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Harmonic retrieval in the presence of non-circular Gaussian multiplicative noise: performance bounds

Philippe Ciblat[a,*], Mounir Ghogho[b], Philippe Forster[c], Pascal Larzabal[d]

[a] *Département Communications et Electronique, Ecole Nationale Supérieure des Télécommunications (ENST), 46 rue Barrault, 75013 Paris, France*

[b] *Department of Electrical Engineering, University of Leeds, Leeds, UK*

[c] *Département Electronique, Université Paris X - Nanterre, Ville d'Avray, France*

[d] *Laboratoire SATIE, Ecole Normale Supérieure de Cachan, Cachan, France*

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Abstract

We address the problem of harmonic retrieval in the presence of multiplicative and additive noise sources. In the new context of a complex-valued non-circular Gaussian multiplicative noise, we express the Cramér–Rao bound (CRB) as well as the asymptotic (large sample) CRB in closed form. Below a certain SNR threshold and/or when the number of samples is not large enough, the CRB becomes too optimistic and therefore we also derive the Barankin bound (BB). The new theoretical expressions for the CRB and BB are then used to study the behavior of the performance bound with respect to the signal parameters. We especially describe the region (in terms of SNR and number of samples) for which the CRB and the BB differ. Finally we compare the performance of the square-power-based frequency estimate, which is equivalent to the non-linear least-squares-based estimate, to these bounds.

Keywords: Non-circularity; Outliers effect; Harmonic retrieval; Asymptotic behavior; Gaussian Cramér–Rao bound; Barankin bound

1. Introduction

Harmonic retrieval in multiplicative and additive noise has received increasing attention during the last decade [1–4]. The signal model is given by

$$y(n) = a(n)e^{2i\pi(\phi_0+\phi_1 n)} + b(n) \quad \text{for } n = 0, \ldots, N-1, \tag{1}$$

where ϕ_0 and ϕ_1 are the phase and frequency shifts to be estimated with the sole knowledge of a finite number of samples, say N, of the discrete-time process $y(n)$. The random process $a(n)$ represents, for example, (i) in digital

*Corresponding author. Tel.: +33 1 4581 7728; fax: +33 1 4589 0020.

E-mail addresses: philippe.ciblat@enst.fr (P. Ciblat), m.ghogho@leeds.ac.uk (M. Ghogho), philippe.forster@cva.u-paris10.fr (P. Forster), larzabal@satie.ens-cachan.fr (P. Larzabal).

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communications, the convolution of the symbol stream with the transmit/receive filters and the physical channel, or (ii) in the context of DOA estimation, the Fourier coefficients of the source signal. The random process $b(n)$ is an additive noise.

The frequency and phase are the parameters to be estimated whereas the other parameters, including the multiplicative noise vector $\mathbf{a}_N = [a(0), \ldots, a(N-1)]^{\mathrm{T}}$ (where the superscript T stands for transposition) are considered nuisance parameters. The literature about the performance lower bound of such an estimation problem is prolific. We first discuss the Cramér–Rao bound (CRB), which is the most popular performance bound. By making different assumptions on the nuisance parameter vector $\mathbf{a}_N$, several CRBs can be obtained [5]. The unconditional CRB (UCRB) is obtained when $\mathbf{a}_N$ is modelled as a random vector. Unfortunately, in most practical problems, no interpretable closed-form expressions exist for the UCRB. To overcome this difficulty, other CRBs have been studied. The Gaussian CRB (GCRB) is obtained when a Gaussianity assumption is imposed on $\mathbf{a}_N$. The GCRB coincides with the UCRB only when the Gaussian assumption is satisfied. The conditional CRB is obtained when the nuisance parameters are considered to be deterministic and are estimated jointly with the parameters of interest. The modified CRB (MCRB) is obtained when the nuisance parameters are considered to be known. It is worth pointing out that different research communities seem to prefer different types of CRBs. The MCRB is mostly used by the Digital Communications community in the context of synchronization, although the UCRB has recently received some attention, but only in the low SNR scenario. The GCRB is mostly used by the Signal Processing community mainly in the context of DOA estimation [2–4,6–8].

In this paper, we focus on the GCRB. Notice that this bound may be too pessimistic for the case when the multiplicative noise is not Gaussian since the high-order statistics are not taken into account. Over the last few years, several papers [2–4,6–8] have addressed the problem of evaluating the GCRB for the following scenarios

- $a(n)$ is real-valued [2–4,7],
- $a(n)$ is complex-valued and circular [8].

Most of these papers only derive expressions for the *exact* (i.e. finite-sample) GCRB [2,3,6,7]. These expressions, unless numerically evaluated, do not give insight into the influence of the signal parameters on the GCRB. To overcome this problem, expressions for the asymptotic (large-sample) GCRB are very desirable [4,8]. Our first contribution consists of providing closed-form expressions for the asymptotic GCRB when the multiplicative noise $a(n)$ is assumed to be a complex-valued and *non-circular* Gaussian process. Notice that even in digital communications, previous assumption on the multiplicative noise can be encountered [9].

When the SNR and/or N, the number of available samples, is less than a certain threshold, the CRB is too optimistic. To investigate more accurately the area of low SNR and/or small N, we use the Barankin bound (BB) which is the tightest lower bound for any unbiased estimate [10–16].

For harmonic retrieval in additive noise, several works have addressed the evaluation of such a bound ([12,15,16] and references therein). The derivation of the Barankin bound for harmonic retrieval in multiplicative noise has seldom been investigated. To our knowledge, only [13] and [14] have addressed this issue. These papers focus on DOA estimation, which boils down to estimating the frequency of an exponential harmonic in the presence of a circular multiplicative noise.

Therefore, our second contribution consists of deriving the BB when the multiplicative noise is assumed to be a *non-circular* complex-valued Gaussian process. Our derivations will be shown to be extensions of the results presented in [13,14].

The paper is organized as follows. The exact GCRB and the asymptotic GCRB are presented in Sections 2 and 3, respectively. In Section 4, closed-form expressions for the BB are provided. The threshold beyond which the Barankin bound becomes different from the CRB is analyzed in Section 5. In Section 6, we review the square-power-based frequency estimator, whose performance is close to that of the maximum likelihood-based estimator [3,4,17,18]. Theoretical comparisons with the asymptotic GCRB are also carried out. Finally, Section 7 is devoted to numerical simulations.

2. Exact Gaussian Cramér–Rao bound

Throughout the paper, the model given by Eq. (1) is considered under the following assumptions:

- $a(n)$ is a zero-mean complex-valued Gaussian and non-circular stationary process; let its correlation and conjugate correlation be denoted by $r_a(\tau) = \mathbb{E}[a(n+\tau)\overline{a(n)}]$ and $u_a(\tau) = \mathbb{E}[a(n+\tau)a(n)]$, where the overline stands for complex conjugate. Also, let the spectrum and conjugate spectrum be defined as follows:
$$s_a(\mathrm{e}^{2\mathrm{i}\pi f}) = \sum_{\tau\in\mathbb{Z}} r_a(\tau)\mathrm{e}^{-2\mathrm{i}\pi f\tau}$$
and
$$c_a(\mathrm{e}^{2\mathrm{i}\pi f}) = \sum_{\tau\in\mathbb{Z}} u_a(\tau)\mathrm{e}^{-2\mathrm{i}\pi f\tau}.$$
By construction, one can remark that $c_a(\mathrm{e}^{2\mathrm{i}\pi f}) = c_a(\mathrm{e}^{-2\mathrm{i}\pi f})$.
- The statistics of $a(n)$, i.e. $\{r_a(\tau), u_a(\tau)\}_{\tau\in\mathbb{Z}}$, are entirely captured by a finite number (say K) of unknown real-valued parameters, which we denote by $\mathscr{A} = \{\aleph_k\}_{k=1,\ldots,K}$.
- $b(n)$ is a complex-valued Gaussian and circular stationary process with zero-mean and unknown variance $\sigma^2 = \mathbb{E}[|b(n)|^2]$.

