Blind Feedforward Cyclostationarity-Based Timing Estimation for Linear Modulations

Yan Wang, Erchin Serpedin, and Philippe Ciblat

***Abstract*—By exploiting a general cyclostationary (CS) statistics-based framework, this letter develops a rigorous and unified asymptotic (large sample) performance analysis setup for a class of blind feedforward timing epoch estimators for linear modulations transmitted through time nonselective flat-fading channels. Within the proposed CS framework, it is shown that several estimators proposed in the literature can be asymptotically interpreted as maximum likelihood (ML) estimators applied on a (sub)set of the second- (and/or higher) order statistics of the received signal. The asymptotic variance of these ML estimators is established in closed-form expression and compared with the modified Cramér–Rao bound. It is shown that the timing estimator proposed by Oerder and Meyr achieves asymptotically the best performance in the class of estimators which exploit all the second-order statistics of the received signal, and its performance is insensitive to oversampling rates P as long as $P \geq 3$. Further, an asymptotically best consistent estimator, which achieves the lowest asymptotic variance among all the possible estimators that can be derived by exploiting jointly the second- and fourth-order statistics of the received signal, is also proposed.**

***Index Terms*—Cramér–Rao bound (CRB), cyclostationarity, maximum likelihood (ML), synchronization, timing estimation.**

I. Introduction

Timing estimation is a challenging but very important task for reliable detection in synchronous receivers. For bandwidth efficiency and burst transmission reasons, nondata aided or blind feedforward timing estimation architectures have received much attention in the literature [2], [3], [6]–[9], [12]. Originally in [8], Oerder and Meyr (O&M) proposed a blind feedforward square timing recovery technique for digital data transmission by linear modulation schemes. Several extensions similar in form to the O&M estimator were later reported in [2], [3], and [12]. These estimators employ a second-law nonlinearity (SLN) on the received samples, and exhibit weak performance when operating with narrowband signaling pulses [5] and [7]. With the assumption of low signal-to-noise ratio (SNR) and phase-shift keying (PSK) constellations, [7] proposes an *ad hoc* feedforward SNR-dependent maximum likelihood (ML)-based timing estimator that assumes a logarithmic nonlinearity (LOGN) and is shown to exhibit better performance than the SLN (O&M) estimator. However, good estimates are obtained by fixing the SNR value to 5 dB. Moreover, since its performance analysis is not fully investigated, no thorough conclusions may be drawn [7]. Reference [9] proposes an approximate performance analysis of the SLN, fourth-law (FLN), and absolute value (AVN) nonlinearities-based estimators assuming binary PSK modulations and a stationary statistics framework.

Irrespective of the nonlinearity function used, one of the common features of all the above-mentioned blind feedforward timing estimators is the exploitation of the cyclostationary (CS) statistics induced by oversampling the received signal. Generalizing the previous results of [14], our goal herein is to exploit optimally the CS-statistics of the received signal in order to develop efficient estimators, and rigorous and thorough performance analysis setups for the existing blind timing estimators. We will show that the CS-based estimators can be asymptotically interpreted as ML-estimators applied on a (sub)set of second- (and/or higher) order cyclic correlations of the received signal. The asymptotic variances of these ML estimators, which can serve as the lower bounds of the performance of the CS-based timing estimators, are established in closed-form expressions. (Due to space limitations, our analysis concentrates only on SLN estimators. As for other nonlinearities, since they can be approximated by a finite power series expansion including second-order, fourth-order, and generally up to sixth-order terms, those estimators can also be analyzed similarly in the presented framework.) It is also shown in this letter that the O&M estimator achieves asymptotically the best performance in the class of SLN estimators, and its performance is insensitive to oversampling rates P whenever $P \geq 3$. Further, to assess the best performance achievable by the CS-based estimators, the asymptotically best consistent (ABC) timing estimator will be derived. Computer simulations illustrate that the proposed ABC estimator improves significantly the performance of the existing timing estimators, especially when dealing with strongly bandlimited signaling.

II. System Model and Assumptions

Consider the following standard baseband received signal of a linearly modulated data sequence transmitted through a time nonselective flat-fading channel (see, e.g., [3], [11])

$$x(n) := x(t)|_{t=nT_s} = \sum_l w(l)h(n - lP - \epsilon P) + v(n) \quad (1)$$

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where $\{w(l)\}$ is the sequence of transmitted symbols, $v(n)$ is the oversampled additive noise, the integer $P := T/T_s > 1$ denotes the oversampling factor with T and T_s representing the symbol period and sampling period, respectively, $h(n) := h_c(t)|_{t=nT_s}$ where $h_c(t)$ denotes the convolution of the transmitter's signaling pulse and the receiver filter,[1] and ϵ stands for the unknown (normalized) symbol timing epoch. We invoke the following assumptions.

(AS1) $w(l)$ is a zero-mean independently and identically distributed (i.i.d.) sequence with values drawn from a linearly modulated circular complex constellation with unit variance, i.e., $\sigma_w^2 := E\{|w(l)|^2\} = 1$. This assumption is not at all restrictive since all the derivations can be extended to noncircular modulations and symbol streams with arbitrary correlations.

(AS2) $v(n)$ is a complex-valued circular Gaussian process independent of $w(l)$, with independent real and imaginary components and autocorrelations $r_{2v}(\tau)$, where τ is an integer lag. The variance of $v(n)$ is equal to N_0 and the SNR is defined as SNR $:= 10\log_{10}(1/N_0)$.

