Appendices for paper entitled “Performance Analysis over Slow Fading Channels of a Half-Duplex Single-Relay Protocol: Decode or Quantize and Forward”

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Abstract

In this document, we provide all the proofs associated with paper entitled “Performance Analysis over Slow Fading Channels of a Half-Duplex Single-Relay Protocol: Decode or Quantize and Forward” submitted for publication to IEEE Transactions on Wireless Communications.

APPENDIX A

PROOF OF THEOREM 1

It is known [1] that the capacity of any static relaying protocol is limited by the cut-set upper-bound. In this appendix, we derive the outage gain associated with the cut-set capacity. We prove next that this outage gain is equal to $\xi_{\text{CS-HD}}$ given by (21).

The cut-set upper-bound on the capacity of any half-duplex single-relay protocol from the class $\mathcal{P}_{\text{HD}}(t_0, \alpha_0, \alpha_1)$, with a listening time equal to $t_0T$ and a cooperation time equal to $(1 - t_0)T = t_1T$, is given by

$$
C_{\text{CS-HD}} = \lim_{T \to \infty} \frac{1}{T} \max_{p(X_{00}, X_{01}, X_{11})} \min \left\{ I(X_{00}; Y_{10}, Y_{20}) + I(X_{01}; Y_{21} | X_{11}),
I(X_{00}; Y_{20}) + I(X_{01}, X_{11}; Y_{21}),
\right\}
$$

(33)

where the maximization in (33) is with respect to all the joint distributions of $X_{00}$, $X_{01}$ and $X_{11}$ that satisfy the power constraints (19) and (20). It can be shown that the maximum in (33) is achieved when vectors $X_{00}$, $X_{01}$ and $X_{11}$ are zero-mean i.i.d Gaussian with covariance matrices that satisfy constraints (19)
and (20). The cut-set upper-bound can thus be written as
\[
C_{\text{CS-HD}} = \min \left\{ t_0 \log \left( 1 + \mathbb{E} \left[ |X_0(i)|^2 \right] G_{01} + \mathbb{E} \left[ |X_0(i)|^2 \right] G_{02} \right) + t_1 \log \left( 1 + \mathbb{E} \left[ |X_0(i)|^2 \right] G_{02} \right) + t_1 \log \left( 1 + \mathbb{E} \left[ |X_0(i)|^2 \right] G_{12} \right) \right\}
\]
\[
= \min \{ C_{\text{SIMO}}, C_{\text{MISO}} \},
\]
where \( C_{\text{SIMO}} \) and \( C_{\text{MISO}} \) are defined in order to simplify the presentation of the proof as follows:
\[
C_{\text{SIMO}} = t_0 \log \left( 1 + \mathbb{E} \left[ |X_0(i)|^2 \right] G_{01} + \mathbb{E} \left[ |X_0(i)|^2 \right] G_{02} \right) + t_1 \log \left( 1 + \mathbb{E} \left[ |X_0(i)|^2 \right] G_{02} \right)
\]
\[
C_{\text{MISO}} = t_0 \log \left( 1 + \mathbb{E} \left[ |X_0(i)|^2 \right] G_{02} \right) + t_1 \log \left( 1 + \mathbb{E} \left[ |X_0(i)|^2 \right] G_{02} + \mathbb{E} \left[ |X_1(i)|^2 \right] G_{12} \right).
\]

We now prove that the limit \( \lim_{\rho \to \infty} \rho^2 \Pr[CS_{\text{HD}} \leq R] \) exists and that it is equal to \( \xi_{CS_{\text{HD}}} \) given by (21). For that sake, note that the following holds:
\[
\Pr[CS_{\text{HD}} \leq R] = 1 - \Pr[CS_{\text{HD}} > R]
\]
\[
= 1 - \Pr[C_{\text{SIMO}} > R, C_{\text{MISO}} > R]
\]
\[
\geq 1 - \Pr[C_{\text{SIMO}} > R] \times \Pr[C_{\text{MISO}} > R]
\]
\[
= 1 - (1 - \Pr[C_{\text{SIMO}} \leq R]) \times (1 - \Pr[C_{\text{MISO}} \leq R])
\]

Now define
\[
P_{o,\text{SIMO}} = \Pr[C_{\text{SIMO}} \leq R]
\]
\[
P_{o,\text{MISO}} = \Pr[C_{\text{MISO}} \leq R].
\]

Using these new notations, we conclude that the following lower-bound on \( \Pr[CS_{\text{HD}} \leq R] \) holds:
\[
\Pr[CS_{\text{HD}} \leq R] \geq P_{o,\text{SIMO}} + P_{o,\text{MISO}} - P_{o,\text{SIMO}} P_{o,\text{MISO}}.
\]

In the same way, it is straightforward to show that \( \Pr[CS_{\text{HD}} \leq R] \) can be upper-bounded as follows.
\[
\Pr[CS_{\text{HD}} \leq R] \leq P_{o,\text{SIMO}} + P_{o,\text{MISO}} + P_{o,\text{SIMO}} P_{o,\text{MISO}}.
\]

Now, we can use the same arguments and tools employed in Subsection B to prove that
\[
\lim_{\rho \to \infty} \rho^2 P_{o,\text{SIMO}} = \frac{c_{02} c_{01}}{\alpha_0^2} \int_{\mathbb{R}_+^2} \mathbb{1} \left\{ t_1 \log (1 + u) + t_0 \log (1 + u + v) \leq R \right\} du dv
\]
\[
\lim_{\rho \to \infty} \rho^2 P_{o,\text{MISO}} = \frac{c_{02} c_{12}}{\alpha_0 \alpha_1} \int_{\mathbb{R}_+^2} \mathbb{1} \left\{ t_0 \log (1 + u) + t_1 \log (1 + u + v) \leq R \right\} du dv
\]
\[
\lim_{\rho \to \infty} \rho^2 P_{o,\text{SIMO}} P_{o,\text{MISO}} = 0.
\]
Note that the integrals in the rhs of (37) and (38) coincide with the two integrals in the rhs of (48). We can thus write

\[
\lim_{\rho \to \infty} \rho^2 P_{o,\text{SIMO}} = \frac{c_{o2} c_{o1}}{\alpha_0^2} \left( \frac{1}{2} + \frac{\exp(2R)}{4t_0 - 2} - \frac{t_0 \exp(R/t_0)}{2t_0 - 1} \right) \tag{40}
\]

\[
\lim_{\rho \to \infty} \rho^2 P_{o,\text{MISO}} = \frac{c_{o2} c_{12}}{\alpha_0 \alpha_{o2}} \left( \frac{1}{2} + \frac{\exp(2R)}{4t_1 - 2} - \frac{t_1 \exp(R/t_1)}{2t_1 - 1} \right) \tag{41}
\]

Combining (35), (36), (39), (40) and (41) we conclude that

\[
\lim_{\rho \to \infty} \rho^2 \Pr[C_{\text{CS-HD}} \leq RT] = \xi_{\text{CS-HD}},
\]

where \(\xi_{\text{CS-HD}}\) is the lower-bound defined by (21). Note that since \(C_{\text{CS-HD}}\) is an upper-bound on the capacity of any static half-duplex relaying protocol belonging to the class \(P_{\text{HD}}(t_0, \alpha_0, \alpha_1)\), then \(\xi_{\text{CS-HD}}\) which satisfies \(\xi_{\text{CS-HD}} = \lim_{\rho \to \infty} \rho^2 \Pr[C_{\text{CS-HD}} \leq RT]\) is a lower-bound on the outage gain of any protocol from the class \(P_{\text{HD}}(t_0, \alpha_0, \alpha_1)\). This completes the proof of Theorem 1.

