Training Design for Repetitive-Slot-based CFO estimation in OFDM

Mounir Ghogho, Philippe Ciblat, Ananthram Swami, and Pascal Bianchi

***Abstract*—Carrier frequency offset (CFO) estimation is a key challenge in wireless systems employing OFDM modulation. Often, CFO estimation is carried out using a preamble made of a number, say J, of repetitive-slots (RS). We here focus on the issue of optimal RS preamble design using the Cramér-Rao bound (CRB) averaged over the channel, which is assumed to be Rayleigh. We show that the optimal value of J is a trade-off between the multipath diversity gain and the number of unknowns to be estimated. In the case of correlated channel taps, we also show that uniform power loading of the active subcarriers is not optimal (in contrast with the uncorrelated case) and better power loading schemes are proposed. The proposed power loading schemes consist of allocating more power to activated carriers with higher signal-to-noise ratios. Simulation-based performance results of the maximum likelihood estimator support the CRB-based theoretical results.**

I. Introduction

Orthogonal frequency division multiplexing (OFDM) has become the standard of choice for wireless LAN's such as IEEE 802.11a, and is being considered for several IEEE 802.11 and 802.16 standards. The popularity of OFDM arises from the balanced transceiver complexity, and the time-frequency granularity that it offers. However, synchronization continues to be a critical challenge. Here, we focus on carrier frequency offset (CFO) synchronization, assuming perfect frame and timing synchronization.

CFO estimation techniques may be classified as time-domain (pre-FFT) or frequency-domain (post-FFT) techniques. The latter are usually used to estimate the integer part of the CFO after the fractional part has been identified and corrected. Time-domain methods are typically used to estimate the fractional part of the CFO, although some of these techniques can also estimate the integer part of the CFO. Time-domain methods can be classified into those that exploit the time-diversity provided by the cyclic prefix (see [1] and references therein), those that ignore the cyclic prefix and rely on pilots or null sub-carriers (NSC) [2]-[5], and blind approaches that exploit the non-Gaussianity of the information-bearing symbols [6]-[8]. Techniques based on single-carrier clock recovery algorithms are described in [9].

Data-aided CFO estimation in current OFDM systems employs a preamble made of a number, say J, of repetitive slots (RS) [2]. This preamble is obtained using one OFDM symbol after deactivating all subcarriers except those whose frequencies are integer multiples of J. It has been shown that the RS-based CFO maximum likelihood (ML) estimator is identical to the NSC-based ML estimator in the absence of virtual subcarriers[1] [5]. Here, we address the issue of optimal preamble design using the Cramér-Rao bound (CRB) as a metric. This involves optimizing J and the power loading. We show that the optimal value of J is a trade-off between the multipath diversity gain (in a sense to be defined later in the paper) and the number of unknowns to be estimated. In the case of uncorrelated channel taps, uniform power loading is optimal. In the case of correlated channel taps, we show that uniform power loading of the active subcarriers is no longer optimal and better power loading schemes are proposed.

Notations: Superscripts * and ^{T} will denote conjugate transposition and transposition. $\Re[\cdot]$, $\Im[\cdot]$, and $\mathrm{Tr}\{\cdot\}$ denote the real part, the imaginary part, and the trace operators, respectively. $\mathbb{E}[\cdot]$ stands for the statistical expectation.

II. Signal Model and Preliminaries

The frequency-selective channel is modelled as an FIR filter with impulse response $\mathbf{h} = [h_0, ..., h_{L-1}]^T$, and frequency-domain response $H_k := \sum_{l=0}^{L-1} h_l e^{-j2\pi kl/N}$. In order to analyze the performance of CFO estimation, we will assume the following:
(**A1**) The channel impulse response vector $\mathbf{h}$ is a zero-mean circularly symmetric Gaussian vector with covariance matrix $\mathbf{R}_h = \mathbb{E}[\mathbf{h}\mathbf{h}^H]$.

We assume a standard cyclic prefix (CP) based OFDM system with CP length $L_{cp} \geq L - 1$. Let ν (a real number) denote the CFO normalized to the subcarrier spacing, i.e., the actual frequency offset is $\nu\Delta f$ Hz, where Δf is the subcarrier spacing. In the presence of CFO and noise, the symbol-rate sampled receive signal can, after removing the CP, be written as

$$y(n) = \frac{1}{\sqrt{K}} e^{j2\pi\nu n/N} \sum_{k\in\mathbb{K}} s_k H_k e^{j2\pi nk/N} + w(n) \qquad (1)$$

with $n = 0, \cdots, N-1$, where N denotes the total number of subcarriers, $\mathbb{K}$ is the subset of active subcarriers with K denoting its cardinality, s_k is the pilot symbol transmitted over the kth subcarrier, and $w(.)$ is AWGN. We assume that the power of the transmitted OFDM pilot symbol is fixed and set to one without loss of generality. This implies $\sum_{k\in\mathbb{K}} |s_k|^2 = K$.