(AS3) With no loss of generality, herein the combined filter $h_c(t)$ is assumed to be a raised cosine pulse of bandwidth $[-(1+\rho)/2T, (1+\rho)/2T]$, where the parameter ρ represents the rolloff factor $(0 < \rho \leq 1)$ [11, Ch. 9].

(AS4) In [2], [3], and [6]–[8], the oversampling rate $P \geq 4$ is adopted to avoid certain aliasing effects. Reference [14] points out that when $P = 2$, the aliasing effects have to be taken into account, and a different form for the SLN timing estimator results. To save space and avoid any overlapping with the results of [14], we assume $P \geq 3$.

(AS5) $v(n)$ satisfies the so-called mixing conditions, i.e., its kth-order cumulant is absolutely summable for any k [1], [3].

Assumptions **(AS1)**–**(AS4)** are quite general, and the mixing condition **(AS5)** resumes to the fact that sufficiently separated samples are approximately independent, a condition which ensures the consistency of the estimators proposed herein and is usually satisfied in practice by all finite memory signals [3]. Therefore, the results presented in this letter are quite general and suitable for many applications of practical interest.

In Section III, first we briefly introduce the blind feedforward SLN timing estimators proposed in [2], [3], [8], and [12], and then propose a unifying ML framework that will enable to analyze their asymptotic performance.

III. Second-Order CS Statistics-Based Timing Estimators

A. SLN Timing Estimators

By exploiting (1) and taking into account the assumptions **(AS1)**–**(AS4)**, straightforward calculations show that the cyclic correlations of the nonstationary process $x(n)$ are given for $k = 0, \ldots, P-1$ by the expressions [3], [14]

$$\begin{aligned} R_{2x}(k;\tau) &:= \frac{1}{P}\sum_{n=0}^{P-1} E\{x^*(n)x(n+\tau)\} e^{-2i\pi\frac{kn}{P}} \\ &= \frac{1}{P} e^{i\pi\frac{k\tau}{P}} e^{-2i\pi k\epsilon} G(k;\tau) + N_0 h(\tau)\delta(k) \\ G(k;\tau) &:= \frac{P}{T}\int_{-\frac{P}{2T}}^{\frac{P}{2T}} H_c\left(F - \frac{k}{2T}\right) \\ &\quad \times H_c\left(F + \frac{k}{2T}\right) e^{2i\pi\frac{\tau T F}{P}} dF \end{aligned} \tag{2}$$

where $H_c(F)$ denotes the continuous-time Fourier transform of $h_c(t)$, and the notation $\delta(\cdot)$ stands for the Kronecker's delta. The frequencies k/P (or simply k), for $k = 0, \ldots, P-1$, are referred to as cyclic frequencies or cycles [3]. Due to the symmetry property of the raised-cosine function $h_c(t)$, it is easy to check that $G(k;\tau)$ is real-valued, and $G(k;\tau)$ (thus, $R_{2x}(k;\tau)$) is nonzero only for cycles $k = 0, \pm 1$. Since $R_{2x}(k;\tau) = \exp\{2i\pi k\tau/P\} R_{2x}^*(-k;-\tau)$, it follows that only the subset $\{R_{2x}(1;\tau)\}$, $\forall\tau$, represents all the second-order statistics that may be used for estimating the unknown timing epoch ϵ. In practice, the cyclic correlations $R_{2x}(k;\tau)$ have to be estimated from a finite number of samples N, and the standard sample estimate of $R_{2x}(k;\tau)$ is given by (see, e.g., [1] and [3])

$$\hat{R}_{2x}(k;\tau) = \frac{1}{N}\sum_{n=0}^{N-\tau-1} x^*(n)x(n+\tau)e^{-2i\pi\frac{kn}{P}}, \qquad \tau \geq 0 \tag{3}$$

which, under **(AS5)**, is asymptotically unbiased and mean square sense consistent.

Based on (2), the following general SLN timing estimator may be proposed

$$\hat{\epsilon} = -\frac{1}{2\pi}\arg\left\{\hat{R}_{2x}(1;\tau)e^{-\frac{i\pi\tau}{P}}\right\}. \tag{4}$$

Note that the second-order CS-based timing estimators proposed in the literature choose different values for the timing lag τ in (4) ($\tau = 0$ in [2] and [8], $\tau = 1$ in [3], and $\tau = P$ in [12]). Next, an ML framework is proposed to analyze the performance of the general SLN timing estimator (4) and to possibly design improved performance estimators by exploiting the entire information provided by all the second-order statistics of the received signal.

B. ML Framework and Asymptotic Performance Analysis

Define the vector of cyclic correlations:[2] $\mathbf{R}_{2x} := [R_{2x}(1;-\tau_m)\ldots R_{2x}(1;\tau_m)]^T$, where τ_m denotes an arbitrary nonnegative integer. According to [1], the sample cyclic correlation estimates $\{\hat{R}_{2x}(1;\tau)\}$, $\forall\tau$ are asymptotically jointly complex-valued and normally distributed. Thus, $\sqrt{N}[\hat{\mathbf{R}}_{2x} - \mathbf{R}_{2x}]$ is asymptotically jointly complex normal

[1]The subscript $_c$ is used to denote a continuous-time signal.