**APPENDIX B**

**PROOF OF THEOREM 2**

Recall the definition of \(P_o(\rho)\) given by (13) as the outage probability associated with the DoQF protocol. In order to prove Theorem 2, we need to show that \(\rho^2 P_o(\rho)\) converges as \(\rho \to \infty\) and to derive the outage gain \(\xi_{\text{DoQF}}\) given by \(\xi_{\text{DoQF}} = \lim_{\rho \to \infty} \rho^2 P_o(\rho)\). According to (13), \(P_o(\rho) = P_{o,1}(\rho) + P_{o,2}(\rho) + P_{o,3}(\rho) + P_{o,4}(\rho)\), where \(P_{o,1}(\rho)\), \(P_{o,2}(\rho)\), \(P_{o,3}(\rho)\) and \(P_{o,4}(\rho)\) are defined by (14), (15), (16) and (17) respectively. Therefore, we need to first compute the limits \(\lim_{\rho \to \infty} \rho^2 P_{o,1}(\rho)\), \(\lim_{\rho \to \infty} \rho^2 P_{o,2}(\rho)\), \(\lim_{\rho \to \infty} \rho^2 P_{o,3}(\rho)\) and \(\lim_{\rho \to \infty} \rho^2 P_{o,4}(\rho)\) in order to obtain the outage gain \(\xi_{\text{DoQF}}\). It has been proved in [2] that

\[
\lim_{\rho \to \infty} \rho^2 P_{o,1}(\rho) = \frac{c_{o2} c_{12}}{\alpha_0 \alpha_{o1}} \int_{\mathbb{R}^2} \mathbf{1}\{t_0 \log(1 + u) + t_1 \log(1 + u + v) \leq R\} du dv, \tag{42}
\]

where \(c_{o1}\) and \(c_{12}\) has been defined in Subsection III-A as \(c_{o1} = f_{G_{o1}}(0+)\) and \(c_{12} = f_{G_{12}}(0+)\) respectively. The steps of the proof that (42) holds are very similar to the steps given below for the derivation of \(\lim_{\rho \to \infty} \rho^2 P_{o,2}(\rho)\). Refer to the definition of \(P_{o,2}(\rho)\) given by (15) as

\[
P_{o,2}(\rho) = \Pr\left[ t_1 \log(1 + \alpha_0 \rho G_{o2}) + t_0 \log \left( 1 + \alpha_0 \rho G_{o2} + \frac{\gamma(G_{o1}, \rho) \alpha_0 \rho G_{o1}}{\gamma(G_{o1}, \rho) + \Delta^2(\rho) \sqrt{\gamma(G_{o1}, \rho)}} \right) < R, \right. \\
\left. \mathcal{E}, \mathcal{F}, \mathcal{S} \right], \tag{43}
\]
where $\gamma(G_{01}, \rho) = \frac{(1+\alpha_0\rho G_{01}-\Delta^2(\rho))^2}{(1+\alpha_0\rho G_{01})}$. Plugging the definitions of events $\mathcal{E}$, $\mathcal{S}$ and $\mathcal{F}$ given respectively by (4), (8) and (10) into (43) leads to

$$P_{\alpha,2}(\rho) = \int_{\mathbb{R}_+^2} 1 \left\{ t_1 \log(1 + \alpha_0 \rho x) + t_0 \log \left( 1 + \alpha_0 \rho x + \frac{\gamma(y, \rho)\alpha_0 \rho y}{\gamma(y, \rho) + \Delta^2(\rho)\sqrt{\gamma(y, \rho)}} \right) \leq R \right\}$$

$$\times 1 \left\{ t_0 \log(1 + \alpha_0 \rho y) \leq R \right\} \left\{ 1 + \alpha_0 \rho y > \Delta^2(\rho) \right\}$$

$$\times 1 \left\{ t_1 \log \left( 1 + \frac{\phi(\rho)z}{1 + \alpha_0 \rho x} \right) > t_0 Q(\rho) \right\} f_{G_{o2}}(x) f_{G_{o1}}(y) f_{G_{12}}(z) dx dy dz ,$$

By making the change of variables $u = \alpha_0 \rho x$ and $v = \alpha_0 \rho y$ we obtain

$$\rho^2 P_{\alpha,2}(\rho) = \frac{1}{\alpha_0^2} \int_{\mathbb{R}_+^2} 1 \left\{ t_1 \log(1 + u) + t_0 \log \left( 1 + u + \frac{\gamma(v, \rho)v}{\gamma(v, \rho) + \Delta^2(\rho)\sqrt{\gamma(v, \rho)}} \right) \leq R \right\}$$

$$\times 1 \left\{ t_0 \log(1 + v) \leq R \right\} \left\{ 1 + v > \Delta^2(\rho) \right\}$$

$$\times 1 \left\{ t_1 \log \left( 1 + \frac{\phi(\rho)z}{1 + u} \right) > t_0 Q(\rho) \right\} f_{G_{o2}} \left( \frac{u}{\alpha_0 \rho} \right) f_{G_{o1}} \left( \frac{v}{\alpha_0 \rho} \right) f_{G_{12}}(z) du dv dz . \tag{44}$$

Since $Q(\rho) = \log \left( K/\Delta^2(\rho) \right)$, it is possible and useful to write the last indicator as follows.

$$1 \left\{ t_1 \log \left( 1 + \frac{\phi(\rho)z}{1 + u} \right) > t_0 Q(\rho) \right\} = 1 \left\{ z > (1 + u)\theta(\rho) \right\} , \tag{45}$$

where

$$\theta(\rho) = \frac{K_{\frac{m}{\alpha_0}}}{\phi(\rho) (\Delta^2(\rho))^{\frac{m}{\alpha_0}}} - \frac{1}{\phi(\rho)} . \tag{46}$$

Define the function $\Phi(u, v, z, \rho)$ as the integrand in the rhs of (44) and let $\mathcal{C}$ be the compact subset of $\mathbb{R}_+^2$ defined as $\mathcal{C} = \left\{ (u, v) \in \mathbb{R}_+^2, t_1 \log(1 + u) + t_0 \log \left( 1 + u + \frac{\gamma(v, \rho)v}{\gamma(v, \rho) + \Delta^2(\rho)\sqrt{\gamma(v, \rho)}} \right) \leq R, t_0 \log(1 + v) \leq R \right\}$. As $f_{G_{o2}}$ and $f_{G_{o1}}$ are right continuous at zero, then the function $(u, v) \mapsto f_{G_{o2}} \left( \frac{u}{\alpha_0 \rho} \right) f_{G_{o1}} \left( \frac{v}{\alpha_0 \rho} \right)$ is bounded on $\mathcal{C}$ for $\rho$ large enough i.e., there exist $\rho_0 > 0$ and $M > 0$ such that $\forall \rho \geq \rho_0, f_{G_{o2}} \left( \frac{u}{\alpha_0 \rho} \right) f_{G_{o1}} \left( \frac{v}{\alpha_0 \rho} \right) \leq M$. It is straightforward to verify that the following inequalities hold for all $\rho \geq \rho_0$:

$$\Phi(u, v, z, \rho) \leq M \times 1 \left\{ t_1 \log(1 + u) + t_0 \log \left( 1 + u + \frac{\gamma(v, \rho)v}{\gamma(v, \rho) + \Delta^2(\rho)\sqrt{\gamma(v, \rho)}} \right) \leq R \right\}$$

$$\times 1 \left\{ t_0 \log(1 + v) \leq R \right\} \left\{ 1 + v > \Delta^2(\rho) \right\}$$

$$\times 1 \left\{ z > (1 + u)\theta(\rho) \right\} f_{G_{12}}(z)$$

$$\leq M \times 1 \{ \log(1 + u) \leq R \} \times 1 \{ t_0 \log(1 + v) \leq R \} f_{G_{12}}(z) .$$
The rhs of the latter inequality is an integrable function on $\mathbb{R}^3_+$ and it does not depend on $\rho$. Therefore, we can apply Lebesgue’s Dominated Convergence Theorem (DCT) in order to compute $\lim_{\rho \to \infty} \rho^2 P_{o,2}(\rho)$ in (44). Note first that $\lim_{\rho \to \infty} \Delta^2(\rho) = 0$, $\lim_{\rho \to \infty} \frac{\gamma(v,\rho)}{\gamma(v,\rho)+\Delta^2(\rho)} = 1$ and $\lim_{\rho \to \infty} \theta(\rho) = 0$ due to assumptions (22)-(25). After some algebra, we can easily show that the following result holds.

$$
\lim_{\rho \to \infty} \rho^2 P_{o,2}(\rho) = \frac{c_{02G1}}{\rho^2} \int_{\mathbb{R}_+^3} 1 \{ t_1 \log(1 + u) + t_2 \log(1 + u + v) \leq R \} \, dv .
$$

We now prove that $\lim_{\rho \to \infty} \rho^2 P_{o,3}(\rho) = 0$. First, recall that $P_{o,3}(\rho) = \Pr[t_0 \log(1 + \alpha_0 \rho G_{02}) < R, \mathcal{E}, \mathcal{S} \in \mathcal{F}, \mathcal{F}]$ from (4), (8) and (10) respectively into the latter equation leads to

$$
P_{o,3}(\rho) = \int_{\mathbb{R}_+^3} 1 \{ t_0 \log(1 + u) \leq R \} 1 \{ t_0 \log(1 + v) \leq R \} 1 \{ 1 + v > \Delta^2(\rho) \}
\times 1 \left\{ t_1 \log \left( 1 + \frac{\phi(\rho)z}{1 + \alpha_0 \rho x} \right) \leq t_0 Q(\rho) \right\} f_{G_{02}}(x) f_{G_{01}}(y) f_{G_{12}}(z) \, dx dy dz .
$$

