rho G_{0,N+1} + \frac{\alpha_0 \rho G_{0,N+1}(\alpha_0 \rho G_{0,N+1} + 1)(\alpha_0 \rho G_{0n} + 1) + \alpha_0 \alpha_n \rho^2 G_{0n} G_{n,N+1}}{1 + \alpha_0 \rho G_{0n} + \alpha_n \rho G_{n,N+1}}\right). \tag{25}$$

Denoting by $X_0 = [X_{00}^{\rm T}, \ldots, X_{0,2N}^{\rm T}]^{\rm T}$ and $Y_{N+1} = [Y_{N+1,0}^{\rm T}, \ldots, Y_{N+1,2N}^{\rm T}]^{\rm T}$ the signals sent by the source and received by the destination respectively during the whole frame, the mutual information I for the whole frame is given by

$$I = I(X_{00}; Y_{N+1,0}) + \sum_{n=1}^{N} I_n = t'_0 T \log(1 + \alpha_0 \rho G_{0,N+1}) + \sum_{n=1}^{N} I_n \tag{26}$$

where the I_n are given by Equation (25). Our purpose is to derive and minimize the outage gain factor $\xi_{\rm AF} = \lim_{\rho\to\infty} \rho^{N+1}\mathbb{P}[I \le RT]$ subject to the time and power constraints respectively written as

$$2\sum_{n=1}^{N} t_n \le 1 \quad \text{and} \quad \sum_{n=0}^{N} \beta_n \le 1 \ .$$

Here $t'_0 = 1 - 2\sum_{n=1}^N t_n$, and as usual $\beta_0 = \alpha_0$ and $\beta_n = \alpha_n t_n$ for $n \ge 1$. Writing the outage gain factor as $\xi_{\rm AF} = \xi_{\rm AF}(t_1, \ldots, t_N, \beta_0, \ldots, \beta_N)$, we have the following proposition:

Proposition 4: The outage gain factor for the AF protocol $\xi_{\rm AF} = \lim_{\rho\to\infty} \rho^{N+1}\mathbb{P}[I \le RT]$ described in this section is given by

$$\xi_{\rm AF}(t_1, \ldots, t_N, \beta_0, \ldots, \beta_N) = c_{0,N+1} \sum_{\Theta\subset\{1,\ldots,N\}} \frac{1}{\beta_0^{|\Theta|+1}} \prod_{n\in\Theta} \frac{c_{0,n}}{t_n} \prod_{m\in\overline{\Theta}} \frac{c_{m,N+1}}{\beta_m} \frac{1}{1-\sum_{n\in\Theta} t_n}$$
$$\int_{\mathbb{R}_+^{N+1}} \mathbf{1}\left\{\sum_{i=0}^{N} x_i \le R\right\} \exp\left(\frac{N+1+|\overline{\Theta}|}{1-\sum_{n\in\Theta} t_n} x_0 + \sum_{i=1}^{N} \frac{x_i}{t_i}\right) \prod_{i=0}^{N} dx_i \ . \quad (27)$$

This function is convex on the convex set $\mathcal{S}_N \times (0,\infty)^{N+1}$ where $\mathcal{S}_N$ is the subset of $(0,\infty)^N$ delineated by the constraint $\sum_{n=1}^N t_n \le 1/2$.

We note that the derivation of the integrals at the RHS of (27) is fairly simple thanks to Lemma 1 again. Notice also that when $N = 1$, the sum over Θ reduces to a sum over the two sets $\Theta = \emptyset$ and $\Theta = \{1\}$, and recovering Proposition 3 is straightforward. Proof of Proposition 4 is reported in Appendix III-A.

IV. Numerical Illustrations and Simulations

In this section, some of the results of Propositions 1 to 4 are illustrated. Figure 3 shows an example of the performance of the DF protocol described above in the single relay case. The channel distributions are the Rice distributions, *i.e.*, $H_{ij} = \mathcal{CN}(a_{ij}, \sigma_{ij}^2)$. The decay profile for all channels is described by the equations $|a_{ij}|^2 = C_1 d_{ij}^{-2}$ and $\sigma_{ij}^2 = C_2 d_{ij}^{-3}$ where d_{ij} is the distance between nodes i and j, and the constants C_1 and C_2 are chosen in such a way that $|a_{0,N+1}|^2 = \sigma_{0,N+1}^2 = 1/2$. The relay is on the source-destination line segment at a distance from the source equal to two thirds of the source-destination distance. The sought data rate is equal to 2 bits per channel use. The solid lines show the behavior of $\xi_{\rm DF}\rho^{-1}$ in the following four cases: the powers satisfy $\alpha_0 = \alpha_1 = 2/3$ and $t_1 = 1/2$, powers are optimized while t_1 is kept fixed at $1/2$, t_1 is optimized while powers are kept equal, and finally, powers and slot durations are both optimized. The dashed curves show the simulation results. In this particular configuration, optimizing powers and slot durations results in a gain of nearly 2 dB. This gain is maintained when we leave the asymptotic regime in the SNR.

Figure 4 shows the performance of the AF protocol. The experimental conditions are identical to those of Figure 3. Here, the gain due to optimization is about 1.4 dB. Furthermore, by comparing Figures 3 and 4, we notice that in this configuration, the DF outperforms the AF protocol with a gain of about 2.4 dB after optimization.

Figure 5 shows the SNR gain due optimization of $\xi_{\rm DF}$ and $\xi_{\rm AF}$ as a function of the distance between the relay and the source. Here, channels are Rayleigh channels with $\sigma^2_{ij} \propto d^{-3}_{ij}$. The dashed curves represent the SNR gain obtained by simulation for an outage probability set to 10^{-3}. We notice that the optimization is all the more useful as the relay is far from the source, and this effect is more pronounced when the DF protocol is used.

Figure 6 provides an example for the outage performance in the case $N = 2$ relays. The channels statistical description is identical to the one used for Figure 3. The relays are at one third and two thirds of the source-destination distance on the the source-destination line segment. Here also, the merit of optimization, as well as the merit of using the DF protocol are clear.

V. CONCLUSION

A technique for outage probability minimization has been proposed for wireless relaying protocol with a statistical knowledge of the channels. The minimization problem is a convex problem with respect to powers given to the transmitting nodes and to the slot durations. The proposed method is fairly generic and works for a large number of relaying protocols. Some future research directions are the following: it would be interesting to search for outage minimization techniques suitable for other classes of relaying protocols such as the Dynamic Decode and Forward [21] or th Compress and Forward [24]. Another research direction concerns relay networks with asynchronous relays, and more generally for networks involving frequency selective channels.

APPENDIX I
PROOF OF LEMMA 1

With the change of variables $v_0 = x_0, v_1 = x_0 + x_1, \ldots, v_K = x_0 + x_1 + \cdots + x_K$, the Right Hand Side (RHS) of (17) can be rewritten

$$J_K(a_0, \ldots, a_K, R) = \int_{0 \le v_0 \le v_1 \le \cdots \le v_K \le R} e^{(a_K - a_{K+1})v_K + \cdots + (a_1 - a_2)v_1 + (a_0 - a_1)v_1} \prod_{k=0}^{K} dv_k$$

where we put $a_{K+1} = 0$. Define the sequence of functions $S_i(v)$ as $S_0(v) = \exp((a_0 - a_1)v)$ and

$$S_i(v) = \exp((a_i - a_{i+1})v) \int_0^v S_{i-1}(u)du \quad \text{for } i = 1, \ldots, K \ . \tag{28}$$

Then we have

$$J_K(a_0, \ldots, a_K, R) = \int_0^R S_K(v)dv \tag{29}$$

The functions $S_i(v)$ can be written as $S_i(v) = \sum_{k=0}^{i} \eta(k,i) \exp((a_k - a_{i+1})v)$ where $\eta(0,0) = 1$ and the $\eta(k,i)$ satisfy the recurrence relation. Indeed, injecting this last expression of the $S_i(v)$ into the definition (28), we obtain

$$\sum_{k=0}^{i} \eta(k,i) \exp(a_k - a_{i+1})v = \sum_{k=0}^{i-1} \frac{\eta(k,i-1)}{a_k - a_i} \left(\exp((a_k - a_{i+1})v) - \exp((a_i - a_{i+1})v)\right)$$

which leads to the recurrence relation. Now it remains to develop the integral at the RHS of (29) to recover (18).