[2]The superscripts ^{T} and ^{H} denote transposition and conjugate transposition, respectively.

with zero-mean $\mathbf{0} := [0 \dots 0]^T$, and its covariance and relation matrices are given by

$$\boldsymbol{\Gamma} := \lim_{N\to\infty} NE\left\{(\hat{\mathbf{R}}_{2x} - \mathbf{R}_{2x})(\hat{\mathbf{R}}_{2x} - \mathbf{R}_{2x})^H\right\}$$
$$\widetilde{\boldsymbol{\Gamma}} := \lim_{N\to\infty} NE\left\{(\hat{\mathbf{R}}_{2x} - \mathbf{R}_{2x})(\hat{\mathbf{R}}_{2x} - \mathbf{R}_{2x})^T\right\}$$

respectively, whose closed-form expressions of the (u,v) entries, $u,v = -\tau_m,\dots,\tau_m$, are established in [14], and with the notations adopted so far, can be expressed as

$$\boldsymbol{\Gamma}_{u,v} = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_{2x}(0;f) S_{2x}^*\left(0; f - \frac{1}{P}\right) e^{2i\pi(u-v)f} \mathrm{d}f + \kappa P R_{2x}(1;u) R_{2x}^*(1;v), \tag{5}$$

$$\widetilde{\boldsymbol{\Gamma}}_{u,v} = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_{2x}(1;f) S_{2x}^*\left(-1; f - \frac{1}{P}\right) e^{2i\pi(u+v)f} df + \kappa P R_{2x}(1;u) R_{2x}(1;v) \tag{6}$$

where κ stands for the kurtosis of $w(l)$ and $S_{2x}(k;f) := \sum_\tau R_{2x}(k;\tau)\exp\{-2i\pi\tau f\}$ denotes the cyclic spectrum of $x(n)$ at cyclic frequency k.

Next, we transform the complex Gaussian probability density function (pdf) $\mathcal{CN}(\mathbf{0}, \boldsymbol{\Gamma}, \widetilde{\boldsymbol{\Gamma}})$ into its equivalent algebraic form of the real Gaussian pdf $f_\epsilon(\hat{\mathbf{U}}_{2x})$ by defining the $(4\tau_m+2)\times 1$-dimensional vector:[3] $\mathbf{U}_{2x} := \left[\mathrm{re}(\mathbf{R}_{2x})^T \ \ \mathrm{im}(\mathbf{R}_{2x})^T\right]^T$. Simple calculations show that the covariance matrix of $\hat{\mathbf{U}}_{2x}$ is given by

$$\boldsymbol{\Lambda} := \lim_{N\to\infty} NE\left\{(\hat{\mathbf{U}}_{2x} - \mathbf{U}_{2x})(\hat{\mathbf{U}}_{2x} - \mathbf{U}_{2x})^T\right\} = \frac{1}{2}\begin{bmatrix} \mathrm{re}(\boldsymbol{\Gamma} + \widetilde{\boldsymbol{\Gamma}}) & \mathrm{im}(\widetilde{\boldsymbol{\Gamma}} - \boldsymbol{\Gamma}) \\ \mathrm{im}(\widetilde{\boldsymbol{\Gamma}} + \boldsymbol{\Gamma}) & \mathrm{re}(\boldsymbol{\Gamma} - \widetilde{\boldsymbol{\Gamma}}) \end{bmatrix}.$$

Now define the error vector $\mathbf{e} := \hat{\mathbf{U}}_{2x} - \mathbf{U}_{2x}$ and consider the following nonlinear regression model: $\hat{\mathbf{U}}_{2x} = \mathbf{U}_{2x}(\epsilon) + \mathbf{e}$. The ABC estimator of ϵ for the above model is given by the nonlinear least-squares estimator weighted by the inverse of the asymptotic covariance matrix of the error vector $\mathbf{e}$, and takes the form [10, Ch. 3], [13, pp. 91–95]:

$$\hat{\epsilon} = \arg\min_{\dot{\epsilon}} J(\dot{\epsilon})$$
$$J(\dot{\epsilon}) = \frac{1}{2}\left[\hat{\mathbf{U}}_{2x} - \mathbf{U}_{2x}(\dot{\epsilon})\right]^T \boldsymbol{\Lambda}(\dot{\epsilon})^{-1}\left[\hat{\mathbf{U}}_{2x} - \mathbf{U}_{2x}(\dot{\epsilon})\right] \tag{7}$$

and $\dot{\epsilon}$ means the trial value of ϵ. As $\mathbf{e}$ is asymptotically normally distributed, the ABC estimator (7) is nothing else than the asymptotic ML estimator of $\hat{\epsilon}$ in terms of the observations contained in the vector $\hat{\mathbf{U}}_{2x}$.

The ABC estimator is computationally very intensive and may suffer from possible local convergence problems. By exploiting (2), $\mathbf{U}_{2x}(\epsilon)$ takes the following expression: $\mathbf{U}_{2x}(\epsilon) = \boldsymbol{\Phi}\cdot\boldsymbol{\theta}$, where $\boldsymbol{\theta} := [\theta_0 \ \ \theta_1]^T = [\cos(2\pi\epsilon) \ \ \sin(2\pi\epsilon)]^T$, and

$$\boldsymbol{\Phi} := \frac{1}{P}\begin{bmatrix} \boldsymbol{\Phi}_1 & \boldsymbol{\Phi}_2 \\ \boldsymbol{\Phi}_2 & -\boldsymbol{\Phi}_1 \end{bmatrix}$$
$$\boldsymbol{\Phi}_1 := \left[G(1;-\tau_m)\cos\Big(-\frac{\pi\tau_m}{P}\Big) \dots G(1;\tau_m)\cos\Big(\frac{\pi\tau_m}{P}\Big)\right]^T$$
$$\boldsymbol{\Phi}_2 := \left[G(1;-\tau_m)\sin\Big(-\frac{\pi\tau_m}{P}\Big) \dots G(1;\tau_m)\sin\Big(\frac{\pi\tau_m}{P}\Big)\right]^T.$$