In this section, our purpose is to derive the exact GCRB, or equivalently the exact Fisher information matrix $\mathbf{F}$, for the deterministic parameter vector $\theta = [\aleph_1, \ldots, \aleph_K, \sigma^2, \phi_0, \phi_1]$ when N samples of $\{y(n)\}$ are available. Let $\mathbf{y}_N = [y(0), \ldots, y(N-1)]^{\mathrm{T}}$.

In order to use well-known results on the Fisher information matrix [19], we work with real-valued processes. Therefore, we consider $\check{\mathbf{y}}_N = [\Re[\mathbf{y}_N^{\mathrm{T}}], \Im[\mathbf{y}_N^{\mathrm{T}}]]^{\mathrm{T}}$, which is a multi-variate Gaussian variable with zero-mean and covariance matrix $\check{\mathbf{R}}_{\mathrm{y}}$. We obtain
$$\check{\mathbf{R}}_{\mathrm{y}} = \frac{1}{2}\begin{bmatrix} \Re[\mathbf{R}_{\mathrm{y}} + \mathbf{U}_{\mathrm{y}}] & -\Im[\mathbf{R}_{\mathrm{y}} - \mathbf{U}_{\mathrm{y}}] \\ \Im[\mathbf{R}_{\mathrm{y}} + \mathbf{U}_{\mathrm{y}}] & \Re[\mathbf{R}_{\mathrm{y}} - \mathbf{U}_{\mathrm{y}}] \end{bmatrix}$$
with $\mathbf{R}_{\mathrm{y}} = \mathbf{E}[\mathbf{y}_N\mathbf{y}_N^{\mathrm{H}}]$ and $\mathbf{U}_{\mathrm{y}} = \mathbf{E}[\mathbf{y}_N\mathbf{y}_N^{\mathrm{T}}]$. Superscript H stands for the complex conjugate transposition.

Due to the frequency shift, $y(n)$ is stationary with respect to its correlation *but* cyclostationary with respect to its conjugate correlation [18]. Thus, $\check{\mathbf{R}}_{\mathrm{y}}$ is symmetric but not block-Toeplitz. However, we have checked that formula (5.2.1) in [19] holds as long as the covariance matrix is symmetric. This leads to
$$\mathbf{F}_{k,l} = \frac{1}{2}\,\mathrm{Tr}\left(\frac{\partial\check{\mathbf{R}}_{\mathrm{y}}}{\partial\theta_k}\,\check{\mathbf{R}}_{\mathrm{y}}^{-1}\,\frac{\partial\check{\mathbf{R}}_{\mathrm{y}}}{\partial\theta_l}\,\check{\mathbf{R}}_{\mathrm{y}}^{-1}\right),$$
where $\mathbf{F}_{k,l}$ corresponds to the joint Fisher information for parameters (θ_k, θ_l) and where Tr(.) denotes the trace operator.

After straightforward algebraic manipulations, we show that
$$\mathbf{F}_{k,l} = \frac{1}{2}\,\mathrm{Tr}\left(\frac{\partial\widetilde{\mathbf{R}}_{\mathrm{y}}}{\partial\theta_k}\,\widetilde{\mathbf{R}}_{\mathrm{y}}^{-1}\,\frac{\partial\widetilde{\mathbf{R}}_{\mathrm{y}}}{\partial\theta_l}\,\widetilde{\mathbf{R}}_{\mathrm{y}}^{-1}\right),$$
where $\widetilde{\mathbf{R}}_{\mathrm{y}}$ is the covariance matrix of the random variable $\widetilde{\mathbf{y}}_N = [\mathbf{y}_N^{\mathrm{T}}, \mathbf{y}_N^{\mathrm{H}}]^{\mathrm{T}}$, which takes the following form:
$$\widetilde{\mathbf{R}}_{\mathrm{y}} = \begin{bmatrix} \mathbf{R}_{\mathrm{y}} & \mathbf{U}_{\mathrm{y}} \\ \overline{\mathbf{U}}_{\mathrm{y}} & \overline{\mathbf{R}}_{\mathrm{y}} \end{bmatrix}. \tag{2}$$
Model (1) can also be written as follows:
$$\widetilde{\mathbf{y}}_N = \mathbf{\Gamma}\widetilde{\mathbf{a}}_N + \widetilde{\mathbf{b}}_N,$$
where $\widetilde{\mathbf{a}}_N$ and $\widetilde{\mathbf{b}}_N$ are defined in a similar way as $\widetilde{\mathbf{y}}_N$. We get
$$\widetilde{\mathbf{R}}_{\mathrm{y}} = \widetilde{\mathbf{\Gamma}}\widetilde{\mathbf{R}}_{\mathrm{x}}\widetilde{\mathbf{\Gamma}}^{\mathrm{H}}, \tag{3}$$
where
$$\widetilde{\mathbf{\Gamma}} = \begin{bmatrix} \mathbf{\Gamma} & \mathbf{0}_{N,N} \\ \mathbf{0}_{N,N} & \overline{\mathbf{\Gamma}} \end{bmatrix} \tag{4}$$
with
$$\mathbf{\Gamma} = \mathrm{diag}(\mathrm{e}^{2\mathrm{i}\pi(\phi_0+\phi_1 n)},\; n = 0, \ldots, N-1), \tag{5}$$
$$\mathbf{0}_{N,N} = \mathrm{zeros}(N, N),$$

and

$$\widetilde{\mathbf{R}}_{\mathbf{x}} = \widetilde{\mathbf{R}}_{\mathbf{a}} + \sigma^2 \mathbf{I}_{2N},$$

with

$$\widetilde{\mathbf{R}}_{\mathbf{a}} = \begin{bmatrix} \mathbf{R}_{\mathbf{a}} & \mathbf{U}_{\mathbf{a}} \\ \overline{\mathbf{U}}_{\mathbf{a}} & \overline{\mathbf{R}}_{\mathbf{a}} \end{bmatrix}.$$

Notice that $\mathbf{R}_{\mathbf{a}}$ and $\mathbf{U}_{\mathbf{a}}$ are defined like $\mathbf{R}_{\mathbf{y}}$, and $\mathbf{U}_{\mathbf{y}}$, respectively. Notice also that $\widetilde{\mathbf{R}}_{\mathbf{x}}$ does not depend on the phase parameters. Therefore, we obtain the following expressions for the Fisher information matrix:

$$\mathbf{F}_{\aleph_k,\aleph_l} = \frac{1}{2}\,\mathrm{Tr}\left(\frac{\partial \widetilde{\mathbf{R}}_{\mathbf{a}}}{\partial \aleph_k}\,\widetilde{\mathbf{R}}_{\mathbf{x}}^{-1}\,\frac{\partial \widetilde{\mathbf{R}}_{\mathbf{a}}}{\partial \aleph_l}\,\widetilde{\mathbf{R}}_{\mathbf{x}}^{-1}\right),$$

$$\mathbf{F}_{\sigma^2,\sigma^2} = \frac{1}{2}\,\mathrm{Tr}\left(\widetilde{\mathbf{R}}_{\mathbf{x}}^{-2}\right),$$

$$\mathbf{F}_{\aleph_k,\sigma^2} = \frac{1}{2}\,\mathrm{Tr}\left(\frac{\partial \widetilde{\mathbf{R}}_{\mathbf{a}}}{\partial \aleph_k}\,\widetilde{\mathbf{R}}_{\mathbf{x}}^{-2}\right),$$

$$\mathbf{F}_{\phi_k,\phi_l} = 2\pi^2\,\mathrm{Tr}(\mathbf{D}_k\widetilde{\mathbf{R}}_{\mathbf{x}}\mathbf{D}_l\widetilde{\mathbf{R}}_{\mathbf{x}}^{-1} + \mathbf{D}_l\widetilde{\mathbf{R}}_{\mathbf{x}}\mathbf{D}_k\widetilde{\mathbf{R}}_{\mathbf{x}}^{-1} - 2\mathbf{D}_k\mathbf{D}_l),$$

$$\mathbf{F}_{\aleph_k,\phi_k} = \mathrm{i}\pi\,\mathrm{Tr}\left(\frac{\partial \widetilde{\mathbf{R}}_{\mathbf{a}}}{\partial \aleph_k}[\widetilde{\mathbf{R}}_{\mathbf{x}}^{-1}\mathbf{D}_k - \mathbf{D}_k\widetilde{\mathbf{R}}_{\mathbf{x}}^{-1}]\right),$$

$$\mathbf{F}_{\sigma^2,\phi_k} = 0,$$

where $\mathbf{D}_k = [\mathbf{d}_k, \mathbf{0}_{N,N}; \mathbf{0}_{N,N}, -\mathbf{d}_k]$ with $\mathbf{d}_0 = \mathbf{I}_N$ and $\mathbf{d}_1 = \mathrm{diag}([0,\ldots,N-1])$. The above expressions are the extensions to the ones presented in [8]. It is worth pointing out that unlike the case of circular multiplicative noise [8], the phase in our case is identifiable (i.e., $\mathbf{F}_{\phi_0,\phi_0} \neq 0$).