In line with practical OFDM systems, we design the preamble used for CFO estimation as a single OFDM block made of J identical sub-blocks of length $M = N/J$ each, with M an integer. Such a pilot OFDM symbol is obtained by deactivating all subcarriers whose frequencies are not multiple of J, i.e, $\mathbb{K} = \{mJ, m = 0, \cdots, M-1\} - VSC$, where VSC is the set of virtual subcarriers (VSC). The size of $\mathbb{K}$ satisfies $K \leq M$; equality holds in the absence of VSC. The case where the preamble is made up of a sequence of identical OFDM blocks can be treated similarly since, for example, two identical OFDM symbols can be thought of as two half symbols of a $2N$-point OFDM block. In this case, a guard interval is not needed between the identical blocks.

The RS structure of the preamble allows for a simple estimation of the CFO thus avoiding the computational complexity of the joint CFO-channel estimation. Further, for $K < L$, the channel cannot be identified while CFO may still be identified; indeed in this case there would be more unknowns than equations.

Mounir Ghogho is with School of Electronics and Electrical Engineering, Leeds University, Leeds, United Kingdom (m.ghogho@leeds.ac.uk).

Philippe Ciblat is with Département Comélec, ENST, Paris, France (philippe.ciblat@enst.fr).

Ananthram Swami is with Army Research Laboratory, Adelphi, USA (a.swami@ieee.org).

Pascal Bianchi is with Département Télécom, Supélec, Gif-sur-Yvette, France (pascal.bianchi@supelec.fr).

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[1]Virtual subcarriers are the subcarriers at the edges of the allocated frequency band that are deactivated in order to avoid interference with adjacent systems

Using the RS structure, the received signal can be rewritten as (with $n = m + \ell M$ and $m = 0, ..., M-1;\ \ell = 0, ..., J-1$)

$$y(m + \ell M) = a(m)\ e^{j2\pi\nu\ell/J} + w(m + \ell M)\ , \tag{2}$$

where

$$a(m) = e^{j2\pi\nu m/N}\left(\frac{1}{\sqrt{K}}\sum_{k\in\mathbb{K}} s_k H_k e^{j2\pi mk/N}\right).$$

If we ignore the dependance of $a(m)$ upon ν, estimating ν from Eq. (2) is equivalent to harmonic retrieval in additive noise in a multivariate setup. Indeed, by defining $\mathbf{y}(\ell) = [y(\ell M), \cdots, y(M-1+\ell M)]^T$ as the "multivariate" receive signal, $\mathbf{a} = [a(0), ..., a(M-1)]^T$ as the unknown amplitude vector, and $\mathbf{w}(\ell) = [w(\ell M), \cdots, w(M-1+\ell M)]^T$ as the "multivariate" additive noise, we obtain

$$\mathbf{y}(\ell) = \mathbf{a}e^{j2\pi\nu\ell/J} + \mathbf{w}(\ell), \quad \text{for } \ell = 0, \cdots, J-1 \tag{3}$$

In [5], the vector $\mathbf{a}$ was modelled as an unknown $(M \times 1)$ deterministic parameter vector and the following RS-based ML (RS-ML) estimator was derived:

$$\hat{\nu}_{\mathrm{RS}} = \arg\max_{\nu} \sum_{\ell=1}^{J-1} \mathcal{R}\left[r(\ell M)e^{-j2\pi\ell\nu/J}\right] \tag{4}$$

where $r(\tau)$ is the autocorrelation sequence

$$r(\tau) = \sum_{n=0}^{N-\tau-1} y^*(n)y(n+\tau).$$

It is worth pointing out at this stage that the implementation of the RS-ML estimator increases with J since it requires the estimation of J correlation coefficients. This observation may have a role to play in the RS-preamble design discussed in Section IV.

III. Performance Analysis

Here, we analytically assess the performance of the RS-ML estimator using the CRB, which characterize the asymptotic performance of the ML estimator astimator. We derive the conditional CRB (conditioned on the channel) and the average (over the channel) CRB. In deriving these bounds, we assume, as in the RS-ML method, that $\mathbf{a}$ in Eq. (2) is an arbitrary vector. The unknown parameter vector is then $[\mathbf{a}_R^T, \mathbf{a}_I^T, \nu]^T$ where $\mathbf{a}_R = \mathcal{R}[\mathbf{a}]$ and $\mathbf{a}_I = \mathcal{I}[\mathbf{a}]$.

A. Conditional CRB

Here, the unknown parameter vector is considered to be deterministic. Since $w(n)$ is circularly symmetric white Gaussian process, the conditional CRB (CCRB) on CFO estimation is found to be (the proof is straightforward and is omitted here due to page limitation.)

$$\mathrm{CCRB}_{\mathrm{RS}}(\nu) = \frac{1}{\gamma_h}\ \frac{3}{2\pi^2 N(1-1/J^2)} \tag{5}$$

where γ_h is the conditional (on the channel) signal-to-noise ratio (SNR)

$$\gamma_h := \frac{\frac{1}{K}\sum_{k\in\mathbb{K}}|H_k|^2|s_k|^2}{\sigma^2} \tag{6}$$

The CCRB is useful to predict the performance of CFO estimation for a particular channel. Notice that Eq. (5) is an extension of Eq. (18) in [2] which was valid only for AWGN channel.