Hence, $\hat{\mathbf{U}}_{2x}$ can be rewritten as: $\hat{\mathbf{U}}_{2x} = \boldsymbol{\Phi}\cdot\boldsymbol{\theta} + \mathbf{e}$, which means that the determination of the ABC estimate of ϵ reduces to finding a best linear unbiased estimate (BLUE) of $\boldsymbol{\theta}$ for this linear model. It follows that in this case the BLUE estimator of $\boldsymbol{\theta}$ admits the closed-from expression [4, Ch. 6], [13, Ch. 4]: $\hat{\boldsymbol{\theta}} = (\boldsymbol{\Phi}^T\boldsymbol{\Lambda}^{-1}\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^T\boldsymbol{\Lambda}^{-1}\hat{\mathbf{U}}_{2x}$, and the corresponding 2-by-2 asymptotic covariance matrix of $\hat{\boldsymbol{\theta}}$ is given by

$$\boldsymbol{\Theta} := \lim_{N\to\infty} NE\left\{(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})^T\right\} = \begin{bmatrix} \Theta_{0,0} & \Theta_{0,1} \\ \Theta_{0,1} & \Theta_{1,1} \end{bmatrix} = (\boldsymbol{\Phi}^T\boldsymbol{\Lambda}^{-1}\boldsymbol{\Phi})^{-1}. \tag{8}$$

Given the BLUE estimate of $\boldsymbol{\theta}$, according to [4, Th. 7.4], the ABC estimate of ϵ can be expressed as

$$\hat{\epsilon} = \frac{1}{2\pi}\arctan\left(\frac{\hat{\theta}_1}{\hat{\theta}_0}\right). \tag{9}$$

Considering a Taylor series expansion of the right-hand side of (9) and neglecting the terms of magnitude higher than $o(1/\sqrt{N})$, one can derive the asymptotic variance of $\hat{\epsilon}$, which is summarized in the result.

Theorem 1: The asymptotic variance of the timing epoch estimator (9) is given by

$$\mathrm{avar}(\hat{\epsilon}) := \lim_{N\to\infty} NE\{(\hat{\epsilon}-\epsilon)^2\} = \frac{\sin^2(4\pi\epsilon)}{16\pi^2} \times\left\{\frac{\Theta_{0,0}}{\cos^2(2\pi\epsilon)} + \frac{\Theta_{1,1}}{\sin^2(2\pi\epsilon)} - \frac{4\Theta_{0,1}}{\sin(4\pi\epsilon)}\right\}. \tag{10}$$

It turns out that the O&M estimator is just a special case of the general estimator (7)–(9) with $\tau_m = 0$, and based on Theorem 1, the asymptotic variance of the O&M estimator can be expressed as

$$\mathrm{avar}(\hat{\epsilon}) = \frac{P^2}{8\pi^2 G^2(1;0)}\left[\boldsymbol{\Gamma}_{0,0} - \mathrm{re}(e^{4i\pi\epsilon}\widetilde{\boldsymbol{\Gamma}}_{0,0})\right] \tag{11}$$

which coincides with the expression established earlier in [14]. As can be seen from the above derivations, the O&M estimator is an ABC-estimator, i.e., asymptotically an ML estimator that exploits only one cyclic correlation $(\hat{\mathbf{R}}_{2x}(1;0))$. Similar conclusions can be drawn on the other types of SLN timing estimators [2], [3], [12].

[3]The notations "re" and "im" stand for the real and imaginary part, respectively.