Appendix II
Proofs for the DF N-Relay Case (Section II-B)

A. *Proof of Proposition 2*

In the sequel, the notation $a(\rho) \overset{\rho\to\infty}{\sim} b\rho^{-n}$ stands for $\lim_{\rho\to\infty} \rho^n a(\rho) = b$. We begin by studying the behavior of the Right Hand Sides (RHS) of Equations (10) and (12) as $\rho \to \infty$. By a derivation similar to the one that was used to obtain (3) from (2) in the previous paragraph, we notice that the diversity order associated with the outage probability given by (10) is $|\mathcal{R}| + 1$ (indeed, $|\mathcal{R}| + 1$ different channels are at stake in this expression), and

$$\begin{aligned} \mathbb{P}\left[\text{outage}\|\mathcal{E}_{\mathcal{R}}\right] &\overset{\rho\to\infty}{\sim} \frac{c_{0,N+1}}{\alpha_0 \rho^{|\mathcal{R}|+1}} \left(\prod_{i\in\mathcal{R}} \frac{c_{i,N+1}}{\alpha_i}\right) \\ &\times \int_{\mathbb{R}_+^{|\mathcal{R}|+1}} \mathbf{1}\left\{(t_0 + \sum_{m\in\overline{\mathcal{R}}} t_m) \log(1 + x_0) + \sum_{i\in\mathcal{R}} t_i \log(1 + x_0 + x_i) \le R\right\} dx_0 \prod_{i\in\mathcal{R}} dx_i \end{aligned} \tag{30}$$

Let us consider now Equation (12). When $m \in \mathcal{R}$, it is clear that $\mathbb{P}\left[\mathcal{D}_m^{(\mathcal{R})} \| \mathcal{D}_1^{(\mathcal{R})} \cap \cdots \cap \mathcal{D}_{m-1}^{(\mathcal{R})}\right] \xrightarrow[\rho\to\infty]{} 1$. Alternatively, when $m \in \overline{\mathcal{R}}$,

$$\mathbb{P}\left[\mathcal{D}_m^{(\mathcal{R})} \| \mathcal{D}_1^{(\mathcal{R})} \cap \cdots \cap \mathcal{D}_{m-1}^{(\mathcal{R})}\right] \overset{\rho\to\infty}{\sim} \frac{c_{0m}}{\alpha_0 \rho^{1+|\{i\in\mathcal{R}, i<m\}|}} \left(\prod_{i\in\mathcal{R}, i<m} \frac{c_{im}}{\alpha_i}\right)$$
$$\times \int_{\mathbb{R}_+^{|\{i\in\mathcal{R},i<m\}|+1}} \mathbf{1}\left\{(t_0 + \sum_{k\in\overline{\mathcal{R}}, k<m} t_k)\log(1+x_0) + \sum_{i\in\mathcal{R}, i<m} t_i \log(1+x_0+x_i) \le R\right\} dx_0 \prod_{i\in\mathcal{R}, i<m} dx_i \tag{31}$$

Getting back to (9) and (11), and considering the asymptotic expressions (30) and (31), we obtain

$$\begin{aligned} P_{\mathrm{o}} &= \sum_{\mathcal{R}\subset\{1,\ldots,N\}} \mathbb{P}\left[\text{outage}\|\mathcal{E}_{\mathcal{R}}\right] \mathbb{P}\left[\mathcal{D}_1^{(\mathcal{R})}\right] \mathbb{P}\left[\mathcal{D}_2^{(\mathcal{R})} \| \mathcal{D}_1^{(\mathcal{R})}\right] \cdots \mathbb{P}\left[\mathcal{D}_N^{(\mathcal{R})} \| \mathcal{D}_1^{(\mathcal{R})} \cap \cdots \cap \mathcal{D}_{N-1}^{(\mathcal{R})}\right] \\ &\overset{\rho\to\infty}{\sim} \sum_{\mathcal{R}\in\{1,\ldots,N\}} \rho^{-\left(1+|\mathcal{R}|+\sum_{m\in\overline{\mathcal{R}}}(1+|\{i\in\mathcal{R},i<m\}|)\right)} \times (\text{a term independent of } \rho)\ . \end{aligned}$$

The dominating terms in this sum are the terms in $\rho^{-(N+1)}$. They correspond to the sole sets $\mathcal{R}_n$ defined for $n = 1, \ldots, N+1$ by

$$\mathcal{R}_1 = \{1, \ldots, N\}, \cdots, \mathcal{R}_n = \{n, \ldots, N\}, \cdots, \mathcal{R}_N = \{N\}, \mathcal{R}_{N+1} = \emptyset\ .$$

We therefore have

$$P_{\mathrm{o}} \overset{\rho\to\infty}{\sim} \sum_{n=1}^{N+1} \mathbb{P}\left[\text{outage}\|\mathcal{E}_{\mathcal{R}_n}\right] \mathbb{P}\left[\mathcal{D}_1^{(\mathcal{R}_n)}\right] \mathbb{P}\left[\mathcal{D}_2^{(\mathcal{R}_n)} \| \mathcal{D}_1^{(\mathcal{R}_n)}\right] \cdots \mathbb{P}\left[\mathcal{D}_{n-1}^{(\mathcal{R}_n)} \| \mathcal{D}_1^{(\mathcal{R}_n)} \cap \cdots \cap \mathcal{D}_{n-2}^{(\mathcal{R}_n)}\right]\ . \tag{32}$$

Notice that for these sets $\mathcal{R}_n$, the right hand member of (31) has a very simple expression : for $m < n$, we have

$$\mathbb{P}\left[\mathcal{D}_m^{(\mathcal{R}_n)} \| \mathcal{D}_1^{(\mathcal{R}_n)} \cap \cdots \cap \mathcal{D}_{m-1}^{(\mathcal{R}_n)}\right] \overset{\rho\to\infty}{\sim} \frac{c_{0m}}{\alpha_0 \rho} \int_{\mathbb{R}_+} \mathbf{1}\left\{(\sum_{k=0}^{m-1} t_k)\log(1+x) \le R\right\} dx \tag{33}$$

Although the solutions of these integrals are immediate, we shall keep the integrals temporarily in this form to simplify the proof of Proposition 2. From (30), (32), and (33), we obtain

$$\xi_{\mathrm{DF}} = \lim_{\rho\to\infty} \rho^{N+1} P_{\mathrm{o}} = c_{0,N+1} \sum_{n=1}^{N+1} \left(\frac{\prod_{m=1}^{n-1} c_{0,m}}{\alpha_0^n} \left(\prod_{m=1}^{n-1} \int_{\mathbb{R}_+} \mathbf{1}\left\{(\sum_{k=0}^{m-1} t_k)\log(1+x) \le R\right\} dx \right) \right.$$
$$\left. \times \left(\prod_{i=n}^{N} \frac{c_{i,N+1}}{\alpha_i}\right) \int_{\mathbb{R}_+^{N-n+2}} \mathbf{1}\left\{(\sum_{m=0}^{n-1} t_m)\log(1+x_0) + \sum_{i=n}^{N} t_i \log(1+x_0+x_i) \le R\right\} dx_0 \prod_{i=n}^{N} dx_i \right)\ .$$