It is instructive to rewrite the CCRB as follows

$$\mathrm{CCRB}_{\mathrm{RS}}(\nu) = \xi_h f(J)\mathrm{CRB}_{\mathrm{EQ-AWGN}}(\nu) \tag{7}$$

where

$$\xi_h = \mathbb{E}[\gamma_h]/\gamma_h \tag{8}$$

$$f(J) = \frac{1-1/N^2}{1-1/J^2} \tag{9}$$

and

$$\mathrm{CRB}_{\mathrm{EQ-AWGN}}(\nu) = \frac{3}{2\pi^2 N(1-1/N^2)\overline{\gamma}}$$

is the CRB on the estimation of the frequency, ν, of a single exponential, $\{\exp(j2\pi\nu n/N),\ n = 0, \cdots, N-1\}$, in AWGN channel with equivalent SNR given by

$$\overline{\gamma} := \mathbb{E}[\gamma_h] = \frac{\frac{1}{K}\sum_{k\in\mathbb{K}}\sigma_H^2(k)|s_k|^2}{\sigma^2} \tag{10}$$

where $\sigma_H^2(k) = \mathbb{E}[|H_k|^2]$. It is worth pointing out that the average SNR, $\overline{\gamma}$, depends on the power distribution when the channel taps are correlated since the $\sigma_H^2(k)$'s are different from each other in this case.

The parameter ξ_h captures the variations (or randomness) of the channel; its distribution is a function of $\mathbf{R}_h$, J and the power distribution among the active subcarriers. The function $f(J)$ measures the above-mentioned amplitude uncertainty. The latter monotonically decreases with J. If $J = N$, $f(N) = 1$, which is the minimum uncertainty; in this case the amplitude of the noise-free received signal is constant. Note that $f(1) = \infty$; indeed in this case the complex amplitude of the noise-free received signal has no repetitive structure, thus $\mathrm{CCRB}_{\mathrm{RS}}(\nu) = \infty$, i.e., the CFO is non-identifiable if the preamble has no repetitive structure. However, if NSC-based estimation is used, then, the CFO could be identifiable even if there is no repetitive structure provided that some of the subcarriers are deactivated [5]. The CCRB associated with the NSC approach can be found in [4].

B. Average CRB

The average CRB (ACRB) is given by

$$\mathrm{ACRB}_{\mathrm{RS}}(\nu) := \mathbb{E}[\mathrm{CCRB}_{\mathrm{RS}}(\nu)] = \overline{\xi} f(J)\mathrm{CRB}_{\mathrm{EQ-AWGN}}(\nu) \tag{11}$$

where $\overline{\xi} := \mathbb{E}[\xi]$ and the expectation is with respect to the channel. Monte-Carlo simulations can be used to accurately evaluate the ACRB. Deriving closed-form expressions for $\overline{\xi}$ and thus the ACRB does not seem tractable except for the interesting special cases listed below. Nevertheless, we introduce in the next subsection simple closed-from expressions approximating the ACRB in the general case.

1) $\overline{\xi} = \infty$ cases: under assumption (A1), this occurs if $i)$ $J = N$ (i.e. $K = 1$) regardless of L and $\mathbf{R}_h$, $ii)$ $L = 1$ (i.e. flat fading) regardless of J, or $iii)$ $\mathrm{rank}(\mathbf{R}_h) = 1$ (i.e. fully correlated paths) regardless of J. Indeed in all the above case γ_h is exponentially distributed, which implies that $\mathbb{E}[1/\gamma_h] = \infty$ and thus $\mathrm{ACRB}_{\mathrm{RS}}(\nu) = \infty$. Hence, for Rayleigh fading channels, in order for CFO estimation to be consistent, multipath diversity must not only be available (i.e. $L > 1$ and $\mathrm{rank}(\mathbf{R}_h) > 1$) but also captured through the choice of J, which dictates the number of modulated subcarriers.

2) $\mathbf{R}_h = \sigma_h^2\mathbf{I}_L$ and $K = M$: In the absence of virtual subcarriers, i.e. $K = M$, uniform power loading (i.e. $|s_k|^2 = 1,\ k \in \mathbb{K}$) is optimum. In this case, γ_h can be written as $\gamma_h = \sigma^{-2}\sum_{i=0}^{\min(K-1,L-1)}|\sum_{j=0}^{\lfloor(L-1)/K\rfloor}h_{i+jK}|^2$ where $h_\ell = 0$ if $\ell \geq L$. If $K \geq L$, $[2\sigma^2/\sigma_h^2]\gamma_h$ is a chi-square variable with $2(L-1)+2$ degrees of freedom. If $K < L$ and L/K an integer, $[2\sigma^2/(\sigma_h^2(L/K))]\gamma_h$ is a chi-square variable with $2(K-1)+2$ degrees of freedom. The mean for an inverse-chi-square random variable with n $(n > 2)$ degrees of freedom is equal to $1/(n-2)$, Consequently $\overline{\xi}$ is found to be

$$\overline{\xi} = \frac{L}{\min(K-1, L-1)\max(1, L/K)} \tag{12}$$

where $K \geq L$ or $K < L$ but with L/K an integer. The above expression is interesting because it explicitly shows the impact of multipath diversity on CFO estimation through $\min(K-1, L-1)$ which can be interpreted as the multipath diversity order captured by activating the subcarriers in $\mathbb{K}$. The result on multipath diversity can be better illustrated by the cumulative distribution function (CDF) of the CCRB which is obtained as [2]

$$Pr\{CCRB(\nu) < \epsilon\} = e^{-1/(2x)} \sum_{i=0}^{D} \frac{1}{i!(2x)^i} \tag{13}$$

where $D = \min(K-1, L-1)$ and

$$x = \frac{\epsilon \max(1, L/K)}{2Lf(J)\mathrm{CRB}_{\mathrm{EQ-AWGN}}(\nu)} \tag{14}$$

Eq. (13) shows the exponential dependence of the CDF of the CCRB with respect to the multipath diversity order.