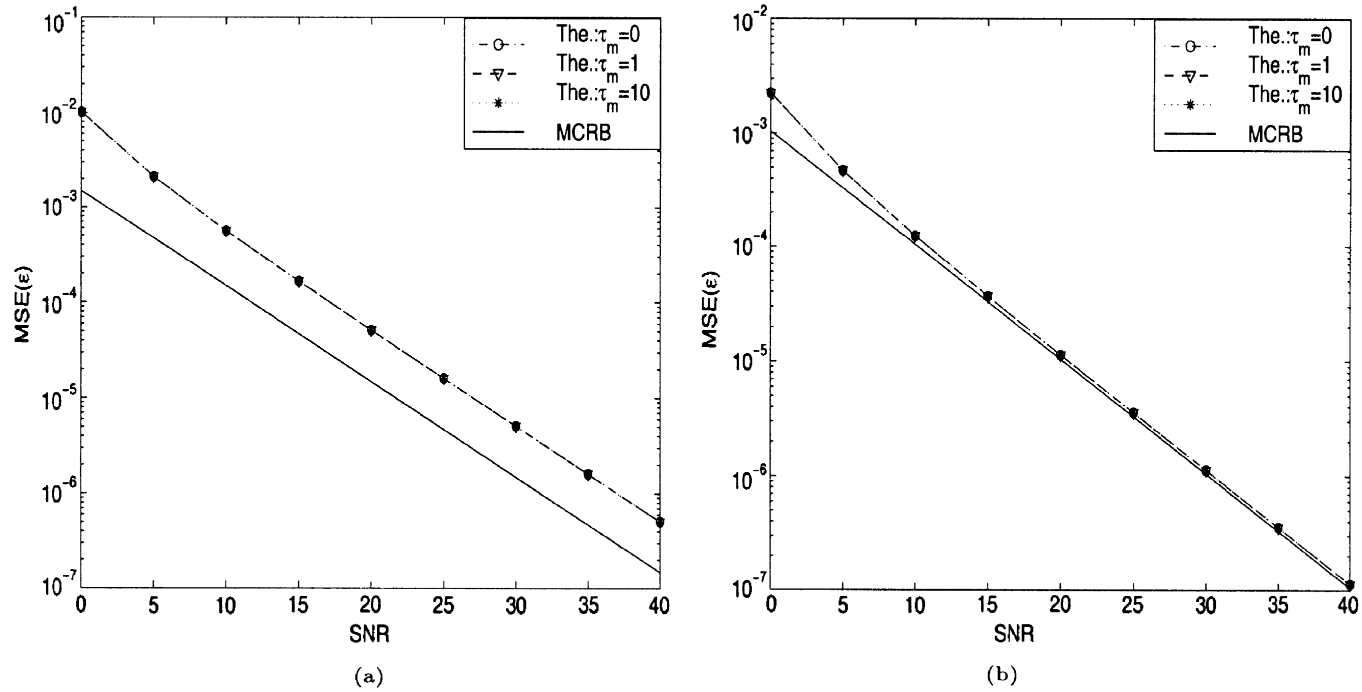

Fig. 1. Theoretical performance of SLN estimate $\hat{\epsilon}$ for different values of τ_m with (a) $\rho = 0.2$ and (b) $\rho = 0.9$.

Now it is of interest to ask whether the performance of the O&M estimator can be improved by exploiting additional second-order statistical information $\hat{\mathbf{R}}_{2x}(1;\tau)$ at lags $\tau \neq 0$. Surprisingly, from the plots shown in Fig. 1(a) and (b), the answer is no. In Fig. 1(a) and (b), we evaluate the theoretical (The.) mean-square error (MSE) of SLN estimate $\hat{\epsilon}$, which asymptotically takes the following form: $\mathrm{MSE}(\hat{\epsilon}) := E\{(\hat{\epsilon}-\epsilon)^2\} = \mathrm{avar}(\hat{\epsilon})/N$, for different values of τ_m in the case of rolloff factors $\rho = 0.2$ and $\rho = 0.9$, respectively, assuming quadrature PSK (QPSK) input symbols, $P = 4$, $\epsilon = 0.3$, and the number of samples $N = 400$ (i.e., the observation length $L = 100$ symbols). The modified Cramér–Rao bound (MCRB) is adopted as a benchmark, and takes the expression $\mathrm{MCRB}(\hat{\epsilon}) = 1/(8\pi^2 L\xi\mathrm{SNR})$, where the parameter ξ, in the case of raised-cosine pulses, is given by [6, p. 65]: $\xi = (1/12) + \rho^2(0.25 - 2/\pi^2)$. The results presented in Fig. 1(a) and (b) are due to the fact that asymptotically, the statistics $\hat{\mathbf{R}}_{2x}(1;\tau)$ at τ other than zero are quite correlated with that at $\tau = 0$, hence, the cost function $J(\dot{\epsilon})$ is totally dominated by $\hat{\mathbf{R}}_{2x}(1;0)$. Therefore, the following conclusion can be drawn: *The performance of all blind feedforward SLN timing estimators which exploit the second-order statistics of the received signal is asymptotically the same as long as the statistical information at timing lag* $\tau = 0$ ($\hat{\mathbf{R}}_{2x}(1;0)$) *has been considered.*

C. Influence of the Oversampling Rate P

In this subsection, we analyze the effect of the oversampling rate P on the SLN timing estimators. Due to the conclusion obtained in the last subsection, we only focus on the O&M estimator, whose asymptotic variance is given by (11). To properly inspect the influence of P, we need to evaluate the theoretical MSE of $\hat{\epsilon}$. Under the assumption of $P \geq 3$, some straightforward calculations yield $\mathrm{MSE}(\hat{\epsilon}) = 8(\zeta_1 - \zeta_2)/\pi^2\rho^2 L$, where

$$\zeta_1 := \zeta_2 + \frac{2N_0}{T^2}\int_{-\frac{\rho}{2T}}^{\frac{\rho}{2T}} H_c^2\left(F+\frac{1}{2T}\right)H_c\left(F-\frac{1}{2T}\right)dF + \frac{N_0^2}{T}\int_{-\frac{\rho}{2T}}^{\frac{\rho}{2T}} H_c\left(F+\frac{1}{2T}\right)H_c\left(F-\frac{1}{2T}\right)dF$$

$$\zeta_2 := \frac{1}{T^3}\int_{-\frac{\rho}{2T}}^{\frac{\rho}{2T}} H_c^2\left(F+\frac{1}{2T}\right)H_c^2\left(F-\frac{1}{2T}\right)dF + \frac{\kappa\rho^2}{64}.$$

Since ζ_1 and ζ_2 do not depend on P, $\mathrm{MSE}(\hat{\epsilon})$ is independent of P whenever $P \geq 3$. One can observe that in the noiseless case ($N_0 \to 0$), $\zeta_1 = \zeta_2$, therefore, the asymptotic variance of the O&M estimate is equal to zero, which means that asymptotically in SNR and N, the variance of the O&M estimate converges to zero faster than $O(1/N)$.

IV. Joint Second- and Fourth-Order CS-Based Timing Estimator

Pulses with small rolloff factors are of interest with bandwidth efficient modulations [7]. SLN timing epoch estimators exhibit bad performance with small rolloffs due to the lack of CS-information and their large self noise, especially in high SNR range [5], [6], [14]. Hence, when dealing with strongly bandlimited pulses, nonlinearities other than the SLN may be considered to improve the performance of timing estimators.