3. Asymptotic Gaussian Cramér–Rao bound

We now focus on the behavior of the Fisher information matrix $\mathbf{F}$ and the GCRB when N becomes large.

Unlike [8], here we cannot apply Whittle's formula [20] to obtain simple asymptotic expressions for the Fisher information matrix because $y(n)$ is cyclostationary. In the sequel, our derivations rely on theorems dealing with the inversion of (large) Toeplitz matrices [21,22].

Let $\{t_k; k = 0, \pm 1, \ldots\}$ be an absolutely summable sequence. Let $\mathbf{t}_N = (t_{m-l})_{-N<l,m<N}$ be a Toeplitz matrix. Define

$$s(\mathrm{e}^{2\mathrm{i}\pi f}) = \sum_{k\in\mathbb{Z}} t_k \mathrm{e}^{-2\mathrm{i}\pi fk} \;\Leftrightarrow\; t_k = \int_0^1 s(\mathrm{e}^{2\mathrm{i}\pi f})\mathrm{e}^{2\mathrm{i}\pi fk}\,\mathrm{d}f.$$

Matrix $\mathbf{t}_N$ is thus entirely captured by $f \mapsto s(\mathrm{e}^{2\mathrm{i}\pi f})$ which justifies the following mapping:

$$\mathbf{t}_N = \mathscr{T}_N(s).$$

Let A_N and B_N be two $(N \times N)$ bounded matrices. $|A_N|$ stands for $(\frac{1}{N}\,\mathrm{Tr}(A_N A_N^{\mathrm{H}}))^{1/2}$. A_N and B_N are said to be asymptotically equivalent (denoted by $\sim$) iff $|A_N - B_N| \to 0$ when $N \to \infty$.

One can remark that $\mathbf{R}_{\mathbf{a}}$ and $\mathbf{U}_{\mathbf{a}}$ are Toeplitz matrices and can be written as

$$\mathbf{R}_{\mathbf{a}} = \mathscr{T}_N(s_a) \quad \text{and} \quad \mathbf{U}_{\mathbf{a}} = \mathscr{T}_N(c_a).$$

This implies that

$$\mathbf{R}_{\mathbf{x}} = \mathscr{T}_N(s) \quad \text{and} \quad \mathbf{U}_{\mathbf{x}} = \mathscr{T}_N(c) \tag{6}$$

with $s(\mathrm{e}^{2\mathrm{i}\pi f}) = s_a(\mathrm{e}^{2\mathrm{i}\pi f}) + \sigma^2$ and $c(\mathrm{e}^{2\mathrm{i}\pi f}) = c_a(\mathrm{e}^{2\mathrm{i}\pi f})$. Furthermore, we get

$$\overline{\mathbf{R}}_{\mathbf{x}} = \mathscr{T}_N(\underline{s}) \quad \text{and} \quad \overline{\mathbf{U}}_{\mathbf{x}} = \mathscr{T}_N(\underline{c}),$$

where $\underline{s}(\mathrm{e}^{2\mathrm{i}\pi f}) = \overline{s(\mathrm{e}^{-2\mathrm{i}\pi f})}$ and $\underline{c}(\mathrm{e}^{2\mathrm{i}\pi f}) = \overline{c(\mathrm{e}^{-2\mathrm{i}\pi f})}$.

To obtain an asymptotic value for $\mathbf{F}$, we first need an asymptotic equivalent for $\widetilde{\mathbf{R}}_{\mathbf{x}}^{-1}$. According to Schur's lemma, we get

$$\widetilde{\mathbf{R}}_{\mathbf{x}}^{-1} = \begin{bmatrix} \mathbf{R}_{\mathbf{x}}^{-1} + \mathbf{R}_{\mathbf{x}}^{-1}\mathbf{U}_{\mathbf{x}}\mathbf{\Delta}^{-1}\overline{\mathbf{U}}_{\mathbf{x}}\mathbf{R}_{\mathbf{x}}^{-1} & -\mathbf{R}_{\mathbf{x}}^{-1}\mathbf{U}_{\mathbf{x}}\mathbf{\Delta}^{-1} \\ -\mathbf{\Delta}^{-1}\overline{\mathbf{U}}_{\mathbf{x}}\mathbf{R}_{\mathbf{x}}^{-1} & \mathbf{\Delta}^{-1} \end{bmatrix}$$

with

$$\mathbf{\Delta} = \overline{\mathbf{R}}_{\mathbf{x}} - \overline{\mathbf{U}}_{\mathbf{x}}\mathbf{R}_{\mathbf{x}}^{-1}\mathbf{U}_{\mathbf{x}}.$$

Thanks to Eq. (6), we have

$$\mathbf{\Delta} = \mathscr{T}_N(\underline{s}) - \mathscr{T}_N(\underline{c})\mathscr{T}_N(s)^{-1}\mathscr{T}_N(c).$$

Since the mapping s is real-valued and does not admit zero over the interval [0,1), $\mathscr{T}_N(s)^{-1} \sim \mathscr{T}_N(s^{-1})$ for large N. Thus

$$\mathbf{\Delta} \sim \mathscr{T}_N(\underline{s}) - \mathscr{T}_N(\underline{c})\mathscr{T}_N(s^{-1})\mathscr{T}_N(c).$$

Since s^{-1} and c are bounded over $[0, 1)$, $\mathscr{T}_N(\underline{c})\mathscr{T}_N(s^{-1})\mathscr{T}_N(c) \sim \mathscr{T}_N(c\,\underline{c}/s)$ for large N, and we finally get

$$\mathbf{\Delta} \sim \mathscr{T}_N([s\,\underline{s} - c\,\underline{c}]/s) := \mathscr{T}_N(\mathscr{X}/s)$$

with $\mathscr{X}(e^{2i\pi f}) = s(e^{2i\pi f})\,\underline{s}(e^{2i\pi f}) - c(e^{2i\pi f})\,\underline{c}(e^{2i\pi f})$. One can see that $\mathscr{X}$ is real-valued and positive. This leads to

$$\mathbf{\Delta}^{-1} \sim \mathscr{T}_N(s/\mathscr{X}).$$

After straightforward manipulations, we conclude that

$$\tilde{\mathbf{R}}_{\mathbf{x}}^{-1} \sim \begin{bmatrix} \mathscr{T}_N(\underline{s}/\mathscr{X}) & -\mathscr{T}_N(c/\mathscr{X}) \\ -\mathscr{T}_N(\underline{c}/\mathscr{X}) & \mathscr{T}_N(s/\mathscr{X}) \end{bmatrix}.$$