By replacing t_0 with $1-\sum_{n=1}^{N} t_n$, and the α_i with the β_i, this last expression can be rewritten

$$\xi_{\mathrm{DF}}(t_1, t_2, \ldots, t_N, \beta_0, \beta_n, \ldots, \beta_N) = \\ c_{0,N+1} \sum_{n=1}^{N+1} \left(\prod_{m=1}^{n-1} c_{0,m} \right) \left(\prod_{m=n}^{N} c_{m,N+1} \right) \chi_n \left(T_n(t_1, t_2, \ldots, t_N, \beta_0, \beta_n, \ldots, \beta_N) \right) \quad (34)$$

where χ_n is the function defined on $(0,\infty)^{2N-n+2}$ for $n \leq N$ as

$$\chi_n(u_1, \ldots, u_N, \beta_0, \beta_n, \beta_{n+1}, \ldots, \beta_N) = \frac{1}{\beta_0^n} \left(\prod_{m=1}^{n-1} \int_{\mathbb{R}_+} \mathbf{1}\{u_m \log(1+x) \leq R\} dx \right) \\ \times \left(\prod_{i=n}^{N} \frac{u_i}{\beta_i} \right) \int_{\mathbb{R}_+^{N-n+2}} \mathbf{1} \left\{ (1 - \sum_{i=n}^{N} u_i) \log(1+x_0) + \sum_{i=n}^{N} u_i \log(1+x_0+x_i) \leq R \right\} dx_0 \prod_{i=n}^{N} dx_i \quad (35)$$

and T_n is the affine transformation

$$T_n(t_1, t_2, \ldots, t_N, \beta_0, \beta_n, \ldots, \beta_N) = \\ \left(1 - \sum_{k=1}^{N} t_k, 1 - \sum_{k=2}^{N} t_k, \ldots, 1 - \sum_{k=n-1}^{N} t_k, t_n, t_{n+1}, \ldots, t_N, \beta_0, \beta_n, \ldots, \beta_N \right) . \quad (36)$$

The following lemma is proven in Appendix II-B:

Lemma 2: The function χ_n defined by (35) can also be written

$$\chi_n = \int_{\mathbb{R}_+^{N+1}} \frac{1}{\beta_0^n \prod_{i=n}^{N} \beta_i} \left(\prod_{m=1}^{n-1} \frac{1}{u_m} \exp\left(\frac{v_m}{u_m} \right) \mathbf{1}\{v_m \in [0,R]\} \right) \\ \mathbf{1}\left\{ \sum_{m=n}^{N+1} v_m \leq R \right\} \exp\left(\frac{v_n}{u_n} + \cdots + \frac{v_N}{u_N} + (N-n+2) v_{N+1} \right) \prod_{m=1}^{N+1} dv_m \ . \quad (37)$$

Moreover, this function is convex on $(0,\infty)^{2N-n+2}$.

Equations (15)–(16) in the statement of Proposition 2 follow directly from (34–37). Concerning the convexity of ξ_{DF}, recall that the composition $f \circ T$ of a convex function f and an affine function T is convex. It results that ξ_{DF} is convex. This concludes the proof of Proposition 2.

B. Proof of Lemma 2

We begin with the following lemma:

Lemma 3: Let $\phi_1(x), \phi_2(x), \ldots, \phi_n(x)$ be real, positive and twice differentiable functions. If the functions ϕ_i satisfy $\phi_i \phi_i'' \geq (\phi_i')^2$ for all $i = 1, \ldots, n$, then the function $f : \mathbb{R}^n \to \mathbb{R}_+$ defined as $f(x_1, \ldots, x_n) = \prod_{i=1}^{n} \phi_i(x_i)$ is convex.

Proof: Define the sequence of functions $f_m : \mathbb{R}^m \to \mathbb{R}_+$ as $f_m(x_1,\ldots,x_m) = \prod_{i=1}^m \phi_i(x_i)$ for $m = 1,\ldots,n$. We shall show by recurrence that the functions f_m satisfy the matrix inequality $f_m\nabla^2 f_m - \nabla f_m \nabla^{\mathrm{T}} f_m \geq \mathbf{0}$ where ∇ is the gradient and ∇^2 is the Hessian matrix of f_m with respect to the variables $(x_1,\ldots,x_m)$, and the inequality is with respect to the symmetric nonnegative matrices ordering. As $f \equiv f_n$, we will then have $f\nabla^2 f - \nabla f \nabla^{\mathrm{T}} f \geq \mathbf{0}$, which implies that $\nabla^2 f \geq \mathbf{0}$, in other words, that f is convex.

By assumption, $f_1 \equiv \phi_1$ satisfies $f_1 f_1'' \geq (f_1')^2$. Assume that f_m satisfies $f_m\nabla^2 f_m - \nabla f_m \nabla^{\mathrm{T}} f_m \geq \mathbf{0}$. Let us show that f_{m+1} satisfies $f_{m+1}\nabla^2 f_{m+1} - \nabla f_{m+1} \nabla^{\mathrm{T}} f_{m+1} \geq \mathbf{0}$. The gradient of $f_{m+1}(x_1,\ldots,x_{m+1}) = f_m(x_1,\ldots,x_m)\phi_{m+1}(x_{m+1})$ writes

$$\nabla f_{m+1} = \begin{bmatrix} \phi_{m+1}\nabla f_m \\ f_m \phi'_{m+1} \end{bmatrix}$$

and the Hessian matrix $\nabla^2 f_{m+1}$ writes

$$\nabla^2 f_{m+1} = \begin{bmatrix} \phi_{m+1}\nabla^2 f_m & \phi'_{m+1}\nabla f_m \\ \phi'_{m+1}\nabla^{\mathrm{T}} f_m & \phi''_{m+1} f_m \end{bmatrix} .$$

We therefore have

$$f_{m+1}\nabla^2 f_{m+1} - \nabla f_{m+1}\nabla^{\mathrm{T}} f_{m+1} = \begin{bmatrix} \phi_{m+1}^2\left(f_m\nabla^2 f_m - \nabla f_m \nabla^{\mathrm{T}} f_m\right) & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & f_m^2\left(\phi_{m+1}\phi''_{m+1} - (\phi'_{m+1})^2\right) \end{bmatrix}$$

which is a nonnegative matrix by the recurrence assumption and $\phi_{m+1}\phi''_{m+1} - \left(\phi'_{m+1}\right)^2 \geq 0$. ■

Proof of Lemma 2: We begin by putting the integrals $\int_{u_m \log(1+x)\leq R} dx$ at the RHS of (35) under the form

$$\int_{\mathbb{R}_+} \mathbf{1}\{u_m \log(1+x) \leq R\} dx = \int_0^R \frac{1}{u_m} \exp\left(\frac{v}{u_m}\right) dv \tag{38}$$

by making the change of variable $v = u_m \log(1+x)$. Furthermore, the function

$$I(u_n,\ldots,u_N,\beta_n,\ldots,\beta_N) \stackrel{\text{def}}{=} \left(\prod_{i=n}^N \frac{u_i}{\beta_i}\right) \int_{\mathbb{R}_+^{N-n+2}} \mathbf{1}\left\{ \quad \right\} dx_0 \prod_{i=n}^N dx_i$$

(see the RHS of (35)) writes

$$\begin{aligned} I &= \left(\prod_{i=n}^N \frac{u_i}{\beta_i}\right) \int \mathbf{1}\left\{\sum_{i=n}^N u_i \log(1+\frac{x_i}{1+x_0}) + \log(1+x_0) \leq R\right\} dx_0 \prod_{i=n}^N dx_i \\ &= \frac{1}{\prod_{i=n}^N \beta_i} \int_{\mathbb{R}_+^{N-n+2}} \mathbf{1}\{\sum_{i=n}^{N+1} v_i \leq R\} \exp\left(\frac{v_n}{u_n} + \cdots + \frac{v_N}{u_N} + (N-n+2)v_{N+1}\right) \prod_{i=n}^{N+1} dv_i \end{aligned} \tag{39}$$

where the second equality is due to the change of variables

$$\begin{aligned} v_i &= u_i \log\left(1 + \frac{x_i}{1+x_0}\right) \quad \text{for } i = n, \ldots, N \\ v_{N+1} &= \log(1+x_0) \ . \end{aligned}$$

By using in addition the Identity (38), we recover the expression (37).