TABLE I
(a) OPT-ESTIMATE OF $\hat{\alpha}_1$ VERSUS SNR AND (b) OPT-ESTIMATE OF $\hat{\alpha}_1$ VERSUS ϵ

SNR(dB)	0	5	10	15	20	25	30	35	40
$\hat{\alpha}_1$	−0.0386	−0.0880	−0.1326	−0.1559	−0.1649	−0.1680	−0.1692	−0.1700	−0.1717

(A)

ϵ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$\hat{\alpha}_1$	−0.1649	−0.1649	−0.1649	−0.1649	−0.1649	−0.1649	−0.1649	−0.1649	−0.1649

(B)

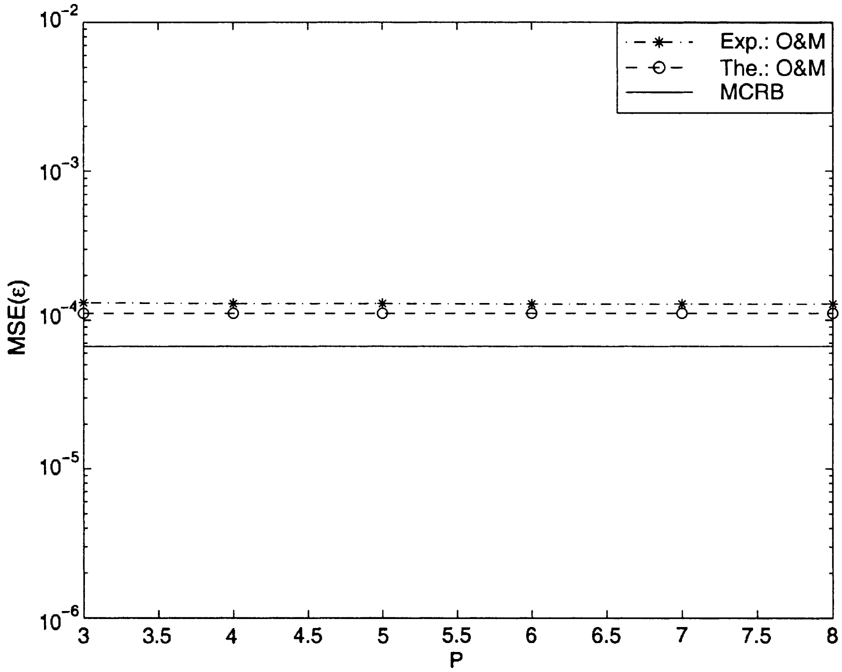

Fig. 2. MSE of the O&M estimator versus P ($\rho = 0.5$, SNR $= 10$ dB).

The most commonly used one is the FLN nonlinearity, and the corresponding estimator takes the form

$$\hat{\epsilon} = -\frac{1}{2\pi}\arg\left\{\hat{R}_{4x}(1;0,0,0)\right\}$$
$$\hat{R}_{4x}(1;0,0,0) = \frac{1}{N}\sum_{n=0}^{N-1}|x(n)|^4 e^{-2i\pi\frac{n}{P}} \tag{12}$$

whose asymptotic variance can be established in a similar expression to (11).

Although the FLN estimator has a better performance than SLN in medium and high SNR ranges with small rolloffs, it is inferior to the latter at low SNRs. Estimators (4) and (12) suggest designing a new optimal (OPT) ABC timing estimator of the following form:

$$\hat{\epsilon} = -\frac{1}{2\pi}\arg\{\boldsymbol{\alpha}^T\hat{\mathbf{R}}_x\}$$
$$\boldsymbol{\alpha} := [1 \quad \alpha_1]^T, \quad \hat{\mathbf{R}}_x := \left[\hat{R}_{2x}(1;0) \quad \hat{R}_{4x}(1;0,0,0)\right]^T. \tag{13}$$

to improve the performance of both SLN and FLN estimators. The real-valued parameter α_1 is to be chosen so that the asymptotic variance of $\hat{\epsilon}$ in (13) is minimized. By adopting the derivation presented in Section III, one can obtain the closed-form expression for the asymptotic variance of $\hat{\epsilon}$ in (13), and hence, α_1 can be found accordingly. Due to the space limitations, their detailed derivations will not be presented in this letter, and the interested reader is suggested to consult [15].

The computation of the OPT estimate $\hat{\alpha}_1$ requires the knowledge of the operating SNR and the value (or estimate) of timing epoch ϵ, which makes the OPT estimator impractical. Fortunately, for most applications of interest, this difficulty can be circumvented with very little performance penalty by fixing α_1 to a constant. Next, we present a case study which illustrates how to select α_1. Consider an i.i.d. QPSK modulated symbol sequence corrupted by additive circular white Gaussian noise with variance N_0. Assuming $\rho = 0.1$, $P = 4$ and $\epsilon = 0.3$, the OPT estimate $\hat{\alpha}_1$ is given in Table I(a) for various SNR levels. Table I(b) shows the optimal value of $\hat{\alpha}_1$ versus the timing epoch ϵ, assuming SNR $= 20$ dB. The results presented in Table I(a) and (b) and extensive simulation experiments suggest that, in this application, we can always fix α_1 to a value in the range $[-0.13, -0.17]$ for implementing the estimator (13) without incurring any performance loss.