Once again, after simple but tedious calculations, the Fisher information matrix expresses as follows:

$$\lim_{N\to\infty} \frac{1}{N}\,\mathbf{F}_{\aleph_k,\aleph_l} = \frac{1}{2}\,\alpha_{k,l},$$

$$\lim_{N\to\infty} \frac{1}{N}\,\mathbf{F}_{\aleph_k,\sigma^2} = \frac{1}{2}\,\beta_k,$$

$$\lim_{N\to\infty} \frac{1}{N}\,\mathbf{F}_{\aleph_k,\phi_0} = 4\pi\delta_k,$$

$$\lim_{N\to\infty} \frac{1}{N^2}\,\mathbf{F}_{\aleph_k,\phi_1} = 2\pi\delta_k,$$

$$\lim_{N\to\infty} \frac{1}{N}\,\mathbf{F}_{\sigma^2,\sigma^2} = \frac{1}{2}\,\gamma,$$

$$\lim_{N\to\infty} \frac{1}{N}\,\mathbf{F}_{\phi_0,\phi_0} = 16\pi^2\xi,$$

$$\lim_{N\to\infty} \frac{1}{N^2}\,\mathbf{F}_{\phi_0,\phi_1} = 8\pi^2\xi,$$

$$\lim_{N\to\infty} \frac{1}{N^3}\,\mathbf{F}_{\phi_1,\phi_1} = \frac{16\pi^2}{3}\,\xi,$$

where

$$\alpha_{k,l} = \int_0^1 \frac{\dfrac{\partial\mathscr{X}(e^{2i\pi f})}{\partial\aleph_k}\dfrac{\partial\mathscr{X}(e^{2i\pi f})}{\partial\aleph_l}}{\mathscr{X}(e^{2i\pi f})^2} + \frac{\mathscr{V}^{(c)}_{k,l}(e^{2i\pi f}) - \mathscr{V}^{(s)}_{k,l}(e^{2i\pi f})}{\mathscr{X}(e^{2i\pi f})}\,df,$$

$$\beta_k = \int_0^1 \frac{1}{\mathscr{X}(e^{2i\pi f})}\frac{\partial\mathscr{X}(e^{2i\pi f})}{\partial\aleph_k}\,df,$$

$$\gamma = \int_0^1 \frac{s(e^{2i\pi f})^2 + \underline{s}(e^{2i\pi f})^2 + 2\,\underline{c}(e^{2i\pi f})c(e^{2i\pi f})}{\mathscr{X}(e^{2i\pi f})^2}\,df,$$

$$\delta_k = \Im\left[\int_0^1 \frac{\partial c(e^{2i\pi f})}{\partial\aleph_k}\frac{\underline{c}(e^{2i\pi f})}{\mathscr{X}(e^{2i\pi f})}\,df\right],$$

$$\xi = \int_0^1 \frac{c(e^{2i\pi f})\,\underline{c}(e^{2i\pi f})}{\mathscr{X}(e^{2i\pi f})}\,df,$$

with the following mapping

$$\mathscr{V}^{(s)}_{k,l}(e^{2i\pi f}) = \frac{\partial s(e^{2i\pi f})}{\partial\aleph_k}\frac{\partial\,\underline{s}(e^{2i\pi f})}{\partial\aleph_l} + \frac{\partial\,\underline{s}(e^{2i\pi f})}{\partial\aleph_k}\frac{\partial s(e^{2i\pi f})}{\partial\aleph_l},$$

and $f \mapsto \mathscr{V}^{(c)}_{k,l}(e^{2i\pi f})$ is defined similarly.

Firstly we consider that the receiver has a knowledge of $\mathscr{A} = [\aleph_1, \ldots, \aleph_K]$ and σ^2, i.e., the statistics of the multiplicative and additive noise sources. In this case, we obtain

$$\mathrm{GCRB}(\phi_0)_{|(\mathscr{A},\sigma^2)\ \mathrm{known}} \sim \frac{1}{4\pi^2\xi N} \tag{7}$$

and

$$\mathrm{GCRB}(\phi_1)_{|(\mathscr{A},\sigma^2)\ \mathrm{known}} \sim \frac{3}{4\pi^2\xi N^3}. \tag{8}$$

Secondly, in the case where $\mathscr{A} = [\aleph_1, \ldots, \aleph_K]$ and σ^2 are unknown at the receiver, we obtain

$$\mathrm{GCRB}(\phi_0)_{|(\mathscr{A},\sigma^2)\ \mathrm{unknown}} = \mathrm{GCRB}(\phi_0)_{|(\mathscr{A},\sigma^2)\ \mathrm{known}} + \frac{\mu}{16\pi^2\xi^2 N},$$

where μ is a bounded scalar given by

$$\mu = \boldsymbol{\delta}^{\mathrm{T}}(\boldsymbol{\alpha}/2 - \boldsymbol{\delta}\boldsymbol{\delta}^{\mathrm{T}}/\xi - \boldsymbol{\beta}\boldsymbol{\beta}^{\mathrm{T}}/(2\gamma))^{-1}\boldsymbol{\delta},$$

where $\boldsymbol{\alpha} = (\alpha_{k,l})_{1\leqslant k,l\leqslant K}$, $\boldsymbol{\beta} = (\beta_k)_{1\leqslant k\leqslant K}$, $\boldsymbol{\delta} = (\delta_k)_{1\leqslant k\leqslant K}$. Lastly, we have that

$$\mathrm{GCRB}(\phi_1)_{|(\mathscr{A},\sigma^2)\ \mathrm{unknown}} = \mathrm{GCRB}(\phi_1)_{|(\mathscr{A},\sigma^2)\ \mathrm{known}}. \tag{9}$$

Thanks to the above expressions for the asymptotic CRB, we make the following comments:

- The convergence rates for the phase and frequency estimation are $1/N$ (cf. Eq. (7)) and $1/N^3$ (cf. Eq. (8)), respectively, regardless of the color (or spectrum) of the multiplicative noise. Such rates were obtained in the case of real-valued multiplicative noise [4]. We recall that for circular complex-valued processes, the phase is not identifiable, the frequency may be identifiable only if the multiplicative noise is colored, and the convergence rate is $1/N$ [8]. As far as estimation performance is concerned, the complex-valued non-circular case is closer to the real-valued case than to the complex-valued circular case. Consequently, in terms of performance, the main cut-off is not complex/real but circular/non-circular.[1]

- Surprisingly, the asymptotic frequency estimation performance is the same whether or not the statistics of $a(n)$ are known (cf. Eq. (9)).
- In the noiseless case, we observe a floor effect (i.e., GCRB$\neq 0$ when $\sigma^2 = 0$) for the phase and the frequency estimation performance. For example, the floor for the frequency estimation performance is given by

$$\mathrm{GCRB}(\phi_1)_{|\sigma^2=0} = \frac{3}{4\pi^2\left(\int_0^1 \frac{c_a(\mathrm{e}^{2\mathrm{i}\pi f})\overline{c_a(\mathrm{e}^{-2\mathrm{i}\pi f})}}{s_a(\mathrm{e}^{2\mathrm{i}\pi f})\overline{s_a(\mathrm{e}^{-2\mathrm{i}\pi f})} - c_a(\mathrm{e}^{2\mathrm{i}\pi f})\overline{c_a(\mathrm{e}^{-2\mathrm{i}\pi f})}}\,\mathrm{d}f\right)N^3}.$$

This effect vanishes iff $s_a(\mathrm{e}^{2\mathrm{i}\pi f})\overline{s_a(\mathrm{e}^{-2\mathrm{i}\pi f})} = c_a(\mathrm{e}^{2\mathrm{i}\pi f})\overline{c_a(\mathrm{e}^{-2\mathrm{i}\pi f})}$ for f belonging to a Borelian open of $[0,1)$. For instance, this condition is fulfilled when the multiplicative noise is real-valued.

4. Barankin bound

For simplicity, we here assume that the noise statistics, i.e., $\{r_a(\tau), u_a(\tau)\}_{\tau\in\mathbb{Z}}$ and σ^2, are known at the receiver. This assumption is usually made because of the high computational and analytical complexities of the derivation of the Barankin bound [13,14]. Furthermore, as shown in the previous section, the asymptotic Gaussian CRB for the frequency shift is insensitive to whether the noise statistics are known or not. We can thus expect that the error induced by neglecting the effects of noise statistics estimation may be small, and that our subsequent results will still be relevant in the case of unknown noise statistics.