It is worth pointing out that when $L \gg 1$ and $K \geq L$, the ACRB gets close to the (RS-based) CRB obtained in the case of AWGN channels[3].

In the general case where none of the above scenarios occurs, we propose the following approximation for the ACRB.

C. ACRB approximations

We have that γ_h is a weighted sum of central chi-square distribution of two degrees of freedom. If the number of components in the sum is large (i.e, K large or J small compared to N), then it is well known ([11], [12] and references therein) that its distribution can be well approximated by a central Gamma distribution $p_G(t)$ with standard parameters (b_1, b_2) given by

$$p_G(t) = \frac{b_1^{b_2}}{\Gamma(b_2)} t^{b_2-1} e^{-b_1 t} \mathbf{1}_{t\geq 0}$$

and such that the mean (resp. variance) of a Gamma distribution b_2/b_1 (resp. b_2/b_1^2) satisfy

$$b_2/b_1 = \mathbb{E}[\gamma_h], \quad b_2/b_1^2 = \mathbb{E}[(\gamma_h - \overline{\gamma})^2].$$

After straightforward but tedious algebraic manipulations, we obtain that

$$b_1 = K\sigma^2 \frac{\mathrm{Tr}(\mathbf{R}_H \mathbf{P})}{\mathrm{Tr}(\mathbf{R}_H \mathbf{P} \mathbf{R}_H \mathbf{P})} \tag{15}$$

$$b_2 = b_1 \frac{1}{K\sigma^2} \mathrm{Tr}(\mathbf{R}_H \mathbf{P}). \tag{16}$$

where $\mathbf{R}_H = [\rho_H(m,n)]_{m,n\in\mathbb{K}}$ with $\rho_H(m,n) := \mathbb{E}[H_m^* H_n]$ and $\mathbf{P}$ is a diagonal matrix composed by $\{|s_k|^2, \ k \in \mathbb{K}\}$.

When γ_h is assumed to be Gamma distributed, the expectation of $1/\gamma_h$ is given by $b_1/(b_2-1)$. We thus can deduce the next lemma.

Lemma 1: If K is large, we have

$$\mathbb{E}\left[\frac{1}{\gamma_h}\right] \approx K\sigma^2 \frac{\mathrm{Tr}(\mathbf{R}_H \mathbf{P})}{(\mathrm{Tr}(\mathbf{R}_H \mathbf{P}))^2 - \mathrm{Tr}((\mathbf{R}_H \mathbf{P})^2)}. \tag{17}$$

In Eq. (17), only the term $\mathrm{Tr}((\mathbf{R}_H\mathbf{P})^2)$ depends on the off-diagonal terms of matrix $\mathbf{R}_H$. In our simulation study, we have observed that neglecting the off-diagonal components of $\mathbf{R}_H$ modifies only very slightly the value of the ACRB. Further, in the case where $K = N/J = L$ and the channel taps have equal powers but may be correlated, we have shown analytically that the expression in the Lemma does not depend on the off-diagonal elements of $\mathbf{R}_H$; the derivations, not shown here because of page limitation, are based on the diagonalization of the Toeplitz matrix $\mathbf{R}_h$ using DFT matrices. Therefore the following corollary can be deduced

Corollary 1: If K is large, $\mathbb{E}[1/\gamma_h]$ may be well approximated by

$$\mathbb{E}\left[\frac{1}{\gamma_h}\right] \approx K\sigma^2 \frac{\sum_{k\in\mathbb{K}} \sigma_H^2(k) p_k}{\left(\sum_{k\in\mathbb{K}} \sigma_H^2(k) p_k\right)^2 - \sum_{k\in\mathbb{K}} \sigma_H^4(k) p_k^2} \tag{18}$$

with $p_k = |s_k|^2$ for $k \in \mathbb{K}$.

IV. Repetitive-Slot Pilot Design

This section gives guidelines on how to choose J and the power distribution among the active subcarriers.

A. Power loading design

Here, the number of activated subcarriers and their positions are fixed, i.e., $\mathbb{K}$ (therefore J) is fixed. In the literature, the symbols $\{s_k, \ k \in \mathbb{K}\}$ are always set to have the same magnitude whatever the channel statistics. In the case of correlated scattering, theoretical analysis (presented below) and simulations results (presented in next section) show that the uniform power loading is not optimal. In [13], power loading for CFO estimation was proposed but the channel realization was assumed to be known at the receiver and at the transmitter, so the power loading was channel-dependent. Here, the power loading is channel statistic dependent.