Defining $u = \alpha_0 \rho x$ and $v = \alpha_0 \rho y$, we get

$$
P_{o,3}(\rho) = \frac{1}{\alpha_0^2 \rho^2} \int_{\mathbb{R}_+^3} 1 \{ t_0 \log(1 + u) \leq R \} 1 \{ t_0 \log(1 + v) \leq R \} 1 \{ 1 + v > \Delta^2(\rho) \}
\times 1 \left\{ t_1 \log \left( 1 + \frac{\phi(\rho)z}{1 + u} \right) \leq t_0 Q(\rho) \right\} f_{G_{02}} \left( \frac{u}{\alpha_0 \rho} \right) f_{G_{01}} \left( \frac{v}{\alpha_0 \rho} \right) f_{G_{12}}(z) \, du dv dz .
$$

As we did in (45), we write the last indicator as follows.

$$
1 \left\{ t_1 \log \left( 1 + \frac{\phi(\rho)z}{1 + u} \right) \leq t_0 Q(\rho) \right\} = 1 \{ z \leq (1 + u) \theta(\rho) \} ,
$$

where $\theta(\rho)$ is defined by (46). In analogy with the approach we used to compute $\lim_{\rho \to \infty} \rho^2 P_{o,2}(\rho)$, we define $\mathcal{E}_1$ as the compact subset of $\mathbb{R}^3_+$ satisfying $\mathcal{E}_1 = \{(u, v, z) \in \mathbb{R}^3_+, t_0 \log(1 + u) \leq R, t_0 \log(1 + v) \leq R, z \leq (1 + u) \theta(\rho) \}$. Next, we use the fact that $f_{G_{02}}, f_{G_{01}}$ and $f_{G_{12}}$ are right continuous at zero, along with $\lim_{\rho \to \infty} \theta(\rho) = 0$, to show that the function $(u, v, z) \mapsto f_{G_{02}} \left( \frac{u}{\alpha_0 \rho} \right) f_{G_{01}} \left( \frac{v}{\alpha_0 \rho} \right) f_{G_{12}}(z)$ is bounded on $\mathcal{E}_1$ for $\rho$ large enough i.e., there exist $\rho_1 > 0$ and $M_1 > 0$ such that $\forall \rho \geq \rho_1$, $f_{G_{02}} \left( \frac{u}{\alpha_0 \rho} \right) f_{G_{01}} \left( \frac{v}{\alpha_0 \rho} \right) f_{G_{12}}(z) \leq M_1$. It follows that the following inequalities hold for all $\rho \geq \rho_1$:

$$
\rho^2 P_{o,3}(\rho) \leq \frac{M_1}{\alpha_0^2} \int_{\mathbb{R}_+^3} 1 \{ 1 + u \leq e^u \} 1 \{ z \leq (1 + u) \theta(\rho) \} \, du dz
\leq \frac{M_1}{\alpha_0^2} \int_{\mathbb{R}_+} 1 \{ z \leq e^z \theta(\rho) \} \, dz \leq \frac{M_1}{\alpha_0^2} \int_{0}^{e^\rho \theta(\rho)} \, dz = \frac{M_1}{\alpha_0^2} e^\rho \theta(\rho) .
$$

Now since $\lim_{\rho \to \infty} \theta(\rho) = 0$ due to assumptions (22)-(25), it follows that $\lim_{\rho \to \infty} \rho^2 P_{o,3}(\rho) = 0$. We can prove in the same way and without difficulty that $\lim_{\rho \to \infty} \rho^2 P_{o,4}(\rho) = 0$. 

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Putting all pieces together,

\[
\lim_{\rho \to \infty} \rho^2 P_o = \lim_{\rho \to \infty} \rho^2 P_{o,1}(\rho) + \lim_{\rho \to \infty} \rho^2 P_{o,2}(\rho) + \lim_{\rho \to \infty} \rho^2 P_{o,3}(\rho) + \lim_{\rho \to \infty} \rho^2 P_{o,4}(\rho)
\]

\[
= \frac{c_{0,2}c_{12}}{\alpha_0 \alpha_1} \int_{\mathbb{R}_+^2} 1 \{ t_0 \log(1 + u) + t_1 \log(1 + u + v) \leq R \} dudv
\]

\[
+ \frac{c_{0,2}c_{0,1}}{\alpha_0^2} \int_{\mathbb{R}_+^2} 1 \{ t_1 \log(1 + u) + t_0 \log (1 + u + v) \leq R \} dudv.
\]

(48)

The remaining task is to prove that the rhs of (48) is equal to the rhs of (21). This can be done by making the change of variables \(x = \log(1 + u)\) and \(y = \log \left(1 + \frac{v}{1+u}\right)\) in (48). The details of the proof can be found in [2]. The proof of Theorem 2 is thus complete.

**APPENDIX C**

**PROOF OF THEOREM 3**

The outage probability associated with the DoQF protocol was given by (13) as

\[
P_o(\rho) = P_{o,1}(\rho) + P_{o,2}(\rho) + P_{o,3}(\rho) + P_{o,4}(\rho),
\]

(49)

where probabilities \(P_{o,1}(\rho), \ P_{o,2}(\rho), \ P_{o,3}(\rho)\) and \(P_{o,4}(\rho)\) are respectively defined by (14), (15), (16) and (17). Inserting (49) into the definition of the DMT \(d(t_0, \delta, r)\) given by (29) leads to

\[
d(t_0, \delta, r) = - \lim_{\rho \to \infty} \frac{\log (P_{o,1}(\rho) + P_{o,2}(\rho) + P_{o,3}(\rho) + P_{o,4}(\rho))}{\log \rho}
\]

\[
= \min \{ d_1(t_0, r), d_2(t_0, \delta, r), d_3(t_0, \delta, r), d_4(t_0, \delta, r) \},
\]

(50)

where

\[
d_i(t_0, \delta, r) = - \lim_{\rho \to \infty} \frac{\log P_{o,i}(\rho)}{\log \rho},
\]

(51)

for \(i = 1, 2, 3, 4\). Note that \(d_1(t_0, r)\) is the only term in (50) that does not depend on parameter \(\delta\). The derivation of the DMT associated with the DoQF protocol will be thus done as follows:

1) Compute the terms \(d_1(t_0, r), d_2(t_0, \delta, r), d_3(t_0, \delta, r)\) and \(d_4(t_0, \delta, r)\) for fixed values of \(t_0\) and \(\delta\) as given by (51). This is done in this Subsection.

2) Compute \(t_{0,\text{DoQF}}^*(r)\) and \(\delta_{0,\text{DoQF}}^*(r)\) minimizing \(d(t_0, \delta, r)\) defined from (50) as the minimum of \(d_1(t_0, r), d_2(t_0, \delta, r), d_3(t_0, \delta, r)\) and \(d_4(t_0, \delta, r)\).

3) The final DMT of the protocol can be finally obtained by calculating \(d(t_{0,\text{DoQF}}^*(r), \delta_{0,\text{DoQF}}^*(r), r)\).
Derivation of the term $d_1(t_0, r)$, i.e., event $E$ is realized:

Recall the definition given by (14) of $P_{o,1}(\rho)$ as the probability that the destination is in outage and that the event $E$ is realized. It is clear from (4) and (14) that $P_{o,1}(\rho)$ is a function of parameter $t_0$. This is why the DMT term $d_1(t_0, r)$ associated with $P_{o,1}(\rho)$ is also a function of this parameter. Following the steps used in Appendix D-A, one can show that the following result holds.

$$d_1(t_0, r) = \begin{cases} 
2(1-r)^+ & \text{for } t_0 \leq 0.5 \\
2 - \frac{r}{1-t_0} & \text{for } t_0 > 0.5 \text{ and } r < 1 - t_0 \\
(1-r)^+ & \text{for } t_0 > 0.5 \text{ and } r \geq 1 - t_0 
\end{cases}$$

(52)

Derivation of the term $d_2(t_0, \delta, r)$, i.e., events $\overline{E}$, $S$ and $F$ are realized:

Note from (10) and (15) that $P_{o,2}(\rho)$ is a function of parameters $t_0$ and $\delta$. This is why the DMT $d_2(t_0, \delta, r)$ associated with $P_{o,2}(\rho)$ is function of $t_0$ and $\delta$.

First, consider the case $t_0 \geq 0.5$.