Our purpose now is to derive the BB for the unknown deterministic vector $\boldsymbol{\phi} = [\phi_0, \phi_1]^{\mathrm{T}}$. We first define the following set of so-called "test-points"$\{\boldsymbol{\psi}(k) = [\psi_0(k), \psi_1(k)]^{\mathrm{T}}\}_{1\leqslant k\leqslant n}$. The BB of order n is defined as follows:

$$\mathrm{BB}_n(\phi_0, \phi_1) = \sup_{\mathscr{E}} S(\mathscr{E}),$$

where

$$S(\mathscr{E}) = \mathscr{E}(\mathbf{B}(\mathscr{E}) - \mathbf{1}_n\mathbf{1}_n^{\mathrm{T}})^{-1}\mathscr{E}^{\mathrm{T}}$$

with $\mathscr{E} = [\boldsymbol{\psi}(1) - \boldsymbol{\phi}, \ldots, \boldsymbol{\psi}(n) - \boldsymbol{\phi}]$, and $\mathbf{1}_n = \text{ones}(n,1)$. Furthermore, $\mathbf{B} = (B_{k,l})_{1\leqslant k,l\leqslant n}$ is the following $n\times n$ matrix,

$$B_{k,l} = \mathbb{E}[L(\widetilde{\mathbf{y}}_N, \boldsymbol{\phi}, \boldsymbol{\psi}(k))L(\widetilde{\mathbf{y}}_N, \boldsymbol{\phi}, \boldsymbol{\psi}(l))],$$

with

$$L(\widetilde{\mathbf{y}}_N, \boldsymbol{\phi}, \boldsymbol{\psi}(k)) = \frac{p(\widetilde{\mathbf{y}}_N|\boldsymbol{\psi}(k))}{p(\widetilde{\mathbf{y}}_N|\boldsymbol{\phi})}$$

and $p(\widetilde{\mathbf{y}}_N|\boldsymbol{\theta})$ is the likelihood function.

It is well known that the mean square error of any unbiased estimator is greater than the BB of any order. From an asymptotic point of view (i.e. as $n\to\infty$), the BB is the tightest lower bound [10,14]. Finally, it is worth remarking that

$$\mathrm{GCRB}(\phi_0, \phi_1) = \lim_{\mathscr{E}\to 0} S(\mathscr{E}). \tag{10}$$

As for the choice of the test-points, it is usual to consider [15,14]

$$\mathscr{E} = \begin{bmatrix} \psi_0 - \phi_0 & 0 \\ 0 & \psi_1 - \phi_1 \end{bmatrix} = \mathrm{diag}(\varepsilon_0, \varepsilon_1). \tag{11}$$

This choice is motivated, on the one hand, for the sake of simplicity and, on the other hand, because other more complex choices do not change significantly the numerical value of the bound.

Our main concern hereafter is to derive a closed-form expression for the matrix $\mathbf{B}$ for the above test-points.

Let us first introduce some notations. The covariance matrix $\widetilde{\mathbf{R}}_{\boldsymbol{\phi}}$ of the multivariate process $\widetilde{\mathbf{y}}_N$ can be written as

$$\widetilde{\mathbf{R}}_{\boldsymbol{\phi}} := \widetilde{\mathbf{R}}_{\mathbf{y}} = \widetilde{\boldsymbol{\Gamma}}_{\boldsymbol{\phi}}\left(\widetilde{\mathbf{R}}_a + \sigma^2\mathbf{I}_{2N}\right)\widetilde{\boldsymbol{\Gamma}}_{\boldsymbol{\phi}}^{\mathrm{H}}, \tag{12}$$

[1] Notice that a real-valued process can be viewed as a special case of a non-circular complex-valued process with the imaginary part equal to zero.

where $\widetilde{\boldsymbol{\Gamma}}_{\boldsymbol{\phi}} := \widetilde{\boldsymbol{\Gamma}}$ (cf. Eq. (4)). Notice that we have re-indexed the matrices with respect to the phase parameters. The probability density function of $\widetilde{\mathbf{y}}_N$ is written as follows:

$$p(\widetilde{\mathbf{y}}_N|\boldsymbol{\psi}) = \frac{1}{\pi^N(\det(\widetilde{\mathbf{R}}_{\boldsymbol{\psi}}))^{1/2}} \exp\left\{-\frac{1}{2}\,\widetilde{\mathbf{y}}_N^{\mathrm{H}}\widetilde{\mathbf{R}}_{\boldsymbol{\psi}}^{-1}\widetilde{\mathbf{y}}_N\right\}.$$

According to Eq. (12), one can see that $\det(\widetilde{\mathbf{R}}_{\boldsymbol{\psi}})$ is independent of $\boldsymbol{\psi}$. This implies that

$$L(\widetilde{\mathbf{y}}_N, \boldsymbol{\phi}, \boldsymbol{\psi}(k)) = \exp\left\{-\frac{1}{2}\,\widetilde{\mathbf{y}}_N^{\mathrm{H}}\left(\widetilde{\mathbf{R}}_{\boldsymbol{\psi}(k)}^{-1} - \widetilde{\mathbf{R}}_{\boldsymbol{\phi}}^{-1}\right)\widetilde{\mathbf{y}}_N\right\}.$$

We now seek to derive the following term:

$$B_{k,l} = \mathbb{E}\left[\exp\left\{-\frac{1}{2}\widetilde{\mathbf{y}}_N^{\mathrm{H}}\mathbf{W}_{k,l}\widetilde{\mathbf{y}}_N\right\}\right]$$

with

$$\mathbf{W}_{k,l} = \widetilde{\mathbf{R}}_{\boldsymbol{\psi}(k)}^{-1} + \widetilde{\mathbf{R}}_{\boldsymbol{\psi}(l)}^{-1} - 2\widetilde{\mathbf{R}}_{\boldsymbol{\phi}}^{-1}.$$

Towards this objective, we first rewrite the above term in terms of $\check{\mathbf{y}}_N$ as follows:

$$B_{k,l} = \mathbb{E}\left[\exp\left\{-\frac{1}{2}\check{\mathbf{y}}_N^{\mathrm{T}}\check{\mathbf{W}}_{k,l}\check{\mathbf{y}}_N\right\}\right],$$

where $\check{\mathbf{W}}_{k,l} = \mathbf{P}^{\mathrm{H}}\mathbf{W}_{k,l}\mathbf{P}$ with $\mathbf{P} = [\mathbf{I}_N, i\mathbf{I}_N; \mathbf{I}_N, -i\mathbf{I}_N]$. Let $\check{\mathbf{R}}_{\boldsymbol{\phi}} = \mathbf{E}[\check{\mathbf{y}}_N\check{\mathbf{y}}_N^{\mathrm{T}}]$ be the covariance matrix of the real-valued process $\check{\mathbf{y}}_N$. Since $\check{\mathbf{R}}_{\boldsymbol{\phi}}$ is symmetric, it can be diagonalized using eigen-decomposition as follows: $\check{\mathbf{R}}_{\boldsymbol{\phi}} = \mathbf{D}_x^{\mathrm{T}}\boldsymbol{\Lambda}_x\mathbf{D}_x$ where $\mathbf{D}_x$ is an orthogonal matrix and $\boldsymbol{\Lambda}_x$ is a diagonal matrix composed of the eigenvalues of $\check{\mathbf{R}}_{\boldsymbol{\phi}}$. Let $\mathbf{x} = \boldsymbol{\Lambda}_x^{-1/2}\mathbf{D}_x\check{\mathbf{y}}_N$. By construction, vector $\mathbf{x}$ is still Gaussian with covariance matrix $\mathbf{Id}_{2N}$. Thus the components of $\mathbf{x}$ are mutually independent. Thus, we obtain

$$B_{k,l} = \mathbb{E}[\exp\{-\tfrac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{V}_{k,l}\mathbf{x}\}].$$

with $\mathbf{V}_{k,l} = \boldsymbol{\Lambda}_x^{1/2}\mathbf{D}_x\check{\mathbf{W}}_{k,l}\mathbf{D}_x^{\mathrm{T}}\boldsymbol{\Lambda}_x^{1/2}$. Once again, since $\mathbf{V}_{k,l}$ is symmetric, it can be decomposed as follows: $\mathbf{V}_{k,l} = \mathbf{D}^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{D}$ where $\mathbf{D}$ is an orthogonal matrix and $\boldsymbol{\Lambda} = \mathrm{diag}([\lambda_0, \ldots, \lambda_{2N-1}])$ with $\{\lambda_m\}_{0\leqslant m\leqslant 2N-1}$ being the eigenvalues of $\mathbf{V}_{k,l}$. Let $\mathbf{z} = [z_0, \ldots, z_{2N-1}]^{\mathrm{T}} = \mathbf{D}\mathbf{x}$. Vector $\mathbf{z}$ is Gaussian with the identity matrix as covariance matrix, i.e., with independent components. Therefore, we get

$$\begin{aligned} B_{k,l} &= \mathbb{E}\left[\exp\left\{-\frac{1}{2}\mathbf{z}^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{z}\right\}\right] \\ &= \mathbb{E}\left[\exp\left\{-\frac{1}{2}\sum_{m=0}^{2N-1}\lambda_m z_m^2\right\}\right] \\ &= \prod_{m=0}^{2N-1}\mathbb{E}\left[\exp\left\{-\frac{1}{2}\lambda_m z_m^2\right\}\right]. \end{aligned}$$