First, note that the sequence $\{p_k := |s_k|^2, \ k \in \mathbb{K}\}$ that minimizes the CCRB of ν under the constraint of constant transmit power, $\sum_{k\in\mathbb{K}} p_k = K$, is channel dependent. In the rather unrealistic case where the channel is known at the transmitter and where CCRB optimzation makes sense, the optimal design for $\{p_k, \ k \in \mathbb{K}\}$ would be to assign the entire transmit power to the subcarrier at which $|H_k|$ is maximum. In this case, only one subcarrier is active, and we would have a constant signal envelope and maximum signal-to-noise ratio at the receiver. For this scenario to be practical, the channel has to be (quasi) time-invariant and known at the transmitter while the CFO may vary with time. Next, we focus on the more practical case where the channel is unknown at the transmitter.

Since the channel is unknown at the transmitter, an alternative measure of performance is required to solve our design problem. In [10], the worst-case channel CRB was used to derive the optimal statistics of the training sequence in the context of single-carrier systems. For our problem, the worst-case channel CRB is not useful since it is equal to infinity when $K \leq L$. This is obtained when all L zeros of the channel coincide with the activated subcarriers. This is a direct consequence of the loss of multipath diversity in cyclic-prefixed systems.

Therefore, we assume a statistical model for the channel and now concentrate on the ACRB. Unfortunately, no general closed-form expression is available for the ACRB. Although the ACRB can be estimated empirically, its numerical minimization with respect to the K-dimensional parameter set $\{p_k, \ k \in \mathbb{K}\}$ is prohibitive since for each parameter vector candidate, a large number of Monte-Carlo simulations are required to accurately estimate the ACRB.

Recall that in the case of uncorrelated channel taps, uniform power is optimal in the absence of VSCs and simulations have shown that it is nearly optimal when VSCs are present. We can now use the approximation provided in subsection III-C to exhibit relevant power loading in the case of correlated channel taps. One can prove quite easily (by evaluating the Hessian matrix and its positivity) that optimizing the ACRB approximated by Eq. (18) given in Corollary 1 with respect to power loading boils down to a convex optimization problem. In the next Theorem, we obtain the power

[2]The cumulative distribution function of an inverse Chi-square probability density function with n degrees of freedom is given by $\Gamma(n/2, 1/(2x))/\Gamma(n/2)$.

[3]In the case of AWGN channel, the performance of the RS-based CFO estimate is optimum when $J = N$. Choosing $J < N$ simplifies the ML algorithm at the expense of reduced performance, which is quantified by $f(J)$. Indeed, the RS-based CRB for ν in the case of AWGN channels can be expressed as $\mathrm{ACRB}_{\mathrm{RS,AWGN}}(\nu) = f(J)\mathrm{CRB}_{\mathrm{EQ-AWGN}}(\nu)$.

loading $\{p_k,\ k \in \mathbb{K}\}$ subject to $\sum_{k\in\mathbb{K}} p_k = K$ that minimizes Eq. (18). Consequently the obtained power loading will be relevant as soon as K is large enough. Theorem 1 is proved in the Appendix.

Theorem 1: Let $\mathbb{N}_n$ be the set of the n larger $\sigma_H^2(k)$ and let

$$\begin{aligned}\alpha_n &= \frac{1}{n}\sum_{k\in\mathbb{N}_n} 1/\sigma_H^2(k)\\ \beta_n &= \frac{1}{n}\sum_{k\in\mathbb{N}_n} 1/\sigma_H^4(k)\\ \mu_n &= \frac{n\alpha_n + \sqrt{\beta_n - n(\beta_n - \alpha_n^2)}}{n-1}.\end{aligned}$$

The optimal power loading minimizing Eq. (18) is

$$p_k = \frac{K}{\sigma_H^2(k)}\frac{(\mu_n - 1/\sigma_H^2(k))^+}{n(\mu_n\alpha_n - \beta_n)} \tag{19}$$

and n must satisfy $\mathrm{card}(k \mid 1/\sigma_H^2(k) < \mu_n) = n$ where card denotes the cardinality operator.

This power allocation gives greater importance to the frequencies associated with high channel variance, i.e., that are good in average. If a carrier is not good enough, it is not used. Such a power allocation achieves a compromise between the average SNR and multipath diversity gain. Keep in mind that this allocation makes sense when the Gamma distribution approximation holds, namely, when J is small enough. Finally, if the channel statistics are unknown, then distributing the transmit power uniformly across the subcarriers $k \in \mathbb{K}$, seems adequate.

B. Design of J

Using the RS-based method, identifiability of the CFO in the acquisition range $[-J/2, J/2]$ is guaranteed if $\gamma_h \neq 0$. This implies that identifiability is lost if $J = 1$ and $H_k = 0$, $\forall k \in \mathbb{K}$. Setting $J = 1$ (i.e. all subcarriers are modulated) offers maximum multipath diversity gain but this also maximizes the amplitude uncertainty and thus makes the CFO unidentifiable because there would be more unknown parameters to estimate than equations[4]. A necessary and sufficient condition for identifiability *regardless* of the channel realization is, provided $\mathbf{h}$ is not the null vector, given by

$$J \geq 2 \quad \text{and} \quad K \geq L$$

The second part of the above condition guarantees that even when all $(L-1)$ channel zeros coincide with activated subcarriers, $\gamma_h \neq 0$ would hold. For example, if $L = N/4 - 1$ and in the absence of VSC, strict identifiability of the CFO is guaranteed only for $J = 2$ and $J = 4$. However, since the channel impulse response $\mathbf{h}$ is a continuous-valued random vector, the probability of identifiability loss when $K \leq L - 1$ is zero. Therefore, we only focus on the estimation performance when deriving the optimal value of J.