If parameter $\delta$ is chosen such that $0 < \delta \leq 1 - \left(1 - \frac{r}{t_0}\right)^+$, then $d_2(t_0, \delta, r)$ can be written as

$$d_2(t_0, \delta, r) = \begin{cases} 
\left(1-r\right)^+ + \max \left\{ \left(1 - \frac{r}{t_0}\right)^+, 1-r-\delta \right\}, & \frac{r}{t_0} - \left(1 - \frac{r}{t_0}\right)^- - \frac{t_0}{t_1} \delta \leq 1 - r \\
\frac{r}{t_1} - \left(1 - \frac{r}{t_0}\right)^+ - \frac{t_0}{t_1} \delta + \max \left\{ \frac{1-2r}{t_0} + \frac{t_0}{t_1} \left(1 - \frac{r}{t_0}\right)^+, \left(1 - \frac{r}{t_0}\right)^+ \right\}, & \frac{r}{t_1} - \left(1 - \frac{r}{t_0}\right)^+ - \frac{t_0}{t_1} \delta > 1 - r 
\end{cases}$$

(53)

As for the choice $\delta > 1 - \left(1 - \frac{r}{t_0}\right)^+$, we show in Appendix D-A that event $\overline{E} \& S$ cannot be realized in this case for any channel state provided that $\rho$ is sufficiently large. Therefore, there exists $\rho_0 > 0$ such that $\forall \rho \geq \rho_0$, $P_{o,2}(\rho) = 0$. The corresponding DMT $d_2(t_0, \delta, r)$ will have no effect on the final DMT of the protocol. The value $d_2(t_0, \delta, r) = 2(1-r)^+$ is conveniently chosen in this case:

$$d_2(t_0, \delta, r) = 2(1-r)^+ \text{ for } \delta > 1 - \left(1 - \frac{r}{t_0}\right)^+$$

(54)

The proof of (53) and (54) is provided in Appendix D-A. We can show using the same arguments of the latter appendix that

$$d_2(t_0, \delta, r) = 2(1-r)^+, \text{ for } \delta \leq 0$$

(55)
Similarly, we can obtain the expression (56) for \( d_2(t_0, \delta, r) \) in the case \( t_0 < 0.5 \).

\[
d_2(t_0, \delta, r) = \begin{cases} 
(1 - \frac{r}{t_0})^+ + \max \left\{ (1 - r)^+, \frac{1-r}{t_1} - \frac{t_0}{t_1} \left( 1 - \frac{r}{t_0} \right)^+ \right\}, & \text{for } t_0 < 0.5 \text{ and } 2t_0 t_1 \leq r \\
(1 - \frac{r}{t_0})^+ + \frac{r}{t_1} - \left( 1 - \frac{r}{t_0} \right)^+ - \frac{t_0}{t_1} \delta, & \text{for } t_0 < 0.5 \text{ and } 2t_0 t_1 > r
\end{cases}
\]  

(56)

**Derivation of the term \( d_3(t_0, \delta, r) \), i.e., events \( \bar{E}, \bar{S} \) and \( \overline{F} \) are realized:**

By referring to (10) and (16), it becomes clear that \( P_{o,3}(\rho) \) is a function of parameters \( t_0 \) and \( \delta \). This explains the fact that \( d_3(t_0, \delta, r) \) also depends on these two parameters.

The expression given below of \( d_3(t_0, \delta, r) \) can be derived using the approach used in Appendix D-A.

\[
d_3(t_0, \delta, r) = \begin{cases} 
2 \left( 1 - \frac{r}{t_0} \right)^+ + \left( 2 \left( 1 - \frac{r}{t_0} \right)^+ + \frac{t_0}{t_1} \delta - \frac{r}{t_1} \right)^+ & \text{for } \delta \leq 1 - \left( 1 - \frac{r}{t_0} \right)^+ \\
2(1-r)^+ & \text{for } \delta > 1 - \left( 1 - \frac{r}{t_0} \right)^+
\end{cases}
\]  

(57)

Recall that in the case \( \delta > 1 - \left( 1 - \frac{r}{t_0} \right)^+ \), event \( \bar{E} \& \bar{S} \) cannot be realized, as we saw earlier, for any channel realization provided that \( \rho \) is sufficiently large. In this case \( P_{o,3}(\rho) = 0 \) and the corresponding DMT \( d_3(t_0, \delta, r) \) will have no effect on the final DMT of the protocol. This is why the value \( d_3(t_0, \delta, r) = 2(1-r)^+ \) was conveniently chosen in (57) in this case.

**Derivation of the term \( d_4(t_0, \delta, r) \), i.e., events \( \bar{E} \) and \( \bar{S} \) are realized:**

This is the case when the relay does not quantize even if it has not succeeded in decoding the source message. This happens when \( \alpha_0 \rho G_{01} + 1 < \Delta^2(\rho) \) which means that the relay stays inactive. Recall the definition of \( P_{o,4}(\rho) \) as the probability that the destination is in outage and that events \( \bar{E} \) and \( \bar{S} \) are realized. It is straightforward to verify that

\[
d_4(t_0, \delta, r) = \begin{cases} 
(1 - r)^+ + \max \left\{ (1 - \frac{r}{t_0})^+, (1 - \delta)^+ \right\} & \text{for } \delta > 0 \\
2(1-r)^+ & \text{for } \delta \leq 0
\end{cases}
\]  

(58)

Note that in the case \( \delta \leq 0 \), the condition \( \alpha_0 \rho G_{01} + 1 > \Delta^2(\rho) \) is always satisfied for sufficiently large values of \( \rho \) for all channel realizations since \( \Delta^2(\rho) = \rho^3 \leq 1 \). Therefore, there exists in this case \( \rho_0 > 0 \) such that \( \forall \rho \geq \rho_0 \), event \( \bar{S} \) is never realized and \( P_{o,4}(\rho) = 0 \). The corresponding DMT \( d_4(t_0, \delta, r) \) will have therefore no effect on the final DMT of the protocol, and as usual we can assign it conveniently the value \( d_4(t_0, \delta, r) = 2(1-r)^+ \) as done in (58).

**Derivation of the final DMT of the DoQF protocol:**

At this point, the DMT terms \( d_1(t_0, r), d_2(t_0, \delta, r), d_3(t_0, \delta, r) \) and \( d_4(t_0, \delta, r) \) associated with all the possible cases encountered by the destination have been derived. The DMT \( d(t_0, \delta, r) \) associated with
the DoQF protocol for fixed values of $t_0$ and $\delta$ can now be obtained from (29) as the minimum of the above DMT terms. No closed-form expression of $d(t_0, \delta, r)$ is given in this paper. However, Theorem 3 does provide the closed-form expression of $d^*_\text{DoQF}(r)$ obtained by solving the optimization problem $d^*_\text{DoQF}(r) = \sup_{\delta, t_0} d(t_0, \delta, r)$. We derive $d^*_\text{DoQF}(r)$ as follows.

Before proceeding with the proof, we define $t^*_0\text{DoQF}(r)$ and $\delta^*_\text{DoQF}(r)$ as the argument of the supremum in $d^*_\text{DoQF}(r) = \sup_{\delta, t_0} d(t_0, \delta, r)$.

We will first compute $d^*_\text{DoQF}(r)$ in the case $r \leq 0.25$, and then in the case $r > 0.25$.

**The case $r \leq 0.25$**

Let us plug $t_0 = 0.5$ and $\delta = 0$ into (52), (53), (57) and (58) to obtain

$$d_1(t_0, r) = d_2(t_0, \delta, r) = d_4(t_0, \delta, r) = 2(1 - r)^+, \quad (59)$$

$$d_3(t_0, \delta, r) = 2(1 - 2r)^+ + (2(1 - 2r)^+ - 2r)^+ = 2 - 8r. \quad (60)$$

Note that $d_3(t_0, \delta, r)$ is the only term that may be different from $2(1 - r)^+$. However, one can verify by referring to (60) that $d_3(t_0, \delta, r) \geq 2(1 - r)^+ \iff r \leq 0.25$. We conclude that, for $r \leq 0.25$, $d(0.5, 0, r) = 2(1 - r)^+$. We have thus proved that the MISO upper-bound is achieved by the DoQF for $r \leq 0.25$ by choosing $t^*_0\text{DoQF}(r) = 0.5$ and $\delta^*_\text{DoQF}(r) = 0$.