In order to prove the convexity of χ_n, we shall prove that the integrand of the at the RHS of (37) is convex in the variables $(u_1, \ldots, u_N, \beta_0, \beta_n, \beta_{n+1}, \ldots, \beta_N)$ for every value of $(v_1, \ldots, v_{N+1}) \in \mathbb{R}_+^{N+1}$. This integrand is a product of functions which have the u_m or the β_m as parameters. Those functions are of three sorts: $\varphi_{k_m}(\beta_m) = \beta_m^{-k_m}$ where $k_m \in \{1, n\}$, $\psi(v_m, u_m) = \exp(v_m/u_m)$ where $v_m \geq 0$, and $\phi(v_m, u_m) = \exp(v_m/u_m)/u_m$ where $v_m \geq 0$. One can easily verify the following identities:

$$\begin{aligned} \varphi_k(\beta)\varphi_k''(\beta) - \left(\varphi_k'(\beta)\right)^2 &= k\beta^{-2(k+1)} > 0, \\ \psi(v,u)\frac{\partial^2 \psi(v,u)}{\partial u^2} - \left(\frac{\partial \psi(v,u)}{\partial u}\right)^2 &= 2vu^{-3}e^{2v/u} > 0, \text{ and} \\ \phi(v,u)\frac{\partial^2 \phi(v,u)}{\partial u^2} - \left(\frac{\partial \phi(v,u)}{\partial u}\right)^2 &= u^{-4}\left(1 + 2vu^{-1}\right)e^{2v/u} > 0 \ . \end{aligned}$$

It results from these identities that the assumptions of Lemma 3 are satisfied. By consequence, the integrand at the RHS of (37) is convex in the parameters $(u_1, \ldots, u_N, \beta_0, \beta_n, \ldots, \beta_N)$, hence χ_n is convex on $(0, \infty)^{2N-n+2}$. Lemma 2 is proven.

Appendix III
Proofs for the AF N-Relay Case (Section III-B)

A. Proof of Proposition 4

We use a partition of the probability space where certain events within this partition lead to negligible outage probabilities and others do not. Choose a real number $\delta \in (0, 1/(N+1))$ and define the indicator functions $\mathbf{1}_{n,i}$ for $i = 1$ to 4 as

$$\begin{aligned} \mathbf{1}_{n,1} = \mathbf{1}\left\{G_{0n} \leq \rho^{\delta-1}\right\}\mathbf{1}\left\{G_{n,N+1} > \rho^{\delta-1}\right\} \quad &, \quad \mathbf{1}_{n,2} = \mathbf{1}\left\{G_{0n} > \rho^{\delta-1}\right\}\mathbf{1}\left\{G_{n,N+1} \leq \rho^{\delta-1}\right\} \\ \mathbf{1}_{n,3} = \mathbf{1}\left\{G_{0n} > \rho^{\delta-1}\right\}\mathbf{1}\left\{G_{n,N+1} > \rho^{\delta-1}\right\} \quad &, \quad \mathbf{1}_{n,4} = \mathbf{1}\left\{G_{0n} \leq \rho^{\delta-1}\right\}\mathbf{1}\left\{G_{n,N+1} \leq \rho^{\delta-1}\right\} \end{aligned} \tag{40}$$

As $\mathbf{1}_{n,1} + \mathbf{1}_{n,2} + \mathbf{1}_{n,3} + \mathbf{1}_{n,4} = 1$, we have

$$\begin{aligned} \mathbb{P}[I \leq RT] &= \mathbb{E}[\mathbf{1}\{I \leq RT\}] = \mathbb{E}\left[\mathbf{1}\{I \leq RT\}\prod_{n=1}^{N}(\mathbf{1}_{n,1} + \mathbf{1}_{n,2} + \mathbf{1}_{n,3} + \mathbf{1}_{n,4})\right] \\ &= \sum_{\boldsymbol{\nu} \in \{1,2,3,4\}^N} \mathbb{E}\left[\mathbf{1}\{I \leq RT\}\prod_{n=1}^{N}\mathbf{1}_{n,\nu(n)}\right] \end{aligned} \tag{41}$$

with $\boldsymbol{\nu} = (\nu(1), \ldots, \nu(N))$ is an N-uple of indices.

For instance, in the single relay case, we have $\mathbb{P}[I \leq RT] = \mathbb{E}[\mathbf{1}\{I \leq RT\}\mathbf{1}_{11}] + \mathbb{E}[\mathbf{1}\{I \leq RT\}\mathbf{1}_{12}] + \mathbb{E}[\mathbf{1}\{I \leq RT\}\mathbf{1}_{13}] + \mathbb{E}[\mathbf{1}\{I \leq RT\}\mathbf{1}_{14}] = \chi_1(\rho) + \chi_2(\rho) + \chi_3(\rho) + \chi_4(\rho)$. The exclusive events $\mathcal{E}_1$ and $\mathcal{E}_2$ presented in the previous section are more precisely defined as

$$\begin{aligned} \mathcal{E}_1 &= [I \leq RT, G_{01} \leq \rho^{\delta-1}, G_{12} > \rho^{\delta-1}], \\ \mathcal{E}_2 &= [I \leq RT, G_{01} > \rho^{\delta-1}, G_{12} \leq \rho^{\delta-1}], \end{aligned}$$

hence $\chi_1(\rho) = \mathbb{P}[\mathcal{E}_1]$ and $\chi_2(\rho) = \mathbb{P}[\mathcal{E}_2]$. When restricted to the case $N = 1$, the proof below will show that χ_3 and χ_4 are negligible for large ρ, and that χ_1 and χ_2 satisfy (21) and (22) respectively.

We now get back to the general case. Equation (41) comes from a partition of the probability space into 4^N events, each corresponding to a N-uple $\boldsymbol{\nu}$. Some of these events will result is a negligible or even null outage probability for large ρ. For instance, if there exists a relay n_0 for which $\nu(n_0) = 3$, the channels G_{0,n_0} and $G_{n_0,N+1}$ are “good”, and the probability of the event $[I < RT]$ will be zero for large ρ. This is formalized by the following lemma:

Lemma 4: For any $\boldsymbol{\nu} \in \{1, 2, 3, 4\}^N$ for which $\exists n_0,\ \nu(n_0) = 3$, there exists a constant $\rho_0 > 0$ such that

$$\mathbb{E}\left[\mathbf{1}\{I \leq RT\} \prod_{n=1}^{N} \mathbf{1}_{n,\nu(n)}\right] = 0 \quad \text{when } \rho > \rho_0 \ . \tag{42}$$

The proof of this lemma is given in Appendix III-B.