Because the channel is random, with $J = N$ (i.e., $K = 1$) the multipath diversity is of order one. Thus, a deep fade at the single active subcarrier would cause a very low SNR, thus making CFO estimation very difficult. Simultaneous deep fades at several subcarriers are less likely than a fade at one subcarrier. Setting $J = 1$ (i.e. all subcarriers are modulated) offers maximum multipath diversity gain but this also maximizes the amplitude uncertainty and thus makes the CFO unidentifiable as mentioned above. Therefore, there must be a trade-off between these two phenomena. Some remark about J design associated with the BLUE estimator is available in [14].

[4]Since the transmitted symbols are known and the H_n's are parameterized by L coefficients only, the CFO can still be identifiable even when all subcarriers are modulated ($J = 1$). However, this will require joint channel and CFO estimation which complicates the CFO estimation algorithm.

The CCRB leads to a channel-dependent optimal value of J, and is therefore not useful because the channel is unknown at the transmitter. Hence we resort to the ACRB. We first study the case where $\mathbf{R}_h$ is proportional to the identity matrix, before studying the more general cases of uncorrelated channel taps (but with different variances), and correlated channel taps. First, it is worth pointing out that if the channel taps are uncorrelated, uniform power loading is optimal in the absence of VSc, and is nearly optimal in their presence. In what follows, we ignore the effects of VSCs.

1) $\mathbf{R}_h = \sigma_h^2\mathbf{I}_L$: As mentioned above, uniform power loading is optimal in this case. Eq. (12) can be rewritten as

$$\overline{\xi} = \frac{\min(L,K)}{\min(L,K)-1}$$

where L/K is assumed an integer when $K < L$. This implies that if we capture full multipath diversity, i.e, $K \geq L$ (i.e. $J \leq N/L$), then $\overline{\xi} = L/(L-1)$, which is independent of J. As $\mathrm{CRB}_{\mathrm{EQ-AWGN}}(\nu)$ is independent of J in this subsubsection and as $f(J)$ decreases with J, the value for J that minimizes the ACRB in Eq. (11) necessarily satisfies $J \geq N/L$. Further, in this case, the ACRB monotonically decreases with L, which confirms that multipath diversity improves the accuracy of CFO estimation. If $K \leq L$ but L/K is an integer, $\overline{\xi} = N/(N-J)$, which increases with J, whereas $f(J)$ decreases with J. The value of J that minimizes the ACRB in Eq. (11) can be obtained by finding J that minimizes $f(J)/(N-J)$. To get a close form expression for this value, we replace J by a continuous-valued variable, say μ. The value of μ that minimizes $f(\mu)/(N-\mu)$ is obtained by solving a third order equation and is found to be

$$\sqrt[3]{N + \sqrt{N^2 + \frac{1}{27}}} + \sqrt[3]{N - \sqrt{N^2 + \frac{1}{27}}} \approx \sqrt[3]{2N} \tag{20}$$

Hence, by combining the $K \geq L$ and $K \leq L$ cases, we take the optimal value of J to be the integer from the set of possible values of J that is the closest to [5]

$$\mu_o = \max\left(\frac{N}{L}, \sqrt[3]{2N}\right) \tag{21}$$

2) $\mathbf{R}_h = \mathrm{diag}\left(\sigma_h^2(0), \cdots, \sigma_h^2(L-1)\right)$: Again uniform power distribution is optimal in this case. However, we do not have an exact closed-form expression for the ACRB in this case. We use the approximation in Lemma 1 which leads, after some derivations, to

$$\overline{\xi} \approx \left[1 - \frac{\mathrm{Tr}\left(\sum_{i=0}^{Q-1}\mathbf{R}_h^{(i)}\right)^2}{(\mathrm{Tr}(\mathbf{R}_h))^2}\right]^{-1} \tag{22}$$

where $Q = 1$ if $K = N/J \geq L$ and $Q = L/(N/J)$ (integer) if $N/J \leq L$, and $\mathbf{R}_h^{(i)} = \mathrm{diag}\left(\sigma_h^2(iN/J + m),\ m = 0, \cdots, N/J - 1\right),\ i = 0, \cdots, Q-1$. Note that $\overline{\xi}$ depends on J. The value of J that minimizes the ACRB can then be easily obtained using the above expression and that for $f(J)$. However, unlike in the previous case, we do not have a closed-form solution as in eq. (21). It is also worth pointing out that the optimal value for J in this case reduces to that in eq. (21) if the uncorrelated channel taps have equal powers.

3) Correlated channel taps: In this case, we can only design analytically J using numerical evaluation of the approximate ACRB in Lemma 1, with power loading given in Theorem 1. Obtaining a closed form expression similar to that in eq. (21) does not seem tractable. Numerical evaluations and simulations show that the solution in eq. (21) is also appropriate for the general case of correlated and/or unequal power channel taps.