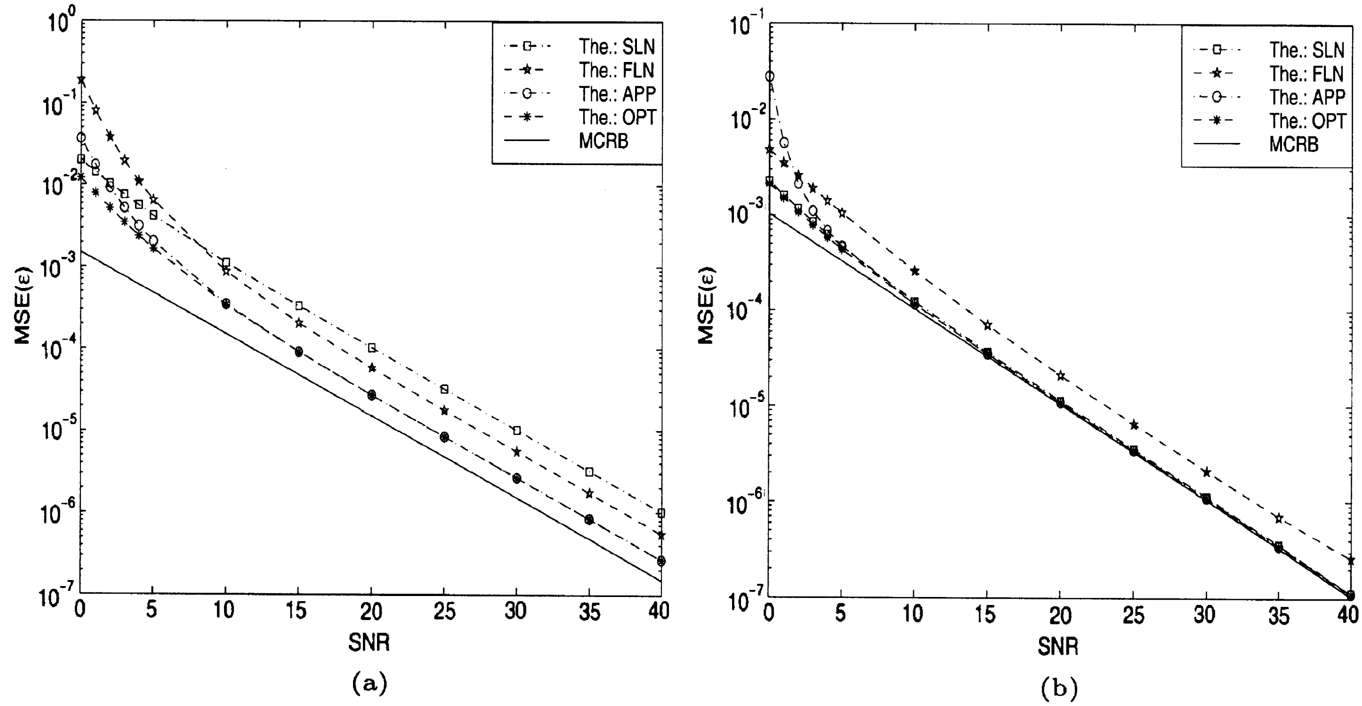

Fig. 3. Comparison of asymptotic variances versus SNR with (a) $\rho = 0.1$ and (b) $\rho = 0.9$.

V. Simulation Experiments

In this section, we conduct computer simulations to confirm the analysis presented in the previous sections and to illustrate the performance of the proposed OPT estimator. All the experimental results are obtained by performing a number of 10^6 Monte Carlo trials assuming a QPSK constellation, and the normalized timing epoch $\epsilon = 0.3$. Unless otherwise noted, the oversampling rate $P = 4$ is adopted.

Experiment 1—Performance of the O&M estimator versus P: By varying the oversampling rate P, we compare the experimental (Exp.) MSE of O&M estimator with its theoretical variance in Fig. 2. The number of symbols is set to $L = 200$, $\rho = 0.5$, and SNR $= 10$ dB. It turns out that increasing P does not improve the performance as long as $P \geq 3$. This result may be also predicted using Shannon's interpolation theorem, since any $P \geq 3$ guarantees the set of obtained statistics to be sufficient within the class of second-order statistics.

Experiment 2—Comparison of asymptotic variances of the O&M, FLN, and OPT estimators: Fig. 3(a) and (b) depicts the asymptotic variances of the SLN, FLN, and OPT estimators, and MCRB, in two extreme cases: a strongly bandlimited pulse $\rho = 0.1$ and a pulse with large bandwidth $\rho = 0.9$. The performance of a practical implementation of the OPT estimator (13) with fixed $\boldsymbol{\alpha}^T = [1 \quad -0.165]^T$, which is just an approximation of the OPT estimator, therefore, termed APP, is also illustrated. It can be seen that when dealing with narrowband pulse shapes, FLN is superior to SLN in medium and high SNR ranges, but worse than the latter at low SNRs. The OPT estimator outperforms both SLN and FLN estimators, and is closer to MCRB. As expected, APP is a satisfying realizable alternative to OPT except at very low SNRs. In the case of large rolloffs, FLN is always inferior to the SLN estimator, while the latter is good enough to approach the performance of the OPT estimator. Fig. 4 shows the improvement exhibited by the OPT estimator with respect to the SLN estimator versus rolloff factor ρ assuming SNR $= 20$ dB. It appears that the improvement is negligible whenever $\rho \geq 0.6$.