**The case $r > 0.25$**

The first step of the proof in this case is to reduce the size of the set of possible values of $t^*_0\text{DoQF}(r)$ and $\delta^*_\text{DoQF}(r)$. For that sake, we first recall that the DMT of (non-orthogonal) DF in the general multiple-relay case has been derived in [4]. Denote by $P_{o,DF}(\rho)$ the outage probability associated with the DF protocol. The DMT of DF for fixed values of $t_0$ can thus be defined as

$$d(t_0, r) = - \lim_{\rho \to \infty} \frac{\log P_{o,DF}(\rho)}{\log \rho}, \quad (61)$$

and the final DMT of the protocol as $d^*_{\text{DF}}(r) = \sup_{t_0} d(t_0, r)$. The closed-form expression of $d^*_{\text{DF}}(r)$ in the case of a single relay is reproduced here by

$$d^*_{\text{DF}}(r) = \begin{cases} 2 - \frac{2}{3 - \sqrt{3}} r & \text{for } 0 \leq r \leq \frac{\sqrt{5} - 1}{\sqrt{5} + 1} \\ (2 - r)(1 - r) & \text{for } \frac{\sqrt{5} - 1}{\sqrt{5} + 1} < r \leq 1. \end{cases} \quad (62)$$

Moreover, the optimal value of $t_0$, as function of $r$, that allows to achieve this DMT is given by

$$t^*_{0,\text{DF}}(r) = \begin{cases} \frac{2}{\sqrt{5} + 1} & \text{for } 0 \leq r \leq \frac{\sqrt{5} - 1}{\sqrt{5} + 1} \\ \frac{1}{2 - r} & \text{for } \frac{\sqrt{5} - 1}{\sqrt{5} + 1} < r \leq 1. \end{cases} \quad (63)$$

given the above results, we will prove in particular that the following three lemmas hold.
Lemma 1. For any \( r \in [0, 1] \), \( d_{\text{DoQF}}^*(r) \geq d_{\text{DF}}^*(r) \).

In other words, Lemma 1 states that the DMT achieved by the DoQF protocol cannot be worse than the DMT achieved by the DF. The proof of Lemma 1 is given in Appendix D-B.

Lemma 2. For any \( r \in [0, 1] \), the following inequalities hold true: \( \max \{0.5, r\} \leq t_{0, \text{DoQF}}^*(r) \leq t_{0, \text{DF}}^*(r) \).

Here, \( t_{0, \text{DF}}^*(r) \) is the value of \( t_0 \) defined by (63) which allows to achieve the DMT of the DF protocol. The proof of Lemma 2 is given in Appendix D-C.

Lemma 3. Assume that \( r > 0.25 \). The following holds true: \( 0 < \delta_{\text{DoQF}}^*(r) < 1 - \left(1 - \frac{r}{t_{0, \text{DoQF}}^*(r)}\right)^+ \).

The proof of Lemma 3 is given in Appendix D-D.

These three lemmas will considerably simplify the derivation of \( d_{\text{DoQF}}^*(r) \). Indeed, with the help of Lemma 2 and Lemma 3, we will derive the DMT of the DoQF firstly in the case when \( 0.25 < r \leq \frac{2(\sqrt{5} - 1)}{9 - \sqrt{5}} \), and secondly in the case when \( \frac{2(\sqrt{5} - 1)}{9 - \sqrt{5}} < r \leq 1 \).

- \( 0.25 < r \leq \frac{2(\sqrt{5} - 1)}{9 - \sqrt{5}} \).

We begin with the simplification of the DMT terms
\[
d_1 \left(t_{0, \text{DoQF}}^*(r), r\right), \quad d_2 \left(t_{0, \text{DoQF}}^*(r), \delta_{\text{DoQF}}^*(r), r\right),
\]
\[
d_3 \left(t_{0, \text{DoQF}}^*(r), \delta_{\text{DoQF}}^*(r), r\right) \quad \text{and} \quad d_4 \left(t_{0, \text{DoQF}}^*(r), \delta_{\text{DoQF}}^*(r), r\right).
\]
The final DMT \( d_{\text{DoQF}}^*(r) \) can then be deduced as the minimum of the above terms. Consider first the derivation of \( d_1 \left(t_{0, \text{DoQF}}^*(r), r\right) \).

Since Lemma 2 states that \( t_{0, \text{DoQF}}^*(r) \leq t_{0, \text{DF}}^*(r) = \frac{2}{\sqrt{5} + 1} \), it follows from (52) that
\[
d_1 \left(t_{0, \text{DoQF}}^*(r), r\right) = 2 - \frac{r}{1 - t_{0, \text{DoQF}}^*(r)}.
\]  
(64)

We now proceed to the simplification of the expression of \( d_2 \left(t_{0, \text{DoQF}}^*(r), \delta_{\text{DoQF}}^*(r), r\right) \). Thanks to Lemma 2 and Lemma 3, we will prove that
\[
d_2 \left(t_{0, \text{DoQF}}^*(r), \delta_{\text{DoQF}}^*(r), r\right) = (1 - r)^+ + \max \left\{1 - \frac{r}{t_{0, \text{DoQF}}^*(r)}, 1 - r - \delta_{\text{DoQF}}^*(r)\right\}.
\]  
(65)

For that sake, refer to (53) and note that proving (65) is equivalent to proving that
\[
\frac{r}{1 - t_{0, \text{DoQF}}^*(r)} - \left(1 - \frac{r}{t_{0, \text{DoQF}}^*(r)}\right)^+ - t_{0, \text{DoQF}}^*(r) - \delta_{\text{DoQF}}^*(r) \leq 1 - r.
\]  
(66)

In order to show that (66) holds, we suppose to the contrary that
\[
\frac{t_{0, \text{DoQF}}^*(r)}{1 - t_{0, \text{DoQF}}^*(r)} - \left(1 - \frac{r}{t_{0, \text{DoQF}}^*(r)}\right)^+ - \delta_{\text{DoQF}}^*(r) > 1 - r.
\]
Since \( \delta_{\text{DoQF}}^*(r) > 0 \) according to Lemma 3, the latter assumption leads to
\[
r > \frac{2t_{0, \text{DoQF}}^*(r) \left(1 - t_{0, \text{DoQF}}^*(r)\right)}{1 + t_{0, \text{DoQF}}^*(r) \left(1 - t_{0, \text{DoQF}}^*(r)\right)}.
\]  
(67)
Moreover, it is straightforward to show that
\[
\min_{0.5 \leq t \leq \frac{2}{\sqrt{3} + 1}} \frac{2t(1 - t)}{1 + t(1 - t)} > \frac{2(\sqrt{5} - 1)}{9 - \sqrt{5}},
\]  
where the restriction to \(0.5 \leq t \leq \frac{t^*_0,\text{DoQF}(r)}{}\) is due to Lemma 2. Now, we can combine (67) and (68) in order to get \(r > \frac{2(\sqrt{5} - 1)}{9 - \sqrt{5}}\), which contradicts the fact that \(r \leq \frac{2(\sqrt{5} - 1)}{9 - \sqrt{5}}\). We conclude that expression (65) holds true.

We can further simplify the expression (65) by proving that \(1 - r - \delta^*_\text{DoQF}(r) \geq 1 - \frac{r}{t^*_0,\text{DoQF}(r)}\). The proof of this point uses the same arguments as above and is thus omitted. The term \(d_2\left(t^*_0,\text{DoQF}(r), \delta^*_\text{DoQF}(r), r\right)\) can finally be written as
\[
d_2\left(t^*_0,\text{DoQF}(r), \delta^*_\text{DoQF}(r), r\right) = 2(1 - r)^+ - \delta^*_\text{DoQF}(r). \tag{69}
\]

As for \(d_3\left(t^*_0,\text{DoQF}(r), \delta^*_\text{DoQF}(r), r\right)\) given by (57), it simplifies to
\[
d_3\left(t^*_0,\text{DoQF}(r), \delta^*_\text{DoQF}(r), r\right) = 4 + \frac{t^*_0,\text{DoQF}(r)}{1 - t^*_0,\text{DoQF}(r)} \delta^*_\text{DoQF}(r) - \left(4 + \frac{t^*_0,\text{DoQF}(r)}{1 - t^*_0,\text{DoQF}(r)}\right) \frac{r}{t^*_0,\text{DoQF}(r)} \tag{70}
\]
The remaining task is to simplify the expression (58) which defines \(d_4\left(t^*_0,\text{DoQF}(r), \delta^*_\text{DoQF}(r), r\right)\). For that sake, we can resort to Lemma 1 to prove that
\[
d_4\left(t^*_0,\text{DoQF}(r), \delta^*_\text{DoQF}(r), r\right) = (1 - r)^+ + (1 - \delta^*_\text{DoQF}(r)).
\]
It follows that \(d_4\left(t^*_0,\text{DoQF}(r), \delta^*_\text{DoQF}(r), r\right) \geq d_2\left(t^*_0,\text{DoQF}(r), \delta^*_\text{DoQF}(r), r\right)\) and that it can thus be dropped from the derivation of the final DMT of the DoQF. Now that the DMT terms \(d_1\left(t^*_0,\text{DoQF}(r), r\right), d_2\left(t^*_0,\text{DoQF}(r), \delta^*_\text{DoQF}(r), r\right)\) and \(d_3\left(t^*_0,\text{DoQF}(r), \delta^*_\text{DoQF}(r), r\right)\) have been expressed as functions of \(t^*_0,\text{DoQF}(r)\) and \(t^*_0,\text{DoQF}(r)\), we can proceed to the determination of \(t^*_0,\text{DoQF}(r)\), \(\delta^*_\text{DoQF}(r)\), and consequently \(d^*_\text{DoQF}(r)\).