[5]Since N is a power of two in practical OFDM systems, J is required to be a power of two in order to obtain J equal length slots.

V. Simulation Results

We consider an OFDM pilot symbol with a total of $N = 64$ subcarriers and no virtual subcarriers. We assume the channel to be static over the OFDM pilot symbol. The channel coefficients are assumed Rayleigh with exponential power delay profile, with decay parameter α, i.e., $\sigma_{h_\ell}^2 := \mathbb{E}[|h_\ell|^2] = ce^{-\alpha\ell}$, and covariance matrix given by $[\mathbf{R}_h]_{i,j} = \sigma_{h_i}\sigma_{h_j}\rho^{|i-j|}$ where $\rho \in [0,1)$ is the correlation factor. The scaling factor c is chosen such that $\mathrm{Tr}\{\mathbf{R}_h\} = 1$. The empirical mean square errors (MSE) of the CFO estimates are estimated using 10,000 Monte-Carlo runs. The SNR is defined as $1/\sigma^2$ and is, unless stated otherwise, set to 10dB.

First, we consider the case of uncorrelated and equally powered channel taps, i.e. $\rho = 0$ and $\alpha = 1$. Figure 1 displays the exact ACRB (evaluated using Monte-Carlo simulations) versus L for different values of J. The results validate our theoretical finding of the optimal value for J in eq. (21). Indeed, as predicted, optimal performance is obtained with $J = N/L$ except when $L = 32$ for which the optimal value is $J = 4$ since it is the closest possible value of J to $\sqrt[3]{128}$.

In what follows, we set $L = 16$ and $\alpha = 2$. Figure 2 shows the ACRB and MSE of the RS-ML estimate versus J for $\rho = 0.5$ when the uniform and proposed power loading schemes are employed. Note that the empirical MSE of RS-ML is in agreement with the ACRB. Consequently, our approach of optimizing the training design using the ACRB is well justified. One can also observe that, although based on approximations, the proposed power loading schemes provide improvement in estimation performance. The theoretical results on the design of J, presented above, predict that the optimal value for J is four, which is in agreement with the result in Figure 2.

Figure 3 illustrates the ACRB and the MSE of RS-ML estimate for the uniform and proposed power loading schemes versus ρ with $J = 4$. We first observe that the MSE associated with uniform power loading increases with ρ because the multipath coding gain of the channel decreases with ρ. Recall that when $\rho = 1$, the taps are fully correlated and the ACRB and the average MSE thus become infinite. Figure 3 also shows that the proposed power loading significantly outperforms the uniform power loading when the channel taps are highly correlated. Moreover when channel statistics are known and judiciously used at the transmitter, the correlation becomes a benefit as long as $\mathbf{R}_h$ remains full-rank.

In Figure 4, we plot the ACRB and the MSE of the RS-ML estimator versus the SNR for the white power distribution and the power loading distribution given in Theorem 1 for $J = 4$ and $\rho = 0.5$. For the same estimation performance, the proposed power distribution provides a 5dB gain in terms of SNR over the white training. Notice also the threshold for the outlier effect is lower when using the proposed training.

VI. Conclusions

We have analyzed the performance of CFO estimators based on a single repetitive-slot pilot symbol in the context of OFDM system. By assuming a Rayleigh channel, we have provided closed-form expressions illustrating the impact of multipath diversity on estimation performance. Using the Cramèr-Rao bounds, we have provided insights into how the preamble should be designed. In the case of correlated Rayleigh fading channels with known statistics at the transmitter, a new power loading scheme was proposed and shown through simulations to outperform the conventional uniform power loading scheme.

Appendix

Proof of Theorem 1

Minimizing Eq. (18) is equivalent to minimizing its opposite inverse. Therefore we focus on the following optimization problem (which is still convex)

$$\min_{p_k \in \mathbb{K}} \frac{\sum_k \sigma_H^4(k) p_k^2}{\sum_k \sigma_H^2(k) p_k} - \sum_k \sigma_H^2(k) p_k$$

subject to $\sum_k p_k = K$ and $-p_k \leq 0$ for all $k \in \mathbb{K}$.

The Lagrangian function can be then written as

$$\mathcal{L} = \frac{\sum_k \sigma_H^4(k) p_k^2}{\sum_k \sigma_H^2(k) p_k} - \sum_k \sigma_H^2(k) p_k + c \sum_k p_k - \sum c_k p_k.$$

Applying the KKT conditions (which correspond to set partial derivative of $\mathcal{L}$ with respect to each p_k to zero), after straightforward but tedious algebraic manipulations, leads to

$$\begin{aligned} p_k &= \frac{K}{\sigma_H^2(k) \mathcal{N}_\mu (\mu \alpha_\mu - \beta_\mu)} (\mu - 1/\sigma_H^2(k))^+ \qquad (23) \\ 2\mu &= \frac{\sum_{k \in \mathcal{N}_\mu} (\mu - 1/\sigma_H^2(k))^2}{\sum_{k \in \mathcal{N}_\mu} (\mu - 1/\sigma_H^2(k))} + \sum_{k \in \mathcal{N}_\mu} (\mu - 1/\sigma_H^2(k)) \end{aligned}$$

where $\mathcal{N}_\mu = \{k \mid 1/\sigma_H^2(k) < \mu\}$.