Thanks to this lemma, we can restrict the range of the admissible N-uples $\boldsymbol{\nu}$ and write

$$\rho^{N+1}\mathbb{P}[I \leq RT] = \sum_{\boldsymbol{\nu} \in \{1,2,4\}^N} \rho^{N+1}\mathbb{E}\left[\mathbf{1}\{I \leq RT\} \prod_{n=1}^{N} \mathbf{1}_{n,\nu(n)}\right] + o_\rho(1)$$

where $o_\rho(1)$ converges to zero as $\rho \to \infty$. Other events can be further neglected: assume there exists a relay n_0 for which $\nu(n_0) = 4$. The network is in outage when the source-destination channel is small and for every relay, at least one of the source-relay or relay-destination channels is small. If we count the source-destination channel and those of the relay channels liable for the outage, we obtain at least N channels excluding relay n_0. As $\nu(n_0) = 4$, both channels of relay n_0 are furthermore small (see (40)), and we end up with $N+2$ small channels at least. This happens with a probability negligible with respect to $\rho^{-(N+1)}$. This fact is formalized by the following lemma which proof is in Appendix III-C:

Lemma 5: For any $\boldsymbol{\nu} \in \{1, 2, 4\}^N$ for which $\exists n_0,\ \nu(n_0) = 4$, the following holds true

$$\lim_{\rho \to \infty} \rho^{N+1}\mathbb{E}\left[\mathbf{1}\{I \leq RT\} \prod_{n=1}^{N} \mathbf{1}_{n,\nu(n)}\right] = 0 \ . \tag{43}$$

Thanks to this lemma, we can write $\rho^{N+1}\mathbb{P}[I \leq RT] = \sum_{\boldsymbol{\nu}\in\{1,2\}^N} \rho^{N+1}\mathbb{E}\left[\mathbf{1}\{I \leq RT\}\prod_{n=1}^{N}\mathbf{1}_{n,\nu(n)}\right] + o_\rho(1)$. This equation can be rewritten

$$\rho^{N+1}\mathbb{P}[I \leq RT] = \sum_{\Theta\subset\{1,\ldots,N\}} \rho^{N+1}\int_{\Delta(\Theta)} f_{G_{0,N+1}}(x)\prod_{n=1}^{N}\left(f_{G_{0n}}(x_n)f_{G_{n,N+1}}(y_n)\right)dx\prod_{n=1}^{N}dx_n dy_n + o_\rho(1)$$

where $\Delta(\Theta)$ is defined as

$$\Delta(\Theta) = \left\{ \begin{array}{c|l} (x, x_1\ldots x_N, y_1\ldots y_N) & \forall n\in\Theta,\ x_n \leq \rho^{\delta-1} \text{ and } y_n > \rho^{\delta-1}, \\ \in \mathbb{R}_+^{2N+1} & \forall n\in\overline{\Theta},\ x_n > \rho^{\delta-1} \text{ and } y_n \leq \rho^{\delta-1}, \\ & \sum_{n=1}^{N} t_n \log\left(1+\alpha_0\rho x + \frac{\alpha_0\rho x(\alpha_0\rho x+1)(\alpha_0\rho x_n+1)+\alpha_0\alpha_n\rho^2 x_n y_n}{1+\alpha_0\rho x_n+\alpha_n\rho y_n}\right) \\ & \quad +t_0'\log(1+\alpha_0\rho x) \leq R \end{array} \right\}$$

and $\overline{\Theta}$ is the complementary set of Θ in $\{1,\ldots,N\}$. Now we make the change of variables $u = \alpha_0\rho x$, $\forall n\in\Theta, u_n = \alpha_0\rho x_n$, and $\forall m\in\overline{\Theta}, v_m = \alpha_m\rho y_m$ to obtain

$$\rho^{N+1}\mathbb{P}[I \leq RT] = \sum_{\Theta\subset\{1,\ldots,N\}} \frac{1}{\alpha_0^{|\Theta|+1}\prod_{m\in\overline{\Theta}}\alpha_m}\int_{\mathbb{R}_+^{2N+1}} J_\Theta\left(u,(u_n,y_n)_{n\in\Theta},(x_m,v_m)_{m\in\overline{\Theta}},\rho\right) du\prod_{n\in\Theta}du_n\,dy_n\prod_{m\in\overline{\Theta}}dx_m\,dv_m + o_\rho(1) \quad (44)$$

where the functions J_Θ are given by

$$\begin{aligned} J_\Theta = & f_{G_{0,N+1}}(u/\rho)\prod_{n\in\Theta}\mathbf{1}\{u_n \leq \rho^\delta\}\mathbf{1}\{y_n > \rho^{\delta-1}\}f_{G_{0n}}(u_n/\rho)f_{G_{n,N+1}}(y_n) \\ & \times\prod_{m\in\overline{\Theta}}\mathbf{1}\{x_m > \rho^{\delta-1}\}\mathbf{1}\{v_m \leq \rho^\delta\}f_{G_{0m}}(x_m)f_{G_{m,N+1}}(v_m/\rho) \\ & \times\mathbf{1}\left\{\sum_{n\in\Theta}t_n\log\left(1+u+\frac{u(u+1)(u_n+1)+\alpha_n\rho u_n y_n}{1+u_n+\alpha_n\rho y_n}\right)\right. \\ & \quad + \sum_{m\in\overline{\Theta}}t_m\log\left(1+u+\frac{u(u+1)(\alpha_0\rho x_m+1)+\alpha_0\rho x_m v_m}{1+\alpha_0\rho x_m+v_m}\right) \\ & \quad \left. + t_0'\log(1+u) \leq R\right\} \end{aligned} \quad (45)$$

One notices that J_Θ has a limit when $\rho \to \infty$ given by

$$\lim_{\rho\to\infty} J_\Theta = c_{0,N+1} \prod_{n\in\Theta} c_{0,n} f_{G_{n,N+1}}(y_n) \prod_{m\in\overline{\Theta}} c_{m,N+1} f_{G_{0,m}}(x_m)$$
$$\mathbf{1}\left\{\sum_{n\in\Theta} t_n \log\left(1+u+u_n\right) + \sum_{m\in\overline{\Theta}} t_m \log\left((1+u)^2+v_m\right) + t_0'\log(1+u) \le R\right\} \quad (46)$$

We have the following lemma which proof is provided in Appendix III-D:

Lemma 6: The following holds true:

$$\rho^{N+1}\mathbb{P}[I \le RT] = c_{0,N+1} \sum_{\Theta\subset\{1,\dots,N\}} \frac{\prod_{n\in\Theta} c_{0,n}}{\alpha_0^{|\Theta|+1}} \prod_{m\in\overline{\Theta}} \frac{c_{m,N+1}}{\alpha_m}$$
$$\int_{\mathbb{R}_+^{N+1}} \mathbf{1}\left\{\sum_{n\in\Theta} t_n \log\left(1+u+u_n\right) + \sum_{m\in\overline{\Theta}} t_m \log\left((1+u)^2+v_m\right) + t_0'\log(1+u) \le R\right\}$$
$$du \prod_{n\in\Theta} du_n \prod_{m\in\overline{\Theta}} dv_m \; + \; o_\rho(1) \quad (47)$$

Recovering Equation (27) from this lemma is a matter of change of variables. The indicator function at the RHS of (47) can be rewritten

$$\mathbf{1}\left\{\sum_{n\in\Theta} t_n \log\left(1+\frac{u_n}{1+u}\right) + \sum_{m\in\overline{\Theta}} t_m \log\left(1+\frac{v_m}{(1+u)^2}\right) + (1-\sum_{n\in\Theta} t_n)\log(1+u) \le R\right\} .$$

By making the change of variables $x_0 = (1-\sum_{n\in\Theta} t_n)\log(1+u)$, $x_n = t_n \log\left(1+\frac{u_n}{1+u}\right)$ for $n\in\Theta$, and $x_m = t_m \log\left(1+\frac{v_m}{(1+u)^2}\right)$ for $m\in\overline{\Theta}$, and by replacing α_i with β_i for all i, we obtain (27).