- **Determination of \(\delta^*_\text{DoQF}(r)\):**
  Assume that \(t^*_0,\text{DoQF}(r)\) has been already determined. It is straightforward to verify that \(d_2\left(t, \delta, r\right)\) given by (69) is decreasing w.r.t \(\delta\), and that \(d_3\left(t, \delta, r\right)\) given by (70) is increasing w.r.t \(\delta\) on \(\mathbb{R}^+\). Furthermore, \(d_2\left(t, 0, r\right) > d_3\left(t, 0, r\right)\). We conclude that
  \[
  d_2\left(t^*_0,\text{DoQF}(r), \delta^*_\text{DoQF}(r), r\right) = d_3\left(t^*_0,\text{DoQF}(r), \delta^*_\text{DoQF}(r), r\right).
  \]
Therefore, \(\delta^*_\text{DoQF}(r)\) can be given as a function of \(t^*_0,\text{DoQF}(r)\) as follows
\[
\delta^*_\text{DoQF}(r) = \left(4 - 3t^*_0,\text{DoQF}(r)\right) \frac{r}{t^*_0,\text{DoQF}(r)} - \left(2 + 2r\right) \left(1 - t^*_0,\text{DoQF}(r)\right), \tag{71}
\]
which leads to
\[
d_2 \left( t_{0, \text{DoQF}}^*, \delta^*_{\text{DoQF}}(r), r \right) = d_3 \left( t_{0, \text{DoQF}}^*(r), \delta^*_{\text{DoQF}}(r), r \right) =
\]
\[
2 - 2r + (2 + 2r) \left( 1 - t_{0, \text{DoQF}}^*(r) \right) - \left( 4 - 3t_{0, \text{DoQF}}^*(r) \right) \frac{r}{t_{0, \text{DoQF}}^*(r)}.
\] (72)

- Determination of \( t_{0, \text{DoQF}}^*(r) \):

We can show in the same way that \( t_{0, \text{DoQF}}^*(r) \) can be obtained by writing
\[
d_1 \left( t_{0, \text{DoQF}}^*(r), r \right) = d_2 \left( t_{0, \text{DoQF}}^*(r), \delta^*_{\text{DoQF}}(r), r \right).
\] (73)

Plugging the expression of \( \delta^*_{\text{DoQF}}(r) \) from (71) and the expression of \( d_2 \left( t_{0, \text{DoQF}}^*(r), \delta^*_{\text{DoQF}}(r), r \right) \) from (72) into (73) leads to equation (32) given in Theorem 3 as
\[
2(1 + r)t_{0, \text{DoQF}}^*(r)^3 - (4 + 5r)t_{0, \text{DoQF}}^*(r)^2 + 2(1 + 4r)t_{0, \text{DoQF}}^*(r) - 4r = 0.
\]

It can be shown after some algebra that the above equation admits a unique solution \( v^*(r) \) on \([0.5, \frac{2}{\sqrt{8}+1}]\) provided that \( r \leq \frac{2(\sqrt{\pi}-1)}{9-\sqrt{5}} \). This explains why the distinction \( r \leq \frac{2(\sqrt{\pi}-1)}{9-\sqrt{5}} \) and \( r > \frac{2(\sqrt{\pi}-1)}{9-\sqrt{5}} \) appears in Theorem 3. Once the solution \( v^*(r) \) to the above equation has been computed, then \( d_{\text{DoQF}}^*(r), t_{0, \text{DoQF}}^*(r) \) and \( \delta^*_{\text{DoQF}}(r) \) can be easily obtained.

- \( \frac{2(\sqrt{\pi}-1)}{9-\sqrt{5}} < r \leq 1 \).

In this case, we need to prove that \( d_{\text{DoQF}}^*(r) = d_{\text{DF}}^*(r) \). To that end, we can show that \( d_{\text{DoQF}}^*(r) > d_{\text{DF}}^*(r) \) leads to a contradiction. The proof of this point is based on Lemmas 1, 2 and 3 and is omitted due to lack of space.

The proof of Theorem 3 is thus completed.

APPENDIX D

DERIVATION OF \( d_2(t_0, \delta, r) \) AND PROOFS OF LEMMAS 1, 2, AND 3

A. Derivation of \( d_2(t_0, \delta, r) \) (for \( t_0 \geq 0.5 \) and \( \delta > 0 \))

First, recall the definition of \( d_2(t_0, \delta, r) \) as
\[
d_2(t_0, \delta, r) = \lim_{\rho \to \infty} \frac{\log \left( P_{\alpha, \beta, \gamma}(\rho) \right)}{\log \rho},
\]
where the probability \( P_{\alpha, \beta, \gamma}(\rho) \) is defined by (15) as
\[
P_{\alpha, \beta, \gamma}(\rho) = \text{Pr} \left[ t_1 \log(1 + \alpha_0 \rho G_{02}) + t_0 \log \left( 1 + \alpha_0 \rho G_{02} + \frac{\gamma(G_{01}, \rho)\alpha_0 \rho G_{01}}{\gamma(G_{01}, \rho) + \Delta^2(\rho)\sqrt{\gamma(G_{01}, \rho)}} \right) \leq \mathcal{R}(\rho), \right. \\
\left. \mathcal{E}, \mathcal{F}, \mathcal{S} \right],
\] (74)
where $\gamma(G_{01}, \rho) = \frac{(1 + \alpha_0 \rho G_{01} - \Delta^2(\rho))^2}{(1 + \alpha_0 \rho G_{01})}$, and where events $E$, $S$ and $F$ are defined by (4), (8) and (10) respectively. Note that $\gamma(G_{01}, \rho)$ is positive since event $S$ i.e., $1 + \alpha_0 \rho G_{01} \geq \Delta^2(\rho)$, is realized. Furthermore, we can check that the following result holds.

$$\frac{\gamma(G_{01}, \rho)}{\gamma(G_{01}, \rho) + \Delta^2(\rho) \sqrt{\gamma(G_{01}, \rho)}} = \frac{1}{1 + \Delta^2(\rho)} = \rho^{-\delta^+}. \quad (75)$$

In the following, we assume that $R(\rho) = r \log \rho$ in accordance with (1), and we define as in [3] the exponential order associated with channel $H_{ij}$ as $a_{ij} = -\frac{\log G_{ij}}{\log \rho}$. We can easily verify that $a_{ij}$ is a Gumbel distributed random variable with the probability density function $f_{a_{ij}}(a) = \log \rho e^{a - e^{-a \log \rho}}$. By plugging $G_{01} = \rho^{-a_{01}}$ into (4), the probability of the event $\bar{E}$ i.e., $t_0 \log(1 + \alpha_0 \rho G_{01}) > R(\rho)$, can be written as

$$\Pr[\bar{E}] = \Pr\left[1 - a_{01}^+ \leq \frac{r}{t_0}\right]. \quad (76)$$

Similarly, we can verify that the probability of event $\bar{F}$ i.e., $t_1 \log \left(1 + \frac{\log \rho G_{12}}{\alpha_0 \rho G_{02} + 1}\right) > Q(\rho) t_0$, satisfies

$$\Pr[\bar{F}] = \Pr\left[\left(1 + \left(1 - \frac{r}{t_0}\right)^+ - a_{12} - (1 - a_{02})^+\right)^+ \leq \frac{r}{t_1} - \frac{t_0}{t_1} \delta\right], \quad (77)$$

and that the probability of $S$ satisfies

$$\Pr[S] = \Pr[\delta \leq (1 - a_{01}^+)]. \quad (78)$$

By plugging $R(\rho) = r \log \rho, G_{01} = \rho^{-a_{01}}, G_{02} = \rho^{-a_{02}}, G_{12} = \rho^{-a_{12}}, (75), (76), (77)$ and (78) into (74), the following high SNR result holds for $\delta > 0$.