Last equation can be re-written as follows

$$(|\mathcal{N}_\mu| - 1)\mu^2 - 2|\mathcal{N}_\mu|\alpha_\mu \mu + (|\mathcal{N}_\mu|\alpha_\mu^2 + \beta_\mu) = 0$$

Solving this second-order equation leads to

$$\mu = \frac{\alpha_\mu |\mathcal{N}_\mu| \pm \sqrt{\delta_\mu}}{|\mathcal{N}_\mu| - 1}.$$

where

$$\delta_\mu = |\mathcal{N}_\mu|\alpha_\mu^2 - \beta_\mu |\mathcal{N}_\mu| + \beta_\mu.$$

Due to its convexity property, the KKT conditions admit a solution and so $\delta_\mu \geq 0$ is necessarily satisfied. One can prove that $\mu = (\alpha_\mu |\mathcal{N}_\mu| - \sqrt{\delta_\mu})/(|\mathcal{N}_\mu| - 1)$ leads to negative p_k which is absurd. Consequently, we get

$$\mu = \frac{\alpha_\mu |\mathcal{N}_\mu| + \sqrt{\beta_\mu - |\mathcal{N}_\mu|(\beta_\mu - \alpha_\mu^2)}}{|\mathcal{N}_\mu| - 1} \qquad (24)$$

One can finally remark that the set $\mathcal{N}_\mu$ is entirely characterized by its cardinal $|\mathcal{N}_\mu|$. By denoting $n = |\mathcal{N}_\mu|$, Eqs. (23) and (24) can be re-written as in Theorem 1.

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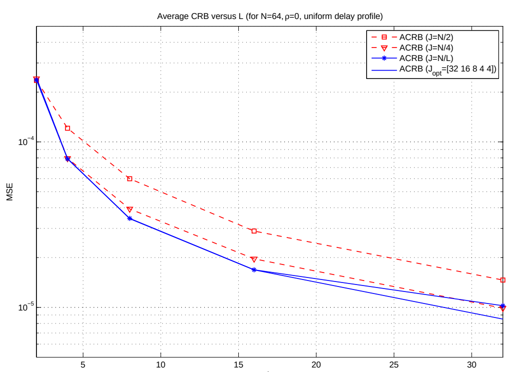

Fig. 1. ACRB vs L in the uncorrelated channel case

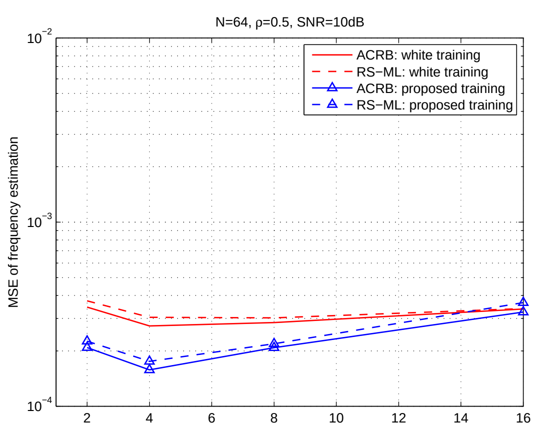

Fig. 2. ACRB and MSE of RS-ML vs J for different training schemes

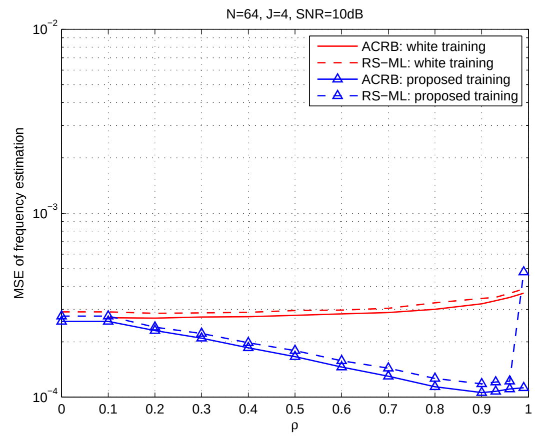

Fig. 3. ACRB and MSE of RS-ML vs ρ for different training schemes

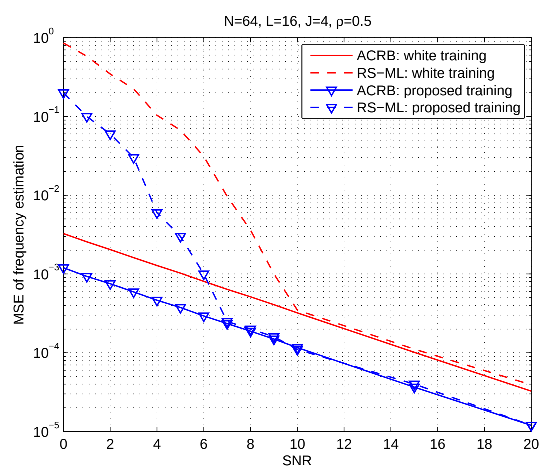

Fig. 4. ACRB and MSE of RS-ML versus SNR for different training schemes