It remains to prove convexity. Fix a set Θ and consider the corresponding summand at the RHS of (27). Put $t_0 = 1-\sum_{n\in\Theta} t_n$. By a method similar to the one used in the proof of Lemma 3 in the appendix, one can easily prove that this summand is convex in $(t_0, t_1, \dots, t_N, \beta_0, \dots, \beta_N)$. Hence, its restriction to $(1-\sum_{n\in\Theta} t_n, t_1, \dots, t_N, \beta_0, \dots, \beta_N)$ is convex. By consequence, ξ_{AF} is a convex function. This concludes the proof of Proposition 4.

B. Proof of Lemma 4

We recall that if two functions g_1 and g_2 satisfy $g_1 \ge g_2$ then $\mathbf{1}\{g_1 \le R\} \le \mathbf{1}\{g_2 \le R\}$, a fact that we shall repeatedly use in this Appendix.

From the Expression (25-26) of the mutual information, it is clear that

$$I \ge t_{n_0} T \log\left(1+\frac{\alpha_0\alpha_{n_0}\rho^2 G_{0n_0} G_{n_0,N+1}}{1+\alpha_0\rho G_{0n_0}+\alpha_{n_0}\rho G_{n_0,N+1}}\right)$$

where n_0 is the relay for which $\nu(n_0) = 3$ as in the assumption. Therefore, the left hand side of (42) satisfies

$$\mathbb{E}\left[\mathbf{1}\{I \leq RT\} \prod_{n=1}^{N} \mathbf{1}_{n,\nu(n)}\right] \leq \mathbb{E}\left[\mathbf{1}\left\{t_{n_0} \log\left(1 + \frac{\alpha_0\alpha_{n_0}\rho^2 G_{0n_0} G_{n_0,N+1}}{1+\alpha_0\rho G_{0n_0} + \alpha_{n_0}\rho G_{n_0,N+1}}\right) \leq R\right\} \mathbf{1}_{n_0,\nu(n_0)}\right]$$
$$= \mathbb{E}\left[\mathbf{1}\left\{\frac{\alpha_0\alpha_{n_0}\rho^2 G_{0n_0} G_{n_0,N+1}}{1+\alpha_0\rho G_{0n_0} + \alpha_{n_0}\rho G_{n_0,N+1}} \leq e^{R/t_{n_0}} - 1\right\} \mathbf{1}\left\{G_{0n_0} > \rho^{\delta-1}\right\} \mathbf{1}\left\{G_{n_0,N+1} > \rho^{\delta-1}\right\}\right] \ . \tag{48}$$

Observe that the function

$$(x, y) \mapsto \log\left(1 + \frac{\alpha_0\alpha_{n_0}\rho^2 xy}{1 + \alpha_0\rho x + \alpha_{n_0}\rho y}\right)$$

defined on $\mathbb{R}_+^2$ is increasing in the variables x and y. Therefore,

$$\text{RHS}(48) \leq \mathbb{E}\left[\mathbf{1}\left\{\frac{\alpha_0\alpha_{n_0}\rho^{2\delta}}{1+\alpha_0\rho^\delta + \alpha_{n_0}\rho^\delta} \leq e^{R/t_{n_0}} - 1\right\}\right] \ .$$

As the function $g(\rho) = \frac{\alpha_0\alpha_{n_0}\rho^{2\delta}}{1+\alpha_0\rho^\delta+\alpha_{n_0}\rho^\delta}$ converges to infinity as $\rho \to \infty$, there exists a constant $\rho_0 > 0$ such that the RHS of the last inequality is zero when $\rho > \rho_0$. By consequence, the left hand side of (42) is zero when $\rho > \rho_0$. Lemma 4 is proven.

C. Proof of Lemma 5

We have $I \geq (t_0' + \sum_{n=1}^N t_n) T \log(1 + \alpha_0\rho G_{0,N+1})$ by inspecting the Expression (25-26) of the mutual information. By consequence,

$$\begin{aligned}\rho^{N+1}\mathbb{E}\left[\mathbf{1}\{I \leq RT\} \prod_{n=1}^{N} \mathbf{1}_{n,\nu(n)}\right] &\leq \rho^{N+1}\mathbb{E}\left[\mathbf{1}\{(t_0' + \sum_{n=1}^{N} t_n)\log(1+\alpha_0\rho G_{0,N+1}) \leq R\} \prod_{n=1}^{N} \mathbf{1}_{n,\nu(n)}\right] \\ &= \rho^{N+1}\mathbb{E}[\mathbf{1}\{\rho G_{0,N+1} \leq C\}] \prod_{n=1}^{N} \mathbb{E}[\mathbf{1}_{n,\nu(n)}]\end{aligned} \tag{49}$$

where $C = \frac{\exp(R/(t_0' + \sum t_n)) - 1}{\alpha_0}$, and the equality is due to the independence of the channels. In the remainder of the proof, K will denote a constant independent of ρ which value can change from line to line. The following facts can be shown as usual by making changes of variables then using the DCT and the right continuity of the channel gains densities:

$$\mathbb{E}[\mathbf{1}\{\rho G_{0,N+1} \leq C\}] = \mathbb{P}[\rho G_{0,N+1} \leq C] \leq K\rho^{-1}$$

If $\nu(n) = 1$, then

$$\mathbb{E}[\mathbf{1}_{n,\nu(n)}] = \mathbb{P}[G_{0,n} \leq \rho^{\delta-1}]\mathbb{P}[G_{n,N+1} > \rho^{\delta-1}] \leq \mathbb{P}[G_{0,n} \leq \rho^{\delta-1}] \leq K\rho^{\delta-1} \ .$$

If $\nu(n) = 2$, then

$$\mathbb{E}[\mathbf{1}_{n,\nu(n)}] = \mathbb{P}[G_{0,n} > \rho^{\delta-1}]\mathbb{P}[G_{n,N+1} \le \rho^{\delta-1}] \le \mathbb{P}[G_{0,N+1} \le \rho^{\delta-1}] \le K\rho^{\delta-1} \ .$$

Finally, if $\nu(n) = 4$, then

$$\mathbb{E}[\mathbf{1}_{n,\nu(n)}] = \mathbb{P}[G_{0,n} \le \rho^{\delta-1}]\mathbb{P}[G_{n,N+1} \le \rho^{\delta-1}] \le K\rho^{2\delta-2} \ .$$

Plugging these inequalities into (49), we obtain

$$\rho^{N+1}\mathbb{E}\left[\mathbf{1}\{I \le RT\} \prod_{n=1}^{N} \mathbf{1}_{n,\nu(n)}\right] \le K\rho^{(N+L)\delta-L}$$

where L is the number of indices $\nu(n) = 4$. As $L \ge 1$ and $\delta < 1/(N+1)$ by assumption, $(N+L)\delta - L < 0$ and the RHS of the last inequality converges to zero. This concludes the proof of Lemma 5.