$$P_{o,2}(\rho) = \Pr\left[t_1 (1 - a_{02})^+ + t_0 (1 - \min(a_{02}, a_{01} + \delta))^+ < r, (1 - a_{01})^+ < \frac{r}{t_0}, \right.$$  

$$\left.\left(1 + \left(1 - \frac{r}{t_0}\right)^+ - a_{12} - (1 - a_{02})^+\right)^+ > \frac{r}{t_1} - \frac{t_0}{t_1} \delta, \delta \leq (1 - a_{01})^+\right], \quad (79)$$

or, equivalently,

$$P_{o,2}(\rho) = \int \mathcal{O} f_{a_{01}}(a_{01}) f_{a_{02}}(a_{02}) f_{a_{12}}(a_{12}) da_{01} da_{02} da_{12}, \quad (80)$$

where $f_{a_{ij}}(.)$ is the probability density function of $a_{ij}$ and

$$\mathcal{O} = \left\{(a_{01}, a_{02}, a_{12}) \in \mathbb{R}^3 \mid t_1 (1 - a_{02})^+ + t_0 (1 - \min(a_{02}, a_{01} + \delta))^+ < r, (1 - a_{01})^+ < \frac{r}{t_0}, \right.$$  

$$\left.\left(1 + \left(1 - \frac{r}{t_0}\right)^+ - a_{12} - (1 - a_{02})^+\right)^+ > \frac{r}{t_1} - \frac{t_0}{t_1} \delta, \delta \leq (1 - a_{01})^+\right\}. \quad (81)$$
Plugging the expression of $f_{a,j}(\cdot)$ given earlier into (80), $P_{\alpha,2}(\rho)$ can be written as

$$
P_{\alpha,2}(\rho) = \int_{O} (\log \rho)^{3} \rho^{-(a_{01} + a_{02} + a_{12})} e^{\rho - a_{01}} e^{\rho - a_{02}} e^{\rho - a_{12}} da_{01} da_{02} da_{12}.
$$

It can be shown (refer to [3]) that the term $(\log \rho)^{3}$ can be dropped from the latter equation without losing its exactness. Moreover, integration in the same equation can be restricted to positive values of $a_{01}, a_{02}$ and $a_{12}$. Define $O_{+} = \mathbb{R}_{+}^{3}$. The probability $P_{\alpha,2}(\rho)$ thus satisfies

$$
P_{\alpha,2}(\rho) = \int_{O_{+}} \rho^{-(a_{01} + a_{02} + a_{12})} da_{01} da_{02} da_{12},
$$

(82)

and the DMT $d_{2}(t_{0}, \delta, r)$ associated with $P_{\alpha,2}(\rho)$ can now be written [3] as

$$
d_{2}(t_{0}, \delta, r) = \inf_{(a_{01}, a_{02}, a_{12}) \in O_{+}} (a_{01} + a_{02} + a_{12}).
$$

(83)

In this appendix, the derivation of $d_{2}(t_{0}, \delta, r)$ will be done only in the case characterized by $t_{0} \geq 0.5$ and $\delta > 0$. The derivation in the case $\delta \leq 0$ or $t_{0} < 0.5$ follows the same approach.

Consider first the case $0 < \delta \leq 1 - \left(1 - \frac{r}{t_{0}}\right)^{+}$. The infimum in (83) can be computed by partitioning $O_{+}$ into subsets according to whether $a_{01}, a_{02}$ are smaller or larger than 1.

- $a_{01} > 1$. In this case, $(1 - a_{01})^{+} = 0$ and the fourth inequality in (81) reduces to $\delta \leq 0$. This result contradicts our assumption that $\delta > 0$. There is therefore no triples $(a_{01}, a_{02}, a_{12}) \in O_{+}$ such that $a_{01} > 1$.

- $a_{01} \leq 1, a_{02} > 1$. Since the third inequality in the definition of $O$ given by (81) contains the term $\left(1 + \left(1 - \frac{r}{t_{0}}\right)^{+} - a_{12} - (1 - a_{02})^{+}\right)^{+}$, then we should consider two categories of triples $(a_{01}, a_{02}, a_{12})$:
  - $1 + \left(1 - \frac{r}{t_{0}}\right)^{+} - a_{12} - (1 - a_{02})^{+} < 0$.
    For triples $(a_{01}, a_{02}, a_{12}) \in O_{+}$ under this category, the third inequality in (81) can be reduced to $\delta > \frac{r}{t_{0}}$, which contradicts the second and the fourth inequalities in (81). This category can be therefore dropped out.
  - $1 + \left(1 - \frac{r}{t_{0}}\right)^{+} - a_{12} - (1 - a_{02})^{+} \geq 0$.

Recall the first inequality in (81) i.e., $t_{1}(1 - a_{02})^{+} + t_{0}(1 - \min(a_{02}, a_{01} + \delta))^{+} < r$. Since $\delta \leq (1 - a_{01})^{+}$ due to the fourth inequality in (81), then $a_{01} + \delta \leq a_{01} + (1 - a_{01})^{+} = 1 \leq a_{02}$. The first inequality in (81) reduces thus to $a_{01} \geq \left(1 - \frac{r}{t_{0}}\right)^{+}$. We conclude that

$$
\inf_{a_{01}: a_{01} \leq 1, a_{02} > 1} (a_{01} + a_{02} + a_{12}) = 1 + \left(1 - \frac{r}{t_{0}}\right)^{+}.
$$

(84)

One can verify after some simple algebra that $\inf_{a_{01}: a_{01} \leq 1, a_{02} > 1} (a_{01} + a_{02} + a_{12}) = 1 + \left(1 - \frac{r}{t_{0}}\right)^{+}$ is always larger than $d_{1}(t_{0}, r)$ given by (52). Therefore, the term $\inf_{a_{01}: a_{01} \leq 1, a_{02} > 1} (a_{01} + a_{02} + a_{12})$ can be dropped from the latter equation without losing its exactness.
\(a_{12}\) never coincides with the minimum in \(d(t_0, \delta, r) = \min \{d_1(t_0, r), d_2(t_0, \delta, r), d_3(t_0, \delta, r), d_4(t_0, \delta, r)\}\). As a result, the argument of the infimum \(\inf_{(a_{01}, a_{02}, a_{12}) \in \mathcal{O}_+} (a_{01} + a_{02} + a_{12})\) coincides necessarily with a triple \((a_{01}, a_{02}, a_{12})\) from the following subset.

- \(a_{01} \leq 1, a_{02} \leq 1\). Two categories of triples \((a_{01}, a_{02}, a_{12})\) should be considered.
  - \(1 + \left(1 - \frac{r}{t_0}\right)^+ - a_{12} - (1 - a_{02})^+ < 0\).
    
    As done before, it is straightforward to verify that there is no triples \((a_{01}, a_{02}, a_{12}) \in \mathcal{O}_+\) that fall under this category.
  - \(1 + \left(1 - \frac{r}{t_0}\right)^+ - a_{12} - (1 - a_{02})^+ \geq 0\).

The third inequality in (81) leads in this case to

\[
a_{02} > \frac{r}{t_1} - \left(1 - \frac{r}{t_0}\right)^+ - \frac{t_0}{t_1} \delta. \tag{85}
\]

In order to evaluate the first inequality in (81), two subcategories of triples \((a_{01}, a_{02}, a_{12})\) should be further examined.

1) \(a_{02} < a_{01} + \delta\). For triples \((a_{01}, a_{02}, a_{12}) \in \mathcal{O}_+\) under this category, the first inequality in (81) leads to \(a_{02} > (1 - r)^+\).

2) \(a_{02} \geq a_{01} + \delta\). The first inequality results in this case in \(a_{02} + \frac{t_0}{t_1} a_{01} > \frac{1-r}{t_1} - \frac{t_0}{t_1} \delta\).

Referring to Figures 1 and 2 reveals that \(\inf_{a_{01} \leq 1, a_{02} \leq 1} (a_{01} + a_{02} + a_{12})\) coincides with the rhs of (53). We have thus proved that \(d_2(t_0, \delta, r)\) is indeed given by (53).

\[a_{12}\]

Now consider the case \(\delta > 1 - \left(1 - \frac{r}{t_0}\right)^+\) in order to prove that (54) holds. To that end, refer to the second
and the fourth inequalities in the definition of \( \emptyset \) given by (81), that is \( (1-a_{01})^+ < \frac{r}{t_o} \) and \( \delta \leq (1-a_{01})^+ \).