One can easily check that z_m^2 follows a Chi-square distribution with one degree of freedom. This leads to [19]

$$B_{k,l} = \begin{cases} \displaystyle\prod_{m=0}^{2N-1}\frac{1}{\sqrt{1+\lambda_m}} & \text{if } (1+\lambda_m) > 0, \forall m, \\ +\infty & \text{otherwise.} \end{cases}$$

The last expression can be compacted as follows:

$$B_{k,l} = \begin{cases} \dfrac{1}{\sqrt{\det(\mathbf{Id}_{2N} + \mathbf{V}_{k,l})}} & \text{if} \quad \mathbf{Id}_{2N} + \mathbf{V}_{k,l} > 0, \\ +\infty & \text{otherwise.} \end{cases}$$

After straightforward algebraic manipulations, we finally obtain

$$B_{k,l} = \begin{cases} \dfrac{1}{\sqrt{\det(\mathbf{Q}_{k,l})}} & \text{if } \mathbf{Q}_{k,l} > 0, \\ +\infty & \text{otherwise,} \end{cases} \tag{13}$$

with

$$\begin{aligned} \mathbf{Q}_{k,l} &= \mathbf{I}_{2N} + \mathbf{W}_{k,l}\widetilde{\mathbf{R}}_{\boldsymbol{\phi}} \\ &= (\widetilde{\mathbf{R}}_{\boldsymbol{\psi}(k)}^{-1} + \widetilde{\mathbf{R}}_{\boldsymbol{\psi}(l)}^{-1})\widetilde{\mathbf{R}}_{\boldsymbol{\phi}} - \mathbf{I}_{2N}. \end{aligned} \tag{14}$$

Let $\mathbf{R}_{\boldsymbol{\phi}} = \mathbf{E}[\mathbf{y}_N\mathbf{y}_N^{\mathrm{H}}]$ and $\mathbf{U}_{\boldsymbol{\phi}} = \mathbf{E}[\mathbf{y}_N\mathbf{y}_N^{\mathrm{T}}]$ be the nonconjugate and conjugate correlation of the received signal respectively. Matrix $\mathbf{U}_{\boldsymbol{\phi}}$ is non-null because of the non-circularity of the signal. In [13] and [14], the expression for $B_{k,l}$ is slightly different from ours: the square root is removed and $\mathbf{Q}_{k,l}$ depends only on $\mathbf{R}_{\boldsymbol{\phi}}$ instead of $\widetilde{\mathbf{R}}_{\boldsymbol{\phi}}$. Actually, our expression is an extension of the one obtained in [13] and [14]. Indeed, by setting $\mathbf{U}_{\boldsymbol{\phi}} = 0$ in Eq. (2), $\widetilde{\mathbf{R}}_{\boldsymbol{\phi}}$ becomes block-diagonal and then our expression reduces to the one introduced in [13] and [14].

In the sequel, we focus on the frequency parameter because the outliers effect particularly affects frequency estimation. For the standard test-points described in Eq. (11), the BB for ϕ_1 takes the following form [14]:

$$\mathrm{BB}(\phi_1) = \sup_{\varepsilon_0,\varepsilon_1} \frac{\varepsilon_1^2}{(\mathbf{B}_{1,1}-1)-(\mathbf{B}_{0,1}-1)(\mathbf{B}_{0,0}-1)^{-1}(\mathbf{B}_{0,1}-1)}.$$

The term $(\mathbf{B}_{0,1}-1)(\mathbf{B}_{0,0}-1)^{-1}(\mathbf{B}_{0,1}-1)$ represents the loss in performance due to joint phase parameter estimation.

5. Threshold analysis

In this section, we derive a closed-form expression for the threshold beyond which the Barankin bound and the CRB are different, in terms of the SNR and N, the number of available samples.

As in [15], we only concentrate on the frequency ϕ_1 by assuming the phase ϕ_0 to be known.

For simplicity, we also consider the multiplicative noise, $\{a(n)\}$, to be white with unit variance. Consequently, the statistics of $a(n)$ is only characterized by the following parameter $\rho = \mathbb{E}[s(n)^2]$. Notice that ρ captures the information about the non-circularity power.

According to Eq. (8), we get

$$\mathrm{GCRB}_a(\phi_1) = \frac{v}{N^3}$$

with

$$v = \frac{3[(1-|\rho|^2)+2\sigma^2+\sigma^4]}{8\pi^2|\rho|^2}.$$

We recall that

$$\mathrm{BB}(\phi_1) = \sup_{\varepsilon} S(\varepsilon),$$

where

$$S(\varepsilon) = \frac{\varepsilon^2}{\left(\frac{1}{\sqrt{\det(Q_\varepsilon)}}-1\right)} \tag{15}$$

with $Q_\varepsilon := Q_{1,1}$ (cf. Eq. (14)).

According to Eq. (13), the determinant of Q_ε can be simplified as follows:

$$\det(Q_\varepsilon) = \begin{cases} \prod_{k=0}^{N-1} \frac{\vartheta_k}{((1+\sigma^2)^2-|\rho|^2)^2} & \text{if } \prod_{k=0}^{N-1}\vartheta_k > 0, \\ +\infty & \text{otherwise,} \end{cases} \tag{16}$$

with

$$\vartheta_k = (1+\sigma^2)^2 - |\rho|^2 - 8|\rho|^2((1+\sigma^2)^2-|\rho|^2) \times \sin^2(2\pi\varepsilon k).$$

The proof of the above formula is straightforward and is thus omitted.

We recall that the conjugate (resp. nonconjugate) correlation of the received signal $y(n)$ takes the following expression $u_a(\tau)\mathrm{e}^{2\mathrm{i}\pi(2\phi_1 n+\phi_1\tau)}$ (resp. $r_a(\tau)\mathrm{e}^{2\mathrm{i}\pi\phi_1\tau}$). Consequently, if $2\phi_1$ belongs to an interval larger than $[-1/2, 1/2]$, then there would be an a priori ambiguity on the frequency ϕ_1. Without loss of generality, we thus choose the search interval for ϕ_1 to be $\mathscr{I} = [-1/4, 1/4]$. Consequently the test point ε is defined in $\mathscr{I}$.

As already observed in [15], we show, after intensive numerical trials, that the maximum of the function $S(\varepsilon)$ is obtained either for $\varepsilon = 0$ or for $\varepsilon \approx 1/4$ (see Fig. 5). Using Eq. (10), we get

$$\mathrm{BB}(\phi_1) = \max(\mathrm{GCRB}(\phi_1), S(1/4)). \tag{17}$$

After some simple but tedious algebraic manipulations, $\det(Q_\varepsilon)$ for $\varepsilon = 1/4$ is found to satisfy

$$\det(Q_{1/4}) = \begin{cases} \vartheta^{N/2} & \text{if } \vartheta > 0, \\ +\infty & \text{otherwise,} \end{cases}$$

with

$$\vartheta = \frac{(1+\sigma^2)^2 - 9|\rho|^2}{(1+\sigma^2)^2-|\rho|^2}.$$

Consequently, the outliers effect surely occurs if

$$S(1/4) \geqslant \mathrm{GCRB},$$

which implies that

$$N \geqslant \sqrt[3]{16v\frac{1-\vartheta^{N/4}}{\vartheta^{N/4}}}. \tag{18}$$

The last equation enables us to predict the value of the threshold below which there is a mismatch between the BB (of order 1) and the CRB. For instance, at $\mathrm{SNR} = -5\,\mathrm{dB}$ and $\rho = 1$, we can assert that if N is less than 60, then a gap between the two bounds appears. This gap prevents any unbiased estimate to reach the CRB. Nevertheless, Eq. (18) only provides a lower bound on the threshold since we have only computed the BB of order one.