D. Proof of Lemma 6

In view of Equations (44) and (46), we only need to prove that $\lim_{\rho\to\infty} \int J_\Theta = \int \lim_{\rho\to\infty} J_\Theta$. To this end, we shall prove that there exists constants $K > 0$ and $\rho_0 > 0$ for which

$$J_\Theta \le \mathbf{1}\left\{(u, (u_n)_{n\in\Theta}, (v_m)_{m\in\overline{\Theta}}) \in [0, K]^{N+1}\right\} \\ f_{G_{0,N+1}}(u/\rho) \prod_{n\in\Theta} f_{G_{0n}}(u_n/\rho) f_{G_{n,N+1}}(y_n) \prod_{m\in\overline{\Theta}} f_{G_{0m}}(x_m) f_{G_{m,N+1}}(v_m/\rho) \ . \quad (50)$$

for all $\rho > \rho_0$. Indeed, assume (50) is true. For ρ large enough, $f_{G_{0,N+1}}(u/\rho)$, $(f_{G_{0n}}(u_n/\rho))_{n\in\Theta}$ and $(f_{G_{m,N+1}}(v_m/\rho))_{m\in\overline{\Theta}}$ are bounded on $[0, K]$ by right continuity at zero. Therefore, J_Θ is dominated by an integrable function for ρ large enough, and it is possible to exchange $\int$ with $\lim_{\rho\to\infty}$ by the DCT.
We now prove (50). Write the last indicator function at the RHS of (45) as $\mathbf{1}\{\mathcal{X}_\Theta \le R\}$. We have

$$\mathcal{X}_\Theta \ge \left(t'_0 + \sum_{n=1}^{N} t_n\right) \log(1 + u)$$

therefore

$$\mathbf{1}\{\mathcal{X}_\Theta \le R\} \le \mathbf{1}\{u \in [0, K_0]\} \quad (51)$$

with $K_0 = \exp(R/(t'_0 + \sum t_n)) - 1$.
For any index $n \in \Theta$, we have by inspecting the expression of $\mathcal{X}_\Theta$:

$$\mathcal{X}_\Theta \ge t_n \log\left(1 + \frac{\alpha_n \rho u_n y_n}{1 + u_n + \alpha_n \rho y_n}\right) \quad (52)$$

hence

$$\mathbf{1}\{\mathcal{X}_\Theta \le R\} \le \mathbf{1}\left\{\frac{\alpha_n \rho u_n y_n}{1 + u_n + \alpha_n \rho y_n} \le C_n\right\}$$

where $C_n = \exp(R/t_n) - 1$. As the function $g(x) = \frac{ax}{c+bx}$ is increasing on $\mathbb{R}_+$ if $ac > 0$, we have

$$\frac{\alpha_n \rho u_n y_n}{1 + u_n + \alpha_n \rho y_n} \mathbf{1}\{y_n > \rho^{\delta-1}\} \geq \frac{\alpha_n \rho^\delta u_n}{1 + u_n + \alpha_n \rho^\delta} \mathbf{1}\{y_n > \rho^{\delta-1}\}$$

hence

$$\mathbf{1}\Big\{\frac{\alpha_n \rho u_n y_n}{1 + u_n + \alpha_n \rho y_n} \leq C_n\Big\} \mathbf{1}\{y_n > \rho^{\delta-1}\} \leq \mathbf{1}\Big\{\frac{\alpha_n \rho^\delta u_n}{1 + u_n + \alpha_n \rho^\delta} \leq C_n\Big\} \mathbf{1}\{y_n > \rho^{\delta-1}\}$$

But $\frac{\alpha_n \rho^\delta u_n}{1+u_n+\alpha_n \rho^\delta} \leq C_n \Leftrightarrow u_n \leq \frac{\alpha_n C_n \rho^\delta + C_n}{\alpha_n \rho^\delta - C_n}$. As the fraction at the RHS converges to C_n as $\rho \to \infty$, there exists a constant $K_n > 0$ for which

$$\mathbf{1}\Big\{\frac{\alpha_n \rho^\delta u_n}{1 + u_n + \alpha_n \rho^\delta} \leq C_n\Big\} \leq \mathbf{1}\{u_n \in [0, K_n]\} \ .$$

when ρ is large enough. In conclusion we have

$$\mathbf{1}\{\mathcal{X}_\Theta \leq R\} \mathbf{1}\{y_n > \rho^{\delta-1}\} \leq \mathbf{1}\{u_n \in [0, K_n]\} \mathbf{1}\{y_n > \rho^{\delta-1}\} \leq \mathbf{1}\{u_n \in [0, K_n]\} \ . \tag{53}$$

Consider now the indices $m \in \overline{\Theta}$. By getting back to the expression of $\mathcal{X}_\Theta$ we can write for any of these indices $\mathcal{X}_\Theta \geq t_m \log\left(1 + \frac{\alpha_0 \rho x_m v_m}{1+v_m+\alpha_0 \rho x_m}\right)$, an inequality similar to (52). By going over the steps that lead to (53) again, we obtain for ρ large enough

$$\mathbf{1}\{\mathcal{X}_\Theta \leq R\} \mathbf{1}\{x_m > \rho^{\delta-1}\} \leq \mathbf{1}\{v_m \in [0, K_m]\} \tag{54}$$

where $K_m > 0$ is a constant. By combining Inequalities (51), (53) and (54), we obtain

$$\mathbf{1}\{\mathcal{X}_\Theta \leq R\} \leq \mathbf{1}\{u \in [0, K_0]\} \prod_{n \in \Theta} \mathbf{1}\{u_n \in [0, K_n]\} \prod_{m \in \overline{\Theta}} \mathbf{1}\{v_m \in [0, K_m]\}$$
$$\leq \mathbf{1}\left\{(u, (u_n)_{n\in\Theta}, (v_m)_{m\in\overline{\Theta}}) \in [0, K]^{N+1}\right\}$$

where $K = \max(\{K_0, \{K_n\}, \{K_m\}\})$. By plugging this inequality into the RHS of (45), we recover Inequality (50). This concludes the proof of Lemma 6.

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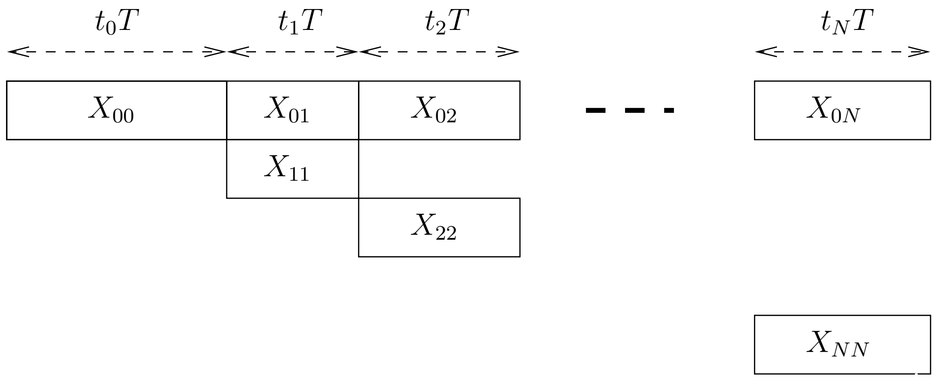