Note that \( (1-a_{01})^+ \leq 1 \) since \( a_{01} > 0 \). A necessary condition for \( a_{01} \) to satisfy the second and the fourth inequalities in (81), and consequently to belong to \( \emptyset_+ \) is thus \( \delta \leq \min \left\{ 1, \frac{r}{t_o} \right\} = 1 - \left( 1 - \frac{r}{t_o} \right)^+ \).

This means that if we choose \( \delta \) such that \( \delta > 1 - \left( 1 - \frac{r}{t_o} \right)^+ \), the set \( \emptyset_+ \) will be empty. In this case, \( P_{o,2}(\rho) = 0 \) for sufficiently large \( \rho \). In other words, there exists \( \rho_0 > 0 \) such that \( \forall \rho \geq \rho_0 \), the event \( \emptyset \& S \) cannot be realized and the relay will not be able to quantize, reducing the DoQF to a classical DF scheme. The corresponding DMT \( d_2(t_0, \delta, r) \) will have no effect in this case on the final DMT of the protocol. We can give it for convenience the value \( d_2(t_0, \delta, r) = 2(1-r)^+ \), which is the upper-bound on the DMT of any single-relay protocol.

**B. Proof of Lemma 1**

Assume that parameters \( t_0 \) and \( \delta \) of the DoQF protocol are fixed such that \( t_0 = t_{0,DF}(r) \) and \( \delta = 1 - \left( 1 - \frac{r}{t_{DF}(r)} \right)^+ = \frac{r}{t_{DF}(r)} \), where \( t_{0,DF}(r) \) is defined by (63). In this case, equations (52), (53), (57) and (58) lead to \( d_1(t_0, r) = d_4(t_0, \delta, r) = d_{0,DF}^*(r) \) and \( d_2(t_0, \delta, r) = d_3(t_0, \delta, r) = 2(1-r)^+ \), meaning that \( d(t_0, \delta, r) = d_{0,DF}^*(r) \).

We conclude that the DoQF can be reduced to have the performance of DF by choosing \( t_0 = t_{0,DF}(r) \) and \( \delta = \frac{r}{t_{0,DF}(r)} \). The final DMT \( d_{0,DF}^*(r) \) of the DoQF is therefore necessarily greater or equal to \( d_{0,DF}^*(r) \). The proof of Lemma 1 is thus completed.

**C. Proof of Lemma 2**

Proving Lemma 2 requires proving that the following three inequalities hold: \( r \leq t_{0,DoQF}^*(r) \), \( t_{0,DoQF}^*(r) \leq t_{0,DF}^*(r) \) and \( 0.5 \leq t_{0,DoQF}^*(r) \). Let us begin with the proof of the inequality \( r \leq t_{0,DoQF}^*(r) \). Assume to the contrary that \( r > t_{0,DoQF}^*(r) \). In this case, \( d_3(t_{0,DoQF}^*(r), \delta_{DoQF}^*(r), r) = 0 \) due to (57). This implies that the DMT of the DoQF satisfies \( d(t_{0,DoQF}^*(r), \delta_{DoQF}^*(r), r) = d_3(t_{0,DoQF}^*(r), \delta_{DoQF}^*(r), r) = 0 \), which is in contradiction with Lemma 1. We conclude that \( r \leq t_{0,DoQF}^*(r) \) holds true.

We now show that the inequality \( t_{0,DoQF}^*(r) \leq t_{0,DF}^*(r) \) also holds true. For that sake, note that the DMT \( d_{0,DF}^*(r) \) of DF given by (62) can be written as a function of \( t_{0,DF}^*(r) \) defined by (63):

\[
d_{0,DF}^*(r) = 2 - \frac{r}{1 - t_{0,DF}^*(r)} = d_1(t_{0,DF}^*(r), r),
\]

where the second equality in (86) can be easily checked by referring to (52). On the other hand,

\[
d_1(t_{0,DoQF}^*(r), r) \geq d_{0,DoQF}^*(r)
\]
due to (50). Furthermore, Lemma 1 states that

$$d_{\text{DoQF}}^*(r) \geq d_{\text{DF}}^*(r).$$

(88)

Combining (86), (87) and (88) leads to $d_1 \left(t_{0,\text{DoQF}}^*(r), r\right) \geq d_1 \left(t_{0,\text{DF}}^*(r), r\right)$. Since $d_1(t_0, r) = 2 - \frac{r}{1-t_0}$, we conclude that $t_{0,\text{DoQF}}^*(r) \leq t_{0,\text{DF}}^*(r)$ holds.

In order to prove that inequality $t_{0,\text{DoQF}}^*(r) \geq 0.5$ holds, we will show that the best DMT that can be achieved with $t_0 < 0.5$ i.e., $\max_{t_0 < 0.5} d(t_0, \delta, r)$, is less or equal to the DMT that can be achieved by choosing $t_0 \geq 0.5$. It can be shown after some algebra that

$$\forall u \geq 0.5, \forall v < 0.5, \quad d_2(v, \delta, r) \leq d_2(u, \delta, r),$$

where $d_2(u, \delta, r)$ is given by (53) and $d_2(v, \delta, r)$ is given by (56). Furthermore, it is straightforward to show that functions $t \mapsto d_3(t, \delta, r)$ and $t \mapsto d_4(t, \delta, r)$ defined respectively by (57) and (58) are increasing w.r.t $t$. Finally, since $d_1(v, r) = 2(1 - r)^+$ for any $v < 0.5$ due to (52), then $d(v, \delta, r) = \min\{d_2(v, \delta, r), d_3(v, \delta, r), d_4(v, \delta, r)\}$. Putting all pieces together, we conclude that

$$\forall u \geq 0.5, \forall v < 0.5, \quad d(v, \delta, r) \leq d(u, \delta, r),$$

which in turn means that $t_{0,\text{DoQF}}^* \geq 0.5$.

**D. Proof of Lemma 3**

Lemma 3 states that the following two inequalities hold true for $r > 0.25$:

$$\delta_{\text{DoQF}}^*(r) < 1 - \left(1 - \frac{r}{t_{0,\text{DoQF}}^*(r)}\right)^+ \quad \text{and} \quad 0 < \delta_{\text{DoQF}}^*(r).$$

Recall from our discussion in Appendix D-A that the first inequality is a necessary condition for the DMT of the DoQF protocol to be greater or equal to the DMT of DF. We thus only need to prove the second inequality. To that end, we will resort to Lemma 1 which implies that

$$d_3 \left(t_{0,\text{DoQF}}^*(r), \delta_{\text{DoQF}}^*(r), r\right) \geq d_{\text{DF}}^*(r),$$

(89)

where $d_3 \left(t_{0,\text{DoQF}}^*(r), \delta_{\text{DoQF}}^*(r), r\right) = 4 + \frac{t_{0,\text{DoQF}}^*(r)}{1-t_{0,\text{DoQF}}^*(r)}\delta_{\text{DoQF}}^*(r) - \left(4 + \frac{t_{0,\text{DoQF}}^*(r)}{1-t_{0,\text{DoQF}}^*(r)}\right)\frac{r}{t_{0,\text{DoQF}}^*(r)}$ due to (70). Consider first the case $\frac{\sqrt{3} - 1}{\sqrt{3} + 1} < r \leq 1$. In this case, $d_{\text{DF}}^*(r) = (1 - r)(2 - r)$ due to [4]. Inequality (89) is therefore equivalent to

$$4 + \frac{t_{0,\text{DoQF}}^*(r)}{1-t_{0,\text{DoQF}}^*(r)}\delta_{\text{DoQF}}^*(r) - \left(4 + \frac{t_{0,\text{DoQF}}^*(r)}{1-t_{0,\text{DoQF}}^*(r)}\right)\frac{r}{t_{0,\text{DoQF}}^*(r)} \geq (1 - r)(2 - r).$$
It is straightforward to show that the above inequality is equivalent to

\[
\frac{t^*_0,DoQF(r)}{1 - t^*_0,DoQF(r)} \delta^*_{DoQF}(r) \geq r^2 + \left( \frac{4}{t^*_0,DoQF(r)} + \frac{1}{1 - t^*_0,DoQF(r)} - 3 \right) r - 2 .
\] (90)

One can check after some algebra that the rhs of (90) is strictly positive for \( \sqrt{5} - 1 < r \leq 1 \). We conclude that \( \delta^*_{DoQF}(r) > 0 \) on this interval. The proof of the strict positivity of \( \delta^*_{DoQF}(r) \) for \( 0.25 < r \leq \frac{\sqrt{5} - 1}{\sqrt{5} + 1} \) can be done without difficulty in the same way, completing the proof of Lemma 3.

REFERENCES