6. Review of the square-power-based estimator

If the multiplicative noise is non-circular, the following estimate, which is related to the squaring loop method [17], makes sense:

$$\hat{\phi}_1^{(N)} = \underset{\phi\in[-1/4,1/4)}{\arg\max}\; J_N(\phi)$$

with

$$J_N(\phi) = \sum_{l=-L}^{L} \left| \frac{1}{N} \sum_{n=0}^{N-1} y(n)y(n+l)\mathrm{e}^{-4\mathrm{i}\pi\phi n} \right|^2. \tag{19}$$

In the real-valued case, this estimate with $L = 0$ is well known ([3,4,17,23] and references therein): surprisingly, the choice $L = 0$ was always made even when the noise was colored. In [3], the performance of this estimate and the GCRB were derived for high SNR. In [4], the asymptotic GCRB was compared with the asymptotic performance of the estimate for each SNR. It was proven that the estimate is asymptotically efficient for high SNR.

The extension of the above estimator to any value of L was presented and analyzed in [18] in the case of a non-circular complex-valued multiplicative. The asymptotic estimation covariance was expressed for any L and any SNR. It was shown in [18] that if the memory of $a(n)$ is finite, say M, the asymptotic covariance is minimized for $L = M$ and takes (for $L = M$) the following form:

$$\gamma_{\phi_1} \sim \frac{3\eta}{4\pi^2 N^3} \tag{20}$$

with

$$\eta = \frac{\int_0^1 |c(\mathrm{e}^{2\mathrm{i}\pi f})|^2 \mathscr{X}(\mathrm{e}^{2\mathrm{i}\pi f})\,\mathrm{d}f}{\left(\int_0^1 |c(\mathrm{e}^{2\mathrm{i}\pi f})|^2\,\mathrm{d}f\right)^2}.$$

The above equation was obtained after simple manipulations of the main equations provided in Theorem 4 of [18].

The comparison between the square-power estimate and the CRB has led to numerous papers ([3,4,23] and references therein). In particular, it was shown via simulations in [23] that the performance of the square-power-based estimate is always close to the CRB except when the outliers effect occurs. More specifically, in [3,4], this estimator was proven to be asymptotically efficient at high SNR regardless of the noise color. Using Eqs. (20) and (8), we hereafter show that the asymptotic covariance of the square-power-based frequency estimate and the asymptotic GCRB are equal only if the mapping $f \mapsto \mathscr{X}(\mathrm{e}^{2\mathrm{i}\pi f})$ is constant. Indeed, by using $c(\mathrm{e}^{2\mathrm{i}\pi f}) = c(\mathrm{e}^{-2\mathrm{i}\pi f})$ and Schwartz's inequality, we have that $\eta \geqslant 1/\xi$, which means that $\gamma_{\phi_1} \geqslant \mathrm{GCRB}$. Equality holds only when $f \mapsto \mathscr{X}(\mathrm{e}^{2\mathrm{i}\pi f})$ is constant. This implies that the square-power-based estimate (assuming $L = M$) is at least asymptotically efficient for any SNR if the multiplicative noise is white.

The cost function to be maximized in Eq. (19) admits numerous local maxima. However, it generally has a particular shape which can be exploited. Indeed, this cost function can usually be depicted as a flat ground-level noise plus a peak around the true value ϕ_1. Therefore, one can proceed in two steps to compute the maximization in Eq. (19):

- The first step, also called *coarse* search, detects the main peak by means of the fast Fourier transform (FFT).
- the second step, also called *fine* search, refines the estimation around the detected peak by means of the gradient-descent algorithm initialized by the coarse estimate provided by the first step.

Generally the outliers effect of the squared-power estimate corresponds to the failure of the coarse search [11].

7. Numerical illustrations

For simplicity, the multiplicative noise is assumed to be MA(1), i.e. $a(n) = s(n) + as(n-1)$ where $\{s(n)\}$ is a white non-circular Gaussian process with $\rho = \mathbb{E}[s(n)^2]$. Notice that a (resp. ρ) captures all the information about the color (resp. the non-circularity) of the multiplicative noise. We also set $\text{SNR} = 10\log_{10}((1+a^2)/\sigma^2)$.

In Fig. 1, the bounds and the MSE of the square-power-based estimate are depicted versus SNR with $N = 100$, $\rho = 0.75$, $a = 0$. We also display the threshold obtained via Eq. (18). First of all, we observe that the GCRB and the asymptotic GCRB are very close. Furthermore the well-known outliers effect obviously occurs at low and medium SNR [12]. We also remark that the expected threshold is in perfect agreement with the beginning of the real gap between both bounds. The threshold corresponding to the square-power-based estimate is much larger than that observed with the BB.

In Fig. 2, the same performance measures are displayed vs. N with $\text{SNR} = 10\,\text{dB}$, $\rho = 0.25$, $a = 0$. The same comments can be drawn. Furthermore the outliers effect vanishes as soon as N is chosen large enough. Finally, the good fitting between the exact CRB and the asymptotic CRB has been observed whatever the value of a for reasonable values of N (e.g., less than 50). Therefore, in the sequel, we omit plotting the asymptotic GCRB.

In Fig. 3, the performance measures are depicted vs. a with $\text{SNR} = 10\,\text{dB}$, $N = 100$, $\rho = 0.75$. It is seen that performance depends only slightly on a. For this configuration, the empirical MSE fits well with the theoretical MSE. The performance of the square-power estimate almost reaches the GCRB even if the multiplicative noise is colored.

In Fig. 4, the performance measures are displayed versus ρ with $\text{SNR} = 10\,\text{dB}$, $N = 64$, $a = 0$. According to Eq. (18), the threshold is equal to $\rho = 0.275$. One can notice that the more $a(n)$ is non-circular (i.e., ρ increases), the better the estimation performance. Furthermore, the outliers effect rapidly degrades the performance if $a(n)$ is not non-circular enough.

In Fig. 5, according to Eqs. (8), (15) and (16), we plot the ratio $S(\varepsilon)$ to GCRB with $N = 64$,

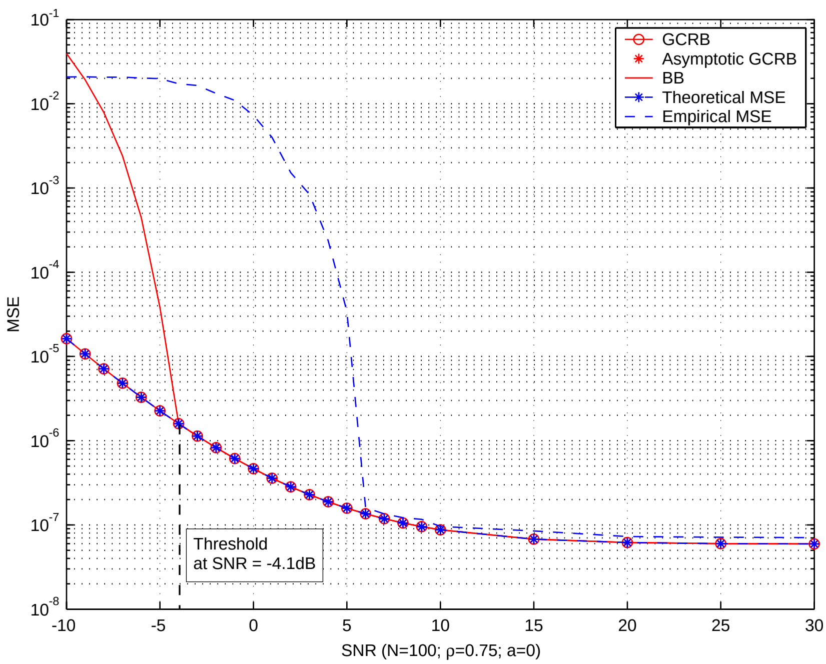

Fig. 1. MSE vs. SNR.