Fig. 1. DF Protocol for N relays

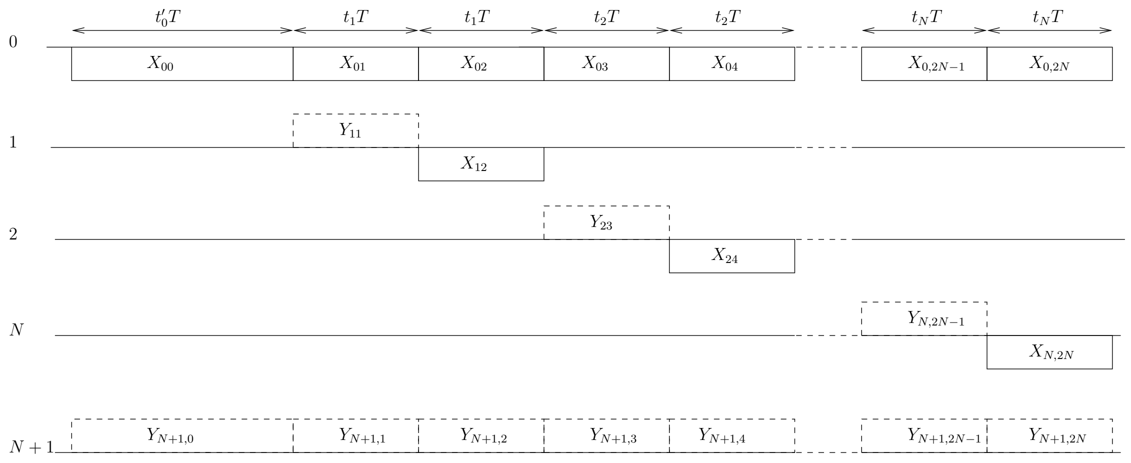

Fig. 2. AF Protocol for N relays

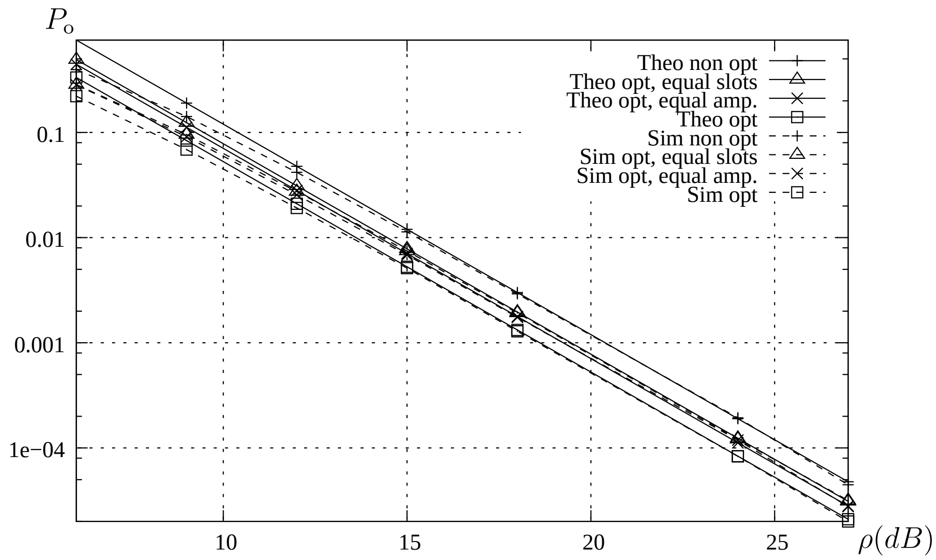

Fig. 3. Outage performance of DF protocol, $N = 1$ relay

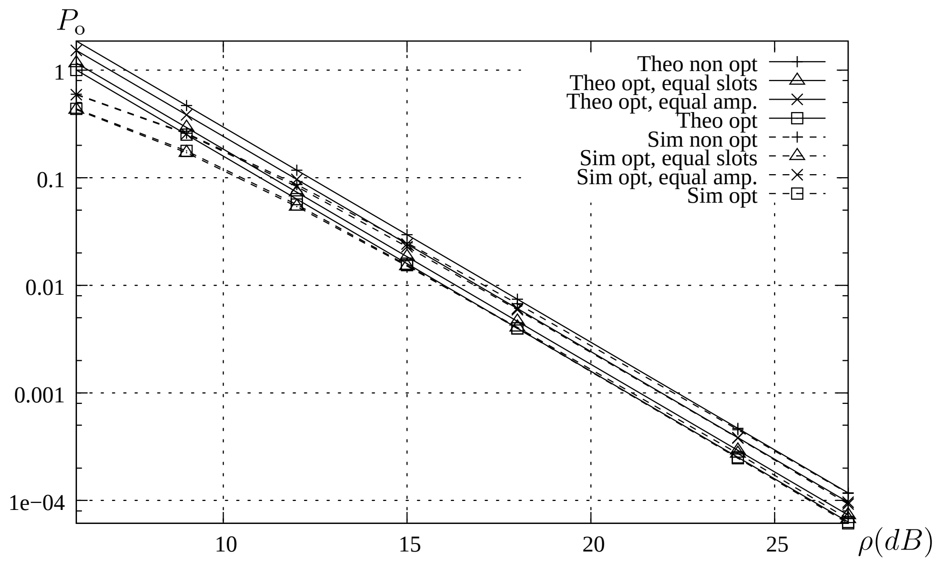

Fig. 4. Outage performance of AF protocol, $N = 1$ relay

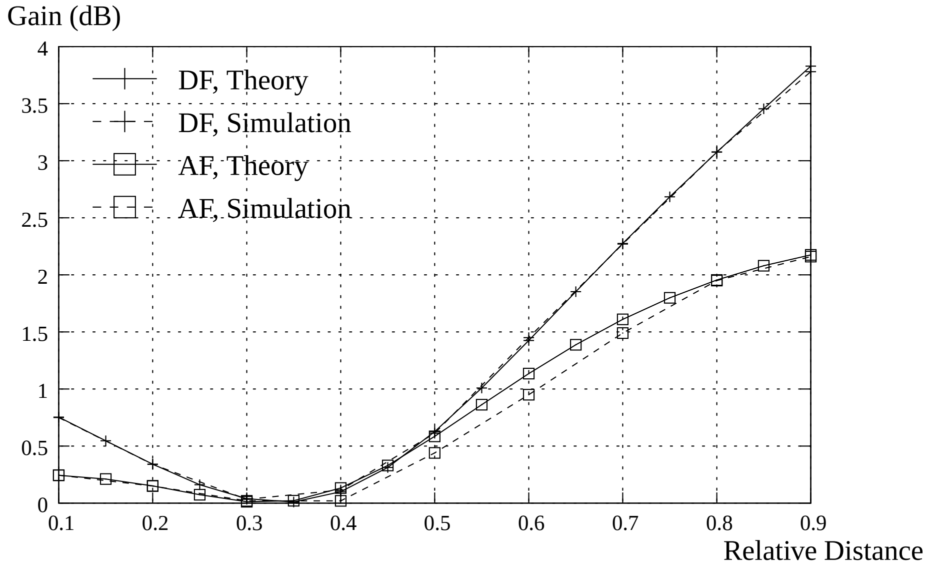

Fig. 5. Merit of optimization, $N = 1$ relay

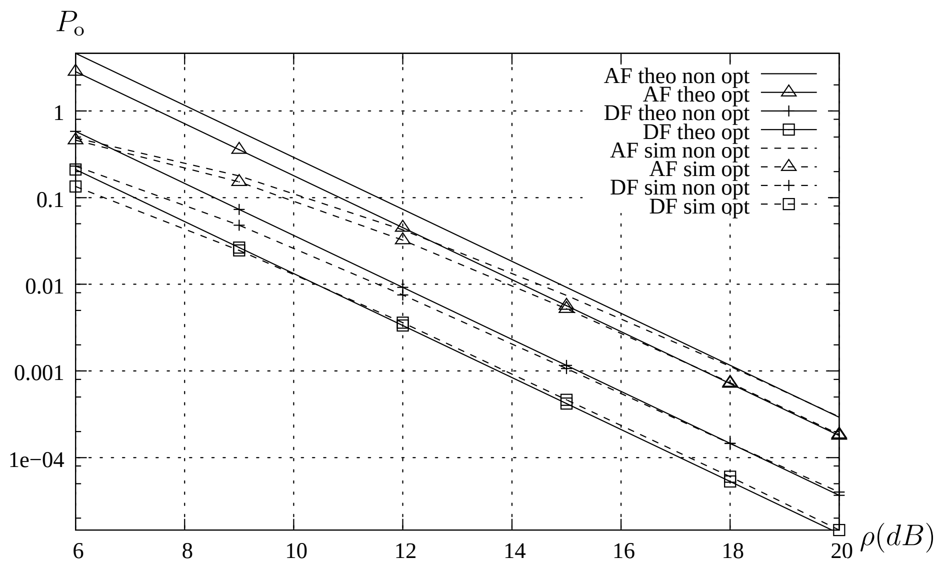

Fig. 6. Outage performance of DF and AF protocols, $N = 2$ relays