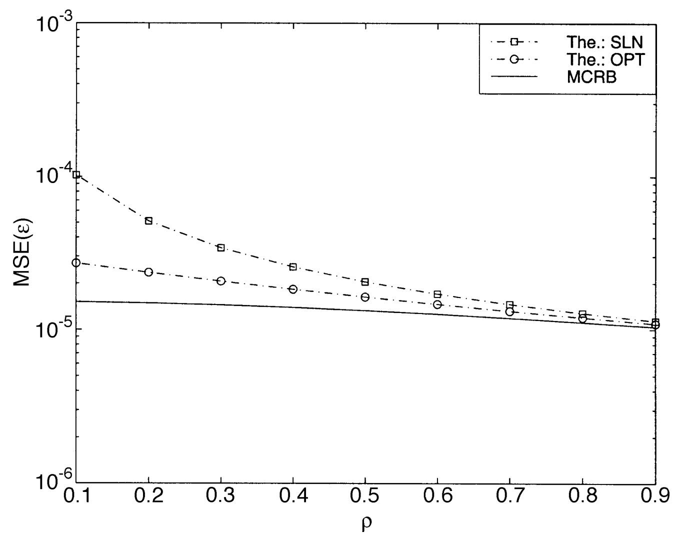

Fig. 4. Improvement of OPT over SLN versus ρ (SNR $= 20$ dB).

Experiment 3—Comparison of the MSE of estimators versus SNR: In Fig. 5(a) and (b), the experimental MSE of the proposed APP estimator is compared with those of the existing methods, assuming $\rho = 0.1$, $L = 400$, and $\rho = 0.9$, $L = 100$, respectively. These figures corroborate the results of Experiment 2 and show again the merit of the APP estimator.

VI. Conclusion

In this letter, we have established a rigorous CS statistics-based ML framework to design and analyze a class of blind feed-

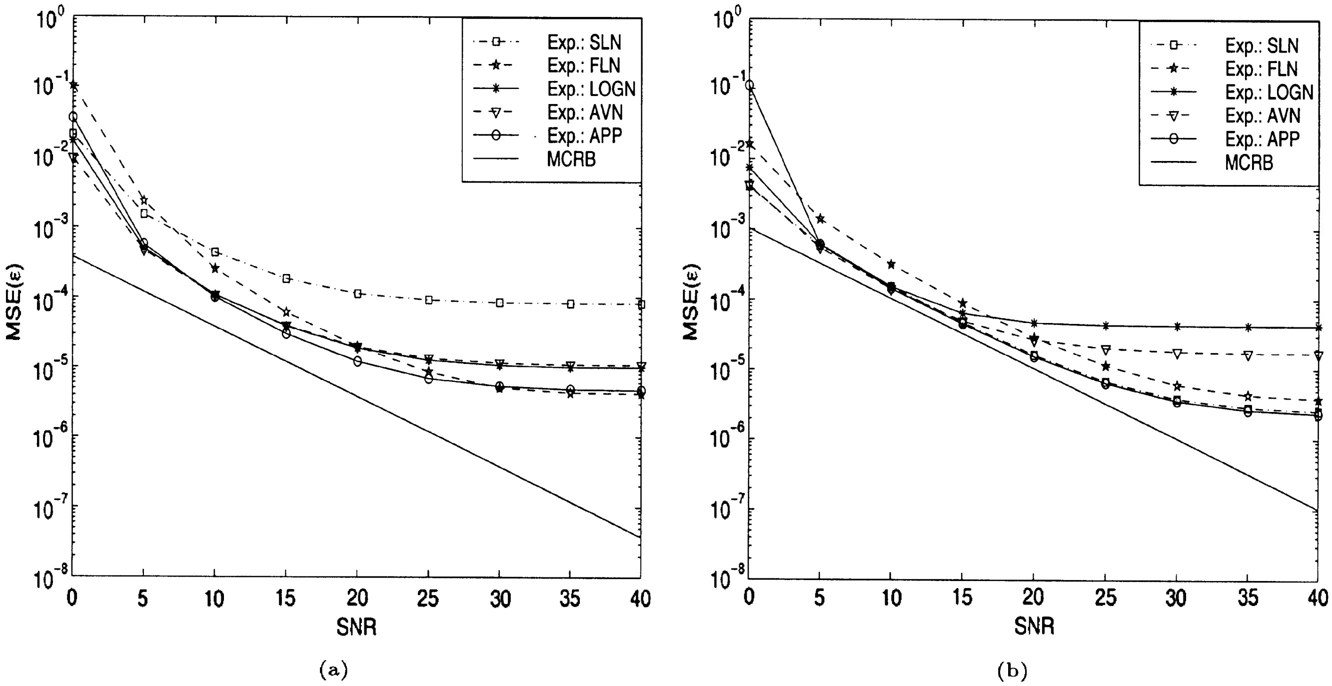

Fig. 5. Comparison of MSEs with (a) $\rho = 0.1$ and (b) $\rho = 0.9$.

forward timing estimators. We have shown that these estimators can be asymptotically interpreted as ML estimators and the O&M estimator achieves asymptotically the best performance in the class of SLN estimators, whose performance is insensitive to the oversampling rate P as long as $P \geq 3$. The asymptotic variance of these ML estimators is derived and can be employed as a benchmark for evaluating the system performance of the timing estimators proposed in the literature. The proposed analysis framework can be extended straightforwardly to the case of correlated symbol streams and time-selective flat-fading channels, and provides a systematic method to design optimal ML timing recovery schemes. Moreover, in this letter, based on the proposed performance analysis, we have introduced an efficient estimator (OPT), which fully exploits the second- and the fourth-order CS statistics of the received signal, that improves significantly the performance of the existing methods, when dealing with narrowband signaling pulses. One may ask whether the performance of timing estimators may be further improved if higher order nonlinearities (i.e., higher than the fourth order) are considered. We conjecture that the improvement is negligible, a fact that is corroborated by the plots depicted in Fig. 5 for the AVN and LOGN estimators, whose Taylor series expansions involve higher order terms.

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