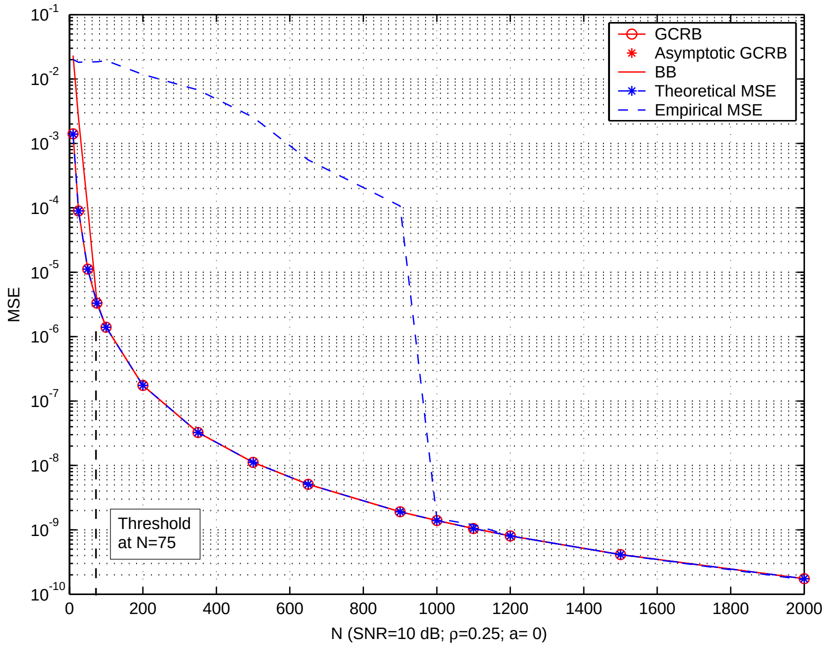

Fig. 2. MSE vs. N.

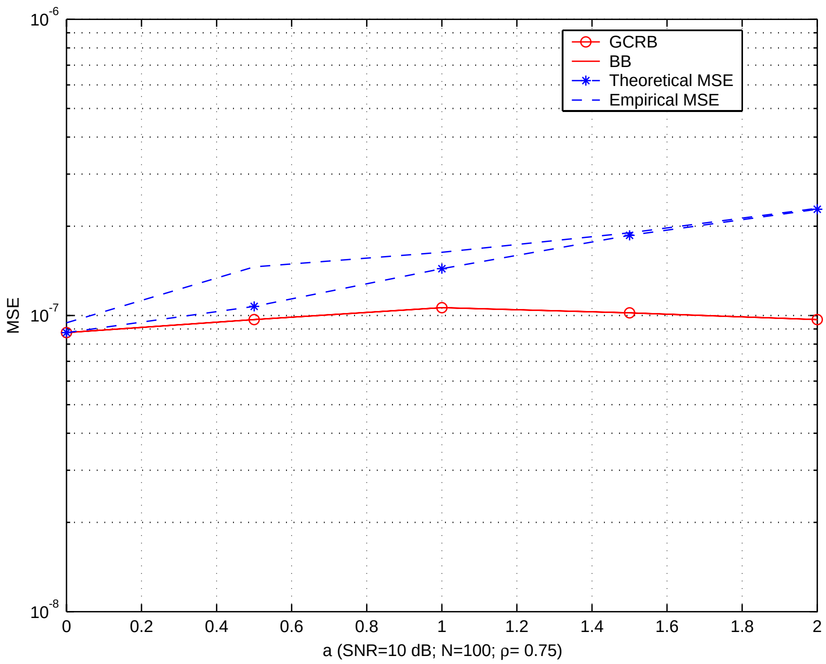

Fig. 3. MSE vs. a.

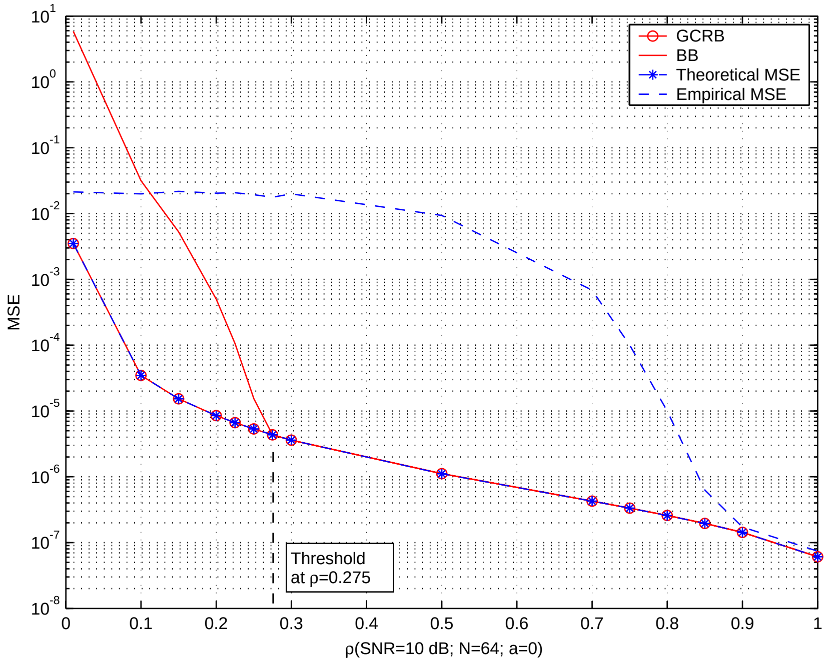

Fig. 4. MSE vs. ρ.

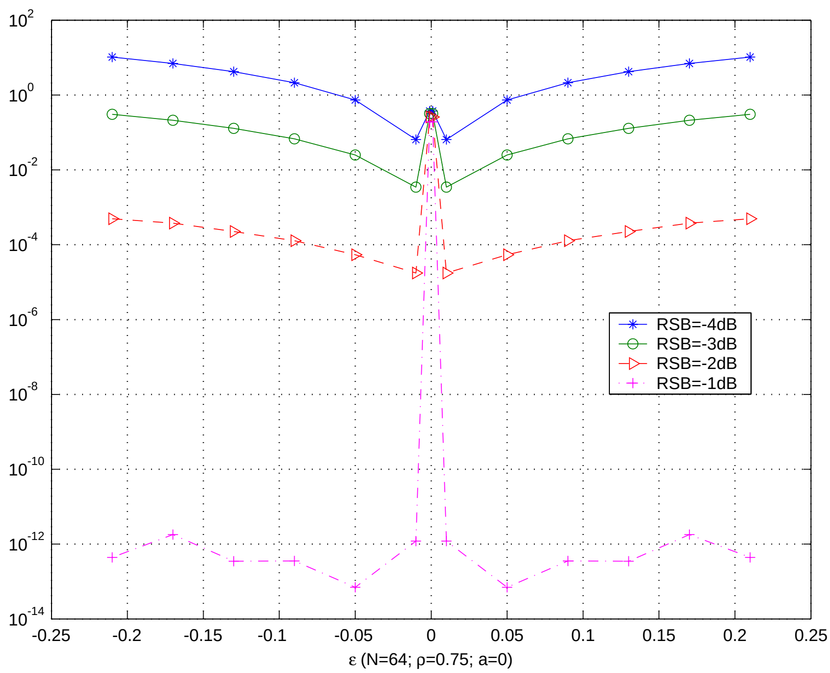

Fig. 5. $S(\varepsilon)/\mathrm{GCRB}$.

$\rho = 0.75$, $a = 0$. This figure confirms that the maximum of the mapping $\varepsilon \mapsto S(\varepsilon)$ is close to 0 or close to 1/4. This supports the approximation done in Eq. (17).

8. Conclusion and future work

In this paper, closed-form expressions for the asymptotic GCRB and the BB were derived. The threshold beyond which the BB and the GCRB differ was also evaluated theoretically.

The threshold provided by the BB of order 1 turned out to be far away from the empirical threshold of the square-power estimate. As observed in [24], the Ziv–Zakaï bound seems to be more accurate for analyzing the empirical threshold of the square-power estimate. Therefore, further work should concentrate on such a bound.

Furthermore, in the context of digital communications, the Gaussian assumption on the multiplicative noise is not justified, particularly in the case of small memory channels. Therefore, other bounds should be derived and compared with the results presented in this paper